This article has been accepted for publication in a future issue of this journal, but has not been fully edited. Content may change prior to final publication. Citation information: DOI10.1109/TAC.2024.3378776, IEEE Transactions on Automatic Control

Recursive State Estimation for Discrete-Time Nonlinear Systems with Binary Sensors: A Locally Minimized Variance Approach

Yi-Bo Huang, Zidong Wang, Yong He and Min Wu

***Abstract*—In this paper, the locally-minimized-variance state estimation problem is investigated for a class of discrete-time nonlinear systems with Lipschitz nonlinearities and binary sensors. The output of each binary sensor takes two possible values (e.g. 0 and 1) in accordance with whether the sensed variable surpasses a prescribed threshold or not. The purpose of this paper is to design a state estimation algorithm such that an upper bound of the estimation error covariance is firstly guaranteed and then minimized at each sampling instant by properly designing the estimator gain. The valid information of sensed variables is extracted from binary measurements, and a novel state estimator is constructed in a recursive form, which is suitable for online computations. Moreover, a sufficient condition is established to ensure the exponential boundedness of the prediction error in the mean square sense. Finally, two examples are presented to verify the effectiveness of the proposed method.**

***Index Terms*—Recursive state estimation, binary sensors, variance constraints, difference equations, nonlinear systems.**

I. Introduction

The past several decades have witnessed a surge of research interest devoted to the estimation/filtering problems from various research communities including control engineering and signal processing, see e.g. [1], [8], [13], [14], [35], [39], [41], [42] and the references therein. Up to now, a large number of effective approaches have been proposed to deal with state estimation issues. According to performance indices, the existing estimation schemes can be generally divided into categories including, but not limited to, Kalman filtering (KF) algorithms [20], set-membership filtering algorithms [5], [23], and H_∞ filtering algorithms [40]. It has been well recognized that the celebrated KF algorithm is able to obtain an optimal estimate in the least mean square sense for linear systems subject to the Gaussian white noises [6], [12]. As for the frequently encountered nonlinearities in reality, the extended Kalman filtering (EKF) algorithm has been shown to be a practical way of tackling nonlinear state estimation problems, see e.g. [19], [21], [22].

In real-world engineering practice, it is often the case that the underlying systems suffer from fading measurements, parameter uncertainties, filter gain perturbations, as well as other network-induced incomplete information [27], [28], [37], [38]. With such kind of incomplete/imperfect information, traditional KF/EKF methods might be no longer applicable to the corresponding state estimation problems, let alone the guarantee of the *optimal* state estimation in the minimum-variance sense. As such, in the past decade, much research effort has been made to develop the so-called *locally-minimized-variance* estimation approach, whose main idea is to ensure an acceptable upper bound of the estimation error variance and then minimize such an upper bound by designing proper estimator parameters.

So far, there has been a rich body of elegant results reported on a locally-minimized-variance (sometimes called variance-constrained) estimation/filtering schemes, see e.g. [7], [18], [26], [29]. Among others, a variance-constrained filter has been designed in [34] for uncertain stochastic systems with missing measurements. In [11], a variance-constrained H_∞ filtering problem has been investigated for a class of nonlinear time-varying systems subject to multiple missing measurements. Moreover, the locally-minimized-variance state estimation problems have been studied in [16] and [17] for complex networks with various networked-induced phenomena.

It should be pointed out that, in the majority of the literature addressing the locally-minimized-variance state estimation problems for nonlinear systems, the nonlinear functions have been linearized by resorting to the Taylor series expansion and then expressed in the form of an uncertain model [21], [24], [45]. Nevertheless, if the process/measurement noise obeys a Gaussian distribution, such an uncertain model might be deficient due to the fact that the Gaussian noise is unbounded, and the resulting upper bound of the estimation error variances might not be accurate which, in turn, poses great limitations on the developed locally-minimized-variance estimation schemes. Therefore, it is of theoretical significance to shed some light on this issue via a rigorous mathematical analysis, which constitutes one of the motivations of our current investigation.

