A Herding Model with Preferential Attachment and Fragmentation

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Abstract

We introduce and solve a model that mimics the herding effect in financial markets when groups of agents share information. The number of agents in the model is growing and at each time step either (i) with probability p an incoming agent joins an existing group, or (ii) with probability $1-p$ a group is fragmented into individual agents. The group size distribution is found to be power-law with an exponent that depends continuously on p. A number of variants of our basic model are discussed. Comparisons are made between these models and other models of herding and random growing networks.

Keywords: Fragmentation; Preferential Attachment; Herding; Financial Markets

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I. INTRODUCTION

Empirical studies of financial price-data on short time scales have revealed that the price variations of various assets, indices and currencies have fat-tails [1,2]. These tails have been shown to be power-law with an exponent in the distribution of returns close to 4 in a number of different markets [3,4]. Beyond this power-law behaviour the distribution crosses over to an exponential decay or to a steeper power-law [3].

It is believed that this behaviour is brought about by a herding effect in which groups of agents all behave in the same way. Cont and Bouchaud [5] introduced a model of randomly connected agents to investigate this herding effect. In [5] agents are connected with probability p and agents that are part of the same group share information and make the same decisions. The parameter p is tuned to be close to the percolation threshold in order to obtain a power-law distribution of group sizes. This in turn leads to a power-law distribution of returns.

In [6] an extension to this picture was introduced in which, instead of being static, the network of agents evolves dynamically as decisions are being made or as agents exchange information. In this model, which was solved exactly in [7], at each time step either two groups of agents shared their information and were aggregated, or a group of agents trade, using their information, and the group is then fragmented.

In this paper we introduce an alternative kinetic model for this phenomenon. In our model, as in [6,7], when a group of agents trade, they use up their shared information, and are *fragmented* to become individual agents again. However, we allow new agents to enter the system and join the existing groups. When this occurs the new agents join a group of size k with a rate proportional to k, so that large groups grow more quickly than small ones. This seems reasonable as one would expect a new agent to be more likely to come into contact with a member of a large group than with a member of a smaller group. This process is called *preferential attachment* in the theory of random growing networks where an incoming node is more likely to connect to a node with a high degree [8,9].

In Sec. 2 we present our model and find an exact analytical expression for the group size distribution. In Sec. 3 we introduce a generalised version of our initial model and a few particular cases are solved. We discuss our work and draw some conclusions in Sec. 4.

II. THE MODEL

We introduce a model in which at each time step one of two events can occur. With probability p an agent is added to the system and joins a group of size k with a rate proportional to k. Alternatively, with probability $q = 1 - p$ a group is selected at random, with a rate independent of the size of the group, and the group is fragmented into individual agents. Consequently the number $n_k(t)$ of groups of size $k > 1$ at time t evolves like

$$\frac{dn_k(t)}{dt} = \frac{p}{M(t)}\left[(k-1)n_{k-1} - kn_k\right] - q\frac{n_k}{N(t)} \tag{1}$$

and the number of groups with only one agent, or equivalently the number of agents in a group on their own, behaves like

$$\frac{dn_1(t)}{dt} = -p\frac{n_1}{M(t)} + \frac{q}{N(t)}\sum_{k=2}^{\infty} kn_k. \tag{2}$$

In these equations

$$N(t) = \sum_{k=1}^{\infty} n_k(t) \tag{3}$$

represents the number of groups and

$$M(t) = \sum_{k=1}^{\infty} kn_k(t) \tag{4}$$

is the number of agents in the system. The first term on the right hand side of Eq.(1) describes the addition of a new agent to an existing group and the last term describes the fragmentation of a group of size k into k groups of size 1. In Eq.(2) the first term on the right hand side is the destruction of free agents caused by the arrival of a new agent and the second, summation, term is the creation of individual agents by the fragmentation of other groups. Using rate equations (1) and (2) it is a simple matter to show that

$$\frac{dN(t)}{dt} = (1-p)\left[\frac{M(t)}{N(t)} - 1\right] \tag{5}$$

and

$$\frac{dM(t)}{dt} = p. \tag{6}$$

Equation (5) represents the fact that with probability $1-p$, *on average*, the number of groups increases by the average group size minus one. Similarly, Eq.(6) indicates that with probability p the number of agents increases by 1. The form of Eqs.(1,2,5,6) suggests that the solution for $n_k(t)$, for $k = 1, 2, ...$, is linear in time for large t. In this limit we can solve Eqs.(5,6) to yield

$$N(t) = \alpha t \qquad \text{and} \qquad M(t) = pt \tag{7}$$

where

$$\alpha = \frac{1-p}{2}\left[\sqrt{4\frac{p}{1-p}+1} - 1\right]. \tag{8}$$

Writing

$$n_k(t) = tc_k \tag{9}$$

we find that for $k > 1$

$$c_k = [(k-1)c_{k-1} - kc_k] - \frac{1-p}{\alpha}c_k. \tag{10}$$

Using an initial condition obtained from Eq.(2) we can solve Eq.(10) to give

$$c_k = \frac{p(1-p)}{\alpha}\Gamma(\beta)\frac{\Gamma(k)}{\Gamma(k+\beta)} \tag{11}$$

where

$$\beta = 2 + \frac{1-p}{\alpha} = 2\left[\frac{\sqrt{4\frac{p}{1-p}+1}}{\sqrt{4\frac{p}{1-p}+1} - 1}\right]. \tag{12}$$

As $k \to \infty$,

$$c_k \sim k^{-\beta}. \tag{13}$$

Varying p in $[0,1]$, we find that β can take any value $\beta > 2$. As $p \to 0$ then $\beta \to \infty$ and when $p \to 1$ then $\beta \to 2$. At $p = q = 1/2$, $\beta = (\sqrt{5}+1)\sqrt{5}/2$.

