# Self-organized Model for Modular Complex Networks : Division and Independence

D.-H. Kim,[1] G. J. Rodgers,[2] B. Kahng,[1] and D. Kim[1]

[1]*School of Physics and Center for Theoretical Physics, Seoul National University, Seoul 151-747, Korea*
[2]*Department of Mathematical Sciences, Brunel University, Uxbridge, Middlesex, UB8 3PH, United Kingdom*
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We introduce a minimal network model which generates a modular structure in a self-organized way. To this end, we modify the Barabási-Albert model into the one evolving under the principle of division and independence as well as growth and preferential attachment (PA). A newly added vertex chooses one of the modules composed of existing vertices, and attaches edges to vertices belonging to that module following the PA rule. When the module size reaches a proper size, the module is divided into two, and a new module is created. The karate club network studied by Zachary is a prototypical example. We find that the model can reproduce successfully the behavior of the hierarchical clustering coefficient of a vertex with degree $k$, $C(k)$, in good agreement with empirical measurements of real world networks.

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Recently, considerable effort has been made to understand complex systems in terms of random graphs, consisting of vertices and edges [1, 2, 3, 4]. Such complex networks exhibit many interesting emerging patterns as follows: First, the degree distribution follows a power-law, $P(k) \sim k^{-\gamma}$, where the degree is the number of edges connecting to a given vertex [5]. Such networks, called scale-free (SF), are ubiquitous in the real world. To illustrate such SF behavior in the degree distribution, Barabási and Albert (BA) [5] introduced an *in silico* model: Initially, fully-connected $m_0$ vertices exist in a system. At each time step, a vertex is newly added and connects to $m$ existing vertices, which are chosen with a probability linearly proportional to the degree of target vertex. Such a selection rule is called the preferential attachment (PA) rule.

Secondly, many real world networks have modular structures within them. Modular structures form geographically in the Internet [6], functionally in metabolic [7] or protein interaction networks [8], or following social activities in social networks [9, 10]. In these modular complex networks, the hierarchical clustering coefficient of a vertex with degree $k$, denoted by $C(k)$, behaves as $C(k) \sim k^{-\beta}$ [7, 11], where the clustering coefficient is defined as the ratio of the number of triangles connected to a given vertex to the number of triples centered on that vertex. Also the clustering coefficient averaged over all vertices is independent of system size $N$. In the BA model, however, $C(k)$ is independent of $k$, but depends on $N$ [2, 11], because the BA model does not contain modules. To understand the behavior of $C(k)$, a deterministic hierarchical model was introduced by Ravasz and Barabási [11], in which $C(k) \sim k^{-1}$ and the clustering coefficient $C$ is independent of $N$ [12]. While it is important to understand the mechanism for the formation of such modular structure through an *in silico* model, few models have been studied, and none in which the modules were generated in a self-organized way. Thus it is our goal of this paper to introduce such a model.

Thirdly, the degree-degree correlation in real world networks is nontrivial. The nontrivial behavior is measured in terms of the mixing coefficient $r$ [13], a Pearson correlation coefficient between the remaining degrees of the two vertices on each side of an edge, where the remaining degree means the degree of that vertex minus one. Complex networks can be classified according to the mixing coefficient $r$ into three types, having $r < 0$, $r \approx 0$, and $r > 0$, called the dissortative, the neutral, and the assortative network, respectively [13]. An assortative or dissorative network can also be identified by a quantity, denoted by $\langle k_{\rm nn}\rangle(k)$, the average degree of a neighboring vertex of a vertex with degree $k$ [14]. For the assortative (dissortative) network, $\langle k_{\rm nn}\rangle(k)$ increases (decreases) with increasing $k$, i.e., a power law $\langle k_{\rm nn}\rangle(k) \sim k^{-\nu}$ is satisfied where $\nu$ is negative (positive) for the assortative (dissortative) network [14].