Owing to their distinctive advantages of low cost and simple installation, binary sensors have found extensive applications in a variety of industrial fields, see e.g. [9], [10], [15] and the references therein. Briefly speaking, a binary sensor can output two possible values (e.g. 0 and 1) according to whether or not the sensed variable surpasses a prescribed threshold, and this implies that the information provided by binary sensors is rather limited. Such a feature of *limited information* brings great challenges to the state estimation problem with binary sensors, and therefore an increasing research interest has been stirred to overcome such challenges.

Up to now, binary measurements have been widely utilized in practice for identification/estimation purposes. For instance, identification problems have been investigated in [32], [33] for systems with binary sensors. In [9], [10], target tracking problems with binary sensor networks have been addressed in the framework of the particle filtering (PF) method. In [44], a distributed fusion Kalman filtering problem under binary sensors has been investigated. For a class of discrete-time nonlinear cyber-physical systems with binary sensors, a secure PF problem has been studied in [30]. In [2], a maximum A posteriori probability approach has been applied to address the state estimation problem with binary measurements. A moving horizon estimation approach has been proposed in [4] for linear discrete-time systems with binary sensors, and stability analysis has also been provided. Moreover, in [43], convex optimization techniques have

This work was supported by the National Natural Science Foundation of China under Grants 61933007 and 62373333, the Royal Society of the UK, and the Alexander von Humboldt Foundation of Germany. *(Corresponding author: Yong He.)*

Yi-Bo Huang is with the College of Control Science and Engineering, Zhejiang University, Hangzhou 310027, China (e-mail: yibo.huang@zju.edu.cn).

Zidong Wang is with the Department of Computer Science, Brunel University London, Uxbridge, Middlesex, UB8 3PH, United Kingdom (e-mail: Zidong.Wang@brunel.ac.uk)

Yong He and Min Wu are with the School of Automation, China University of Geosciences, Wuhan 430074, China; with the Hubei Key Laboratory of Advanced Control and Intelligent Automation for Complex Systems, Wuhan 430074, China; and also with the Engineering Research Center of Intelligent Technology for Geo-Exploration, Ministry of Education, Wuhan 430074, China (e-mails: heyong08@cug.edu.cn; wumin@cug.edu.cn).

This article has been accepted for publication in a future issue of this journal, but has not been fully edited. Content may change prior to final publication. Citation information: DOI10.1109/TAC.2024.3378776, IEEE Transactions on Automatic Control

been employed to solve the fusion estimation problem with binary measurements. Nevertheless, when it comes to the nonlinear state estimation problem with binary sensors, the corresponding results under error variance constraints have been very few if not none.

Motivated by the above discussions, in this paper, we endeavor to design a state estimation scheme for a class of discrete-time nonlinear systems with binary sensors, that is, to estimate system states by utilizing binary observations. The primary purpose is to develop a state estimator such that an upper bound of the estimation error covariance is first obtained and then minimized at each time instant by designing an appropriate estimator gain. It is worth mentioning that such a research topic is nontrivial due to three technical difficulties identified as follows: *1) how to derive an analytical expression of the estimation error covariance (or its upper bound) given the fact that the measurement information provided by binary sensors is quite limited? 2) how to justify the validity of the linearization operation exploited in the locally-minimized-variance nonlinear state estimation schemes via a rigorous mathematical analysis? and 3) how to establish conditions under which the boundedness of the mean-square error is guaranteed?*

The primary novelties of this paper lie in the following three aspects.

1) A locally-minimized-variance approach is, for the first time, introduced to cope with the state estimation problem for discrete-time nonlinear systems with binary sensors.
2) A sufficient condition is established for the validity of the proposed linearization process in dealing with the system nonlinearities.
3) The state estimator gain is given in an explicit form, and a sufficient condition is provided to guarantee the boundedness of the prediction error in the mean square sense.

The remainder of this paper is structured as follows. Section II formulates the locally-minimized-variance nonlinear state estimation problem with binary sensors and gives some preliminaries. Section III presents the design scheme of the proposed state estimator and the corresponding performance analysis. Two examples are given in Section IV to demonstrate the effectiveness of the proposed state estimation algorithm. Finally, some conclusions are drawn in Section V.