The distribution of returns, $R(k)$, is the distribution of the relative difference between the number of buyers and the number of sellers. This can be obtained by realising that a group of agents of size k trades with rate kc_k. If we assume that the traded amount is proportional to the number of agents in the group, then we find that

$$R(k) \sim kc_k \sim k^{-\delta} \tag{14}$$

where $\delta = \beta - 1$. By varying p we can allow δ to take any value greater than 1.

III. GENERALISATIONS

The above models can be generalised by allowing incoming agents to join groups of size k with rate A_k and fragmenting groups of size k with rate B_k. In the previous section we considered the model with $A_k = k$ and $B_k = 1$. The rate equations (Eqs.(1,2)) can be rewritten

$$\frac{dn_k(t)}{dt} = \frac{p}{A(t)}\left[A_{k-1}n_{k-1} - A_k n_k\right] - qB_k\frac{n_k}{B(t)} \tag{15}$$

and

$$\frac{dn_1(t)}{dt} = -pA_1\frac{n_1}{A(t)} + \frac{q}{B(t)}\sum_{k=2}^{\infty} kB_k n_k. \tag{16}$$

In these equations

$$A(t) = \sum_{k=1}^{\infty} A_k n_k(t) \tag{17}$$

and

$$B(t) = \sum_{k=1}^{\infty} B_k n_k(t). \tag{18}$$

The terms in Eqs.(15,16) have the same meaning as those in Eqs.(1,2). Obviously we cannot solve the above equations for general A_k and B_k so instead we consider four simple special cases.

Model A $A_k = B_k = 1$. This is the simplest model in this class. With constant interaction kernels one would expect the power-law behaviour to disappear. This is indeed what happens and we easily find that

$$c_k = (1-p)\left[\frac{p}{\alpha+1}\right]^k \tag{19}$$

with α given in Eq.(8).

Model B $A_k = k + \lambda$ and $B_k = 1$. Here we keep the fragmentation rate constant but adjust the preferential attachment by adding a constant $\lambda > -1$ to the rate. As in the work on random growing networks, [8], we retain a power-law group size distribution, but with a modified exponent

$$\beta = 2\frac{\sqrt{4\frac{p}{1-p}+1}+\lambda}{\sqrt{4\frac{p}{1-p}+1}-1}. \tag{20}$$

As before, varying p in $[0,1]$ gives values of β between 2 and ∞.

Model C $A_k = B_k = k$. In this model the preferential attachment and the fragmentation proceed at the same rate and we find that

$$c_k \sim \frac{\Gamma(k)}{\Gamma(k+p+1)}p^k \tag{21}$$

so that the distribution is a power-law with an exponential cut-off for large k.

Model D $A_k = 1$ and $B_k = 1/k$. Here the ratio $A_k/B_k \sim k$ as in our first model. Hence one might anticipate that the group size distribution would be power-law. In fact

$$c_k = c_1\frac{\Gamma(k+1)\Gamma(2+\frac{\Delta+1-p}{\Delta+p})}{\Gamma(k+\frac{\Delta+1-p}{\Delta+p}+1)}\left[\frac{p}{\Delta+p}\right]^{k-1} \tag{22}$$

where $\Delta(p)$ is determined by

$$\Delta = \sum_{k=1}^{\infty} c_k. \tag{23}$$

As in Model C, this version exhibits a power-law with an exponential cut-off. The cut-off remains for all values of p.

IV. DISCUSSION

We have introduced and solved a kinetic model of herding which exhibits a power-law distribution of group sizes for all its parameter values. The exponent of the power-law can be varied continuously. Other systems introduced to investigate the herding phenomenon, [5–7], required a tuning of their parameters to obtain a power-law. Furthermore, the exponent obtained in [5–7] is much smaller than those obtained empirically for this phenomenon [2–4]. We would require a value of $p \approx 1/3$ to give a distribution of returns with an exponent ≈ 4, Eq.(14), to match the empirical results [2–4].

Our system is open and growing with new agents are continually entering the system. These new agents join an existing group with preferential attachment. This is in contrast to [6,7], where there were a fixed number of agents and the groups increased in size by coagulation. In this sense our model is reminiscent of random growing systems [8,9], where power-law distributions are often found. We have been able to make use of similar solution techniques to those employed on network models [9].

A consideration of a few variants of our model indicates that a delicate balance between fragmentation and attachment is required to obtain power-laws over a range of parameter values. An exponential group size distribution is found for the constant coefficient model and a power-law mediated by an exponential cut-off is obtained when $A_k = A_k = k$ and $A_k = 1, B_k = 1/k$. It is only when we keep the same rate of fragmentation, and enhance the preferential attachment with an additive constant, that power-laws are again recovered. This suggests that preferential attachment is a necessary condition for power-laws in these systems.

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