In this paper, we are interested in modelling modular complex networks, in particular, forming in a self-organized way. In social networks, modules represent the communities each individual belongs to, which may evolve as time passes. The karate club (KC) network, originally proposed by Zachary [15], is an example of a social network containing community structures. Recently, Newman and Girvan [9] studied the KC network to test a new algorithm for clustering communities [9, 16]. Here we notice that the KC network
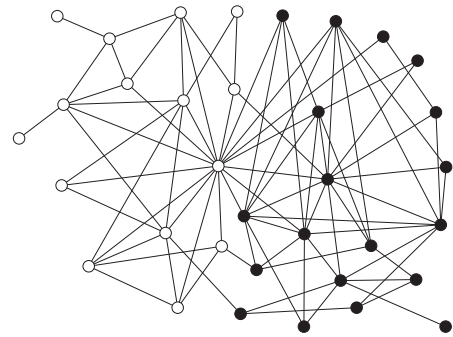


FIG. 1: A snapshot of the model network with parameters $N = 34$, $m_0 = 4$ and $n = 17$, looking similar to the Karate club network proposed by Zachary. Here two groups are identified by ($\circ$) and ($\bullet$).

contains an important ingredient, division and independence, needed for the formation of modular structure, in addition to growth and PA principles as noticed in the BA model. Thus we introduce a network model evolving by such principles, and perform numerical simulations for large system size. Indeed, we find that the model exhibits a characteristic feature of modular structure, $C(k) \sim k^{-1}$ as much as those for empirical data.

To be specific, the main dynamic process of the evolution of the KC network is as follows. In a KC, a disagreement develops between the administrator of the club and the club's instructor as time goes on, ultimately resulting in the instructor leaving (division) and founding a new club (independence), accompanied by about half the original club's members. This generic feature of division and independence can be observed in many other social communities such as schools, companies, churches, clubs, parties, etc. For example, in the coauthorship network, a graduate student publishes papers with her/his thesis advisor, so that they are connected in a coauthorship network. When she/he graduates and becomes a professor in another school (division), she/he also get her/his own students, creating a new group (independence).

To model the evolution of the KC network, we modify the BA model by assigning a color to each vertex. The color assigned to each vertex indicates the group the vertex belongs to. The dynamic rule of our model is as follows:

(i) **BA model (Growth and PA)** : Initially, there exist $m_0$ vertices. They are fully connected. Each vertex $i$ is assigned the same index of color $\mu_i = 1$. Thus the total number of distinct colors $q = 1$. At each time step, a vertex is introduced and connects to $m$ existing vertices following the PA rule. Here $m$ is not fixed, but is distributed uniformly among integers in the range $[1, m_0]$. The new vertex $j$ is also assigned the index of color $\mu_j = 1$ and this process is repeated until the number of vertices reaches $n$, a cutoff of the group size. This process defines the first group $q = 1$.

(ii) **Division and independence** : Then we identify the two vertices $i$ and $j$ among the group $q$ with the largest and the second largest degree, respectively, for division and independence. Then the vertex $j$ declares independence and changes its color to a new one, i.e., $\mu_j = q + 1$. Then, each remaining vertex $k(\neq i, j)$ in the group having the same color as vertex $i$ measures the distances $d(k, i)$ and $d(k, j)$ to the vertices $i$ and $j$, respectively. If $d(k, i) \leq d(k, j)$, then the vertex $k$ keeps the index of color as it is, otherwise, it changes its index of color to that of $j$. Then the system comprises of $q + 1$ different groups, and then $q + 1 \rightarrow q$, by definition. So the newest group has the new color $q$.

(iii) **Growth and PA again** : If $q > 1$, then a newly added vertex $\ell$ chooses one of $q$ colors, say $\mu_\ell$, with equal probability, and $m$, the number of outgoing links, also randomly from the integers $1, \ldots, m_0$. Then $m$ existing vertices are chosen in the group with the color $\mu_\ell$ following the PA rule, and $m$ edges are inserted between them and the new node. This process is repeated until the number of vertices of any group reaches $n$ again. After then, we repeat the step of division and indepen-

dence (ii) in that group only.

The network constructed in this way is shown in FIG. 1 based on the same number of vertices as the empirical data of the KC network. The structure of the model is different from the BA model due to the presence of modular structure. Note that in our model, one vertex may transfer from one group to another as time goes on, that is, a vertex can change its color as it transfers to a new group. This characteristic is different from that of the $q$-component static model proposed by the current authors [17], where each individual belongs concurrently to $q$ different groups such as high school alumni, college alumni, company, etc. Those two models may reflect different aspects of our social community.

Based on the empirical data by Zachary, we obtain topological properties of the KC network, which are listed in TABLE 1 and FIG. 2. Until now, it has been believed that social networks are generally assortative [13, 18]. But, in "division and independence" social networks such as the KC network, each element is connected to the others in a hierarchical way, without any mediator, leading to a dissortative network, as shown in TABLE 1 and FIG. 2. Since different colors represent distinct modules [7, 11] or communities [9], connections are very tight. Thus it is expected that the clustering coefficient $C$ is non-trivially large [18]. TABLE I shows the dissortativity and the highly-clustered nature of the KC network and our model. Agreements between the two are excellent except for the mixing coefficient $r$. Note that the $r$ value of the model is not close to zero although we used the BA-type random attachment rule. It should be noted that the large value of $C$ is obtained in a self-organized way. FIG. 2 shows that the degree distribution, $P(k) \sim k^{-2.7}$, the hierarchical clustering coefficient, $C(k) \sim k^{-1.0}$, and $\langle k_{\mathrm{nn}} \rangle(k) \sim k^{-0.5}$ of the KC network, which are also in good agreement with those obtained from the present model network. Such agreements indicate that our simple model captures the essential topology of the KC network.

More generally, we investigated the topological properties of our model network for large $N$ with various $n$. In FIG. 3, we consider the case of $N = 10000$, $m_0 = 4$, and $n = 500$. FIG. 3(a) shows the degree distribution of our model. It seems that $P(k)$ follows a power law with the exponent $\gamma \approx 3.5$, but that there exists plateau behavior for large $k$. The plateau for large $k$ is caused by the artificially uniform cutoff of the group

| Name | $N$ | $\langle k \rangle$ | $d$ | $r$ | $C$ |
|---|---|---|---|---|---|
| Zachary's | 34 | 4.59 | 2.41 | -0.48 | 0.59 |
| Ours | 34 | 4.61 | 2.54 | -0.19 (-0.22) | 0.56 |

TABLE I: Mean degree $\langle k \rangle$, the diameter $d$, the assortativity coefficient $r$, and the clustering coefficient $C$ obtained from Zachary's KC network and from ours with parameter $N = 34$, $m_0 = 4$ and $n = 17$. All the numerical values for the model are averaged over ten configurations. Note that Zachary presumed that the edge between the administrator and the instructor of the club no longer hold upon division and independence. Following the Zachary's way, we obtain $r = -0.22$ in our model.
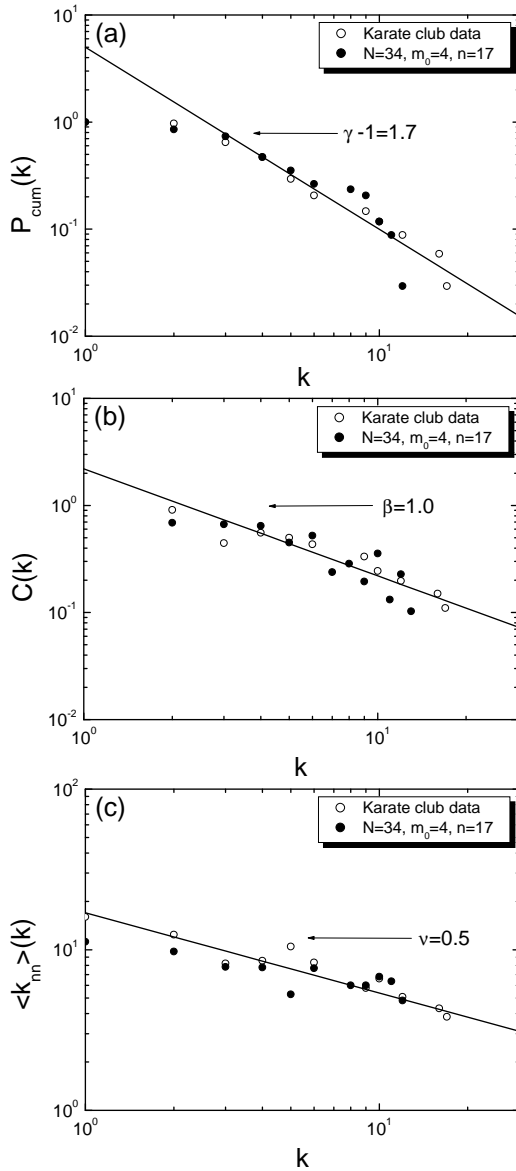
FIG. 2: Plots of the cumulative degree distribution $P_{\text{cum}}(k)$ (a), the clustering coefficient $C(k)$ (b), and $\langle k_{\text{nn}}\rangle(k)$ (c) versus degree $k$. In all, the empirical data and the data from the model are denoted by ($\circ$) and ($\bullet$), respectively. The parameters for the model network are the same as used in FIG. 1. Lines are drawn as a guide to the eye.
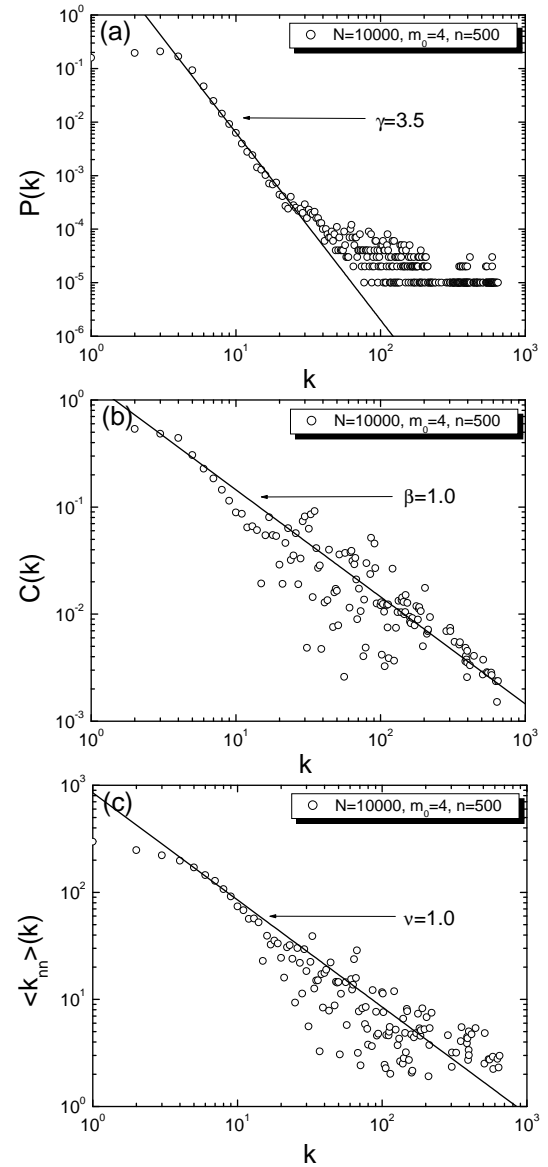
FIG. 3: (a) Plots of the degree distribution $P(k)$ (a), the clustering coefficient $C(k)$ (b), and $\langle k_{\text{nn}}\rangle(k)$ (c) versus degree $k$. The data in all figures are obtained with parameters $N = 10000$, $m_0 = 4$ and $n = 500$. In this case we obtain the mean degree $\langle k \rangle = 4.98$, the diameter $d = 4.87$, the assortativity coefficient $r = -0.24$, and the clustering coefficient $C = 0.42$.

size $n$, which should be modified to fir the empirical data, if available. If $n$ is not uniform, but is made stochastic following, for example, a power law, then the shape of the plateau would change accordingly. FIG. 3(b) shows the hierarchical clustering coefficient $C(k)$ behaving as $\sim k^{-1.0}$, which is in good agreement with the Ravasz-Barabási model [11]. FIG. 3(c) shows $\langle k_{\text{nn}}\rangle(k)$, showing a dissortative behavior. The exponent $\nu$ is somewhat different from the one measured in the small network in FIG. 2(c), because the size of $N = 34$ in FIG. 2 may be too small to measure the exponent $\nu$, as can be seen in small $k$ of FIG. 3(c). Also there occurs a plateau region for large $k$ in $\langle k_{\text{nn}}\rangle(k)$. The dissortative behavior ($\nu > 0$)

is caused by hierarchical organization inside a group.

FIG. 4(a) and (b) show the $n$-dependence of the hierarchical clustering coefficient $C(k)$. When $n$ is very small with respect to network size $N$, $C(k)$ behaves as $\sim k^{-1.0}$, but as $n$ increases to $N$, $C(k)$ deviates from the power law $C(k) \sim k^{-1.0}$. The $n = 10$ case shows the clear power law behavior. For $n = 100$, a scattered behavior occurs in the middle of the power law regime. This is found in the actor network (FIG. 3(a) of Ref. [11]). For $n = 500$, many points are scattered in the middle of the power law regime, which is similar to the empirical results from the Internet au-
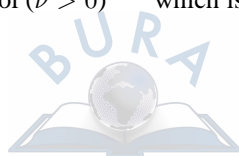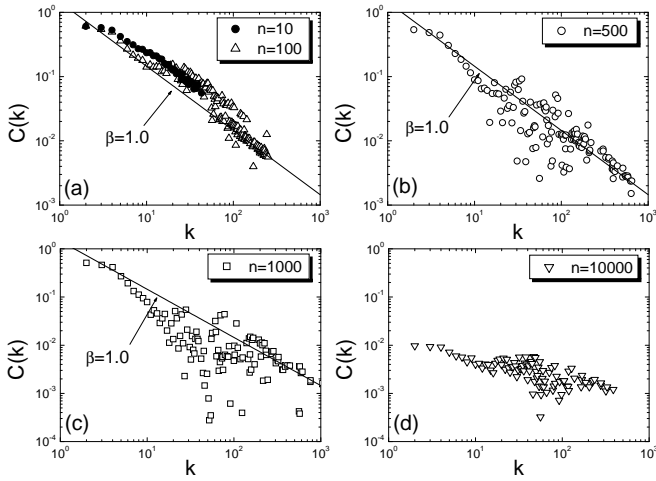
FIG. 4: The clustering coefficient $C(k)$ versus degree $k$ obtained with the parameters $N = 10000$, $m_0 = 4$, and $n = 10$ and $100$ (a), $500$ (b), $n = 1000$ (c) and $10000$ (d).
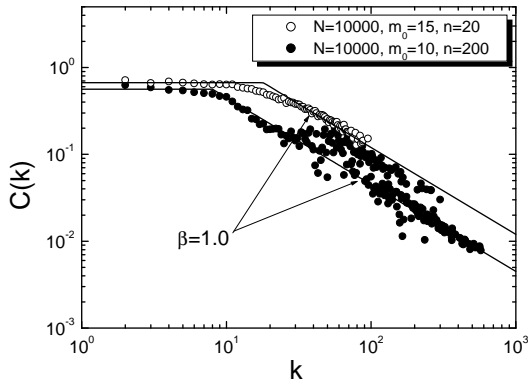


FIG. 5: The clustering coefficient $C(k)$ versus degree $k$ obtained under two selected conditions with large values of $m_0$.

tonomous system (FIG. 3(d) of Ref. [11]). For $n = 1000$, most of the data points for $C(k)$ are scattered more diversely with $k$, which is similar to the results from the World Wide Web (WWW) (FIG. 3(c) of Ref. [11]). When the group size $n$ approaches the network size $N$, $C(k)$ of our model reduces to that of the BA model (FIG. 2(b) of Ref. [11]).

FIG. 5 shows the $m_0$-dependence of the hierarchical clustering coefficient $C(k)$. For large $m_0$ values, we can see clearly that $C(k)$ has both a plateau regime from $k = 2$ to $k \simeq m_0$ and a power law regime satisfying $C(k) \sim k^{-1.0}$ beyond that degree. When $m_0$ approaches the group size $n$, such as when $m_0 = 15$ and $n = 20$, i.e., when vertices inside one module are nearly fully-connected, such a plateau with a $C$ value near 1.0 appears. We can thus say that the actor and language networks (FIG. 3(a) and 3(b) of Ref. [11]) have modules composed of nearly fully-connected vertices. Our model can thus explain most of the hierarchical clustering structures of real world networks qualitatively well, when the two parameters $m_0$ and $n$ are properly selected. As an ex-

ample, the case of $m_0 = 10$ and $n = 200$ of FIG. 5 shows a plateau regime as well as a scattered behavior in the middle of the power law regime, which are very similar to the actor network (FIG. 3(a) of Ref. [11]).

In conclusion, we have generalized the BA model by assigning a color to each vertex for the purpose of modelling modular complex networks in a simple way. The model evolves with time under the principle of division and independence, in a manner reminiscent of the KC network. Through this model, we confirmed the behavior of the hierarchical clustering coefficient, which is in accordance with the ones obtained from the deterministic hierarchical structure and the empirical data such as the Internet, the WWW, and the actor networks [11]. Also it was found that our model exhibits an dissortative mixing behavior as observed in the KC network. Our model can be modified in various ways, for example, diversifying the group size cutoff $n$, to fit real world networks. Finally, we suggest that the principle of division and independence could be used in constructing modular complex networks in various fields, for example, bio-complex networks, where the strong mutation of a gene may correspond to transferring from one group to another [19].

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