Notations: The notations utilized in this paper are fairly standard. $\mathbb{R}^n$ and $\mathbb{R}^{n\times m}$ denote, respectively, the n-dimensional Euclidean space and the set of $n \times m$ real matrices. $\mathbb{S}_n^+$ represents the set of $n\times n$ positive definite matrices. $\mathbb{R}^+$ and $\mathbb{N}$ denote the sets of positive real numbers and non-negative integers, respectively. For a matrix A, A^{T} and A^{-1} represent, respectively, the transpose and the inverse of A. For a square matrix $A \in \mathbb{R}^{n\times n}$, we define $\mathrm{He}(A) = A + A^{\mathrm{T}}$. $\|\cdot\|$ is the notation of Euclidean norm. $\rho(A)$ stands for the maximum eigenvalue of the symmetric matrix A. For any real, symmetric matrices X and Y, the notation $X > Y$ (respectively, $X \geq Y$) means that $X - Y$ is positive definite (respectively, positive semi-definite). $\mathbb{E}\{x\}$ is the mathematical expectation of the stochastic variable x. $\mathrm{diag}\{a_1, a_2, \cdots, a_n\}$ denotes a block diagonal matrix with the ith block being a_i and all other entries being zero. We let δ be the Kronecker delta function, which satisfies that $\delta(a) = 1$ if $a = 0$ and $\delta(a) = 0$ otherwise.

II. Problem Formulation and Preliminaries

A. System Setup

Consider the following discrete-time nonlinear system:

$$x_{k+1} = f(x_k) + w_k \tag{1}$$

where $x_k \in \mathbb{R}^n$ is the state vector at time k. The initial state x_0 is a zero-mean Gaussian random vector with covariance $X_0 \geq 0$. $f(\cdot) : \mathbb{R}^n \to \mathbb{R}^n$ is a known, continuously differentiable nonlinear function with $f(0) = 0$ and satisfies the Lipschitz condition, i.e., there exists a constant l such that for any $a, b \in \mathbb{R}^n$, $\|f(a)-f(b)\| \leq l\|a-b\|$. The process noise $w_k \in \mathbb{R}^n$ is a random vector with zero mean and variance Q.

In this paper, binary sensors are adopted to monitor the system. The measurement equation of the ith sensor is given by

$$y_{i,k} = C_i x_k + v_{i,k}, \quad i = 1, 2, \ldots, m, \tag{2}$$

where $y_{i,k} \in \mathbb{R}$ is the system output taken by sensor i at time k. The measurement noise $v_{i,k} \in \mathbb{R}$ is a random variable with zero mean and variance R_i. The ith sensor processes its measurement according to

$$\breve{y}_{i,k} = \begin{cases} 1 & y_{i,k} \geq \tau_i, \\ 0, & y_{i,k} < \tau_i, \end{cases} \tag{3}$$

where $\breve{y}_{i,k}$ is the output of sensor i at time k, and τ_i is a given threshold. Moreover, throughout the paper, we assume that the stochastic variables w_k and $v_{i,k}$ are white and mutually independent.

Remark 1: To further clarify the relationship between $y_{i,k}$ and $\breve{y}_{i,k}$, let us take a proximity sensor as an illustration. For a target tracking system with a proximity (binary) sensor, system output $y_{i,k}$ denotes the distance between the detected object and the sensor; τ is the radius of the detectable area. If $y_{i,k} < \tau$, indicating that the object is within the sensing area, the output signal is thus $\breve{y}_{i,k} = 0$. Otherwise, $\breve{y}_{i,k} = 1$. Generally speaking, any binary sensor can be regarded as a device that transforms the system output $y_{i,k}$ into binary signal $\breve{y}_{i,k}$.

B. Preliminaries

The main obstacle to state estimation with binary sensors is that the sensor output $\breve{y}_{i,k}$ does not directly reflect the exact value of the system output $y_{i,k}$. Accordingly, unlike KF or EKF, the innovation $z_{i,k} \triangleq y_{i,k} - C_i x_k$ cannot be obtained in the case of binary sensors, which brings difficulties in establishing an explicit relationship between the system state x_k and the binary output $\breve{y}_{i,k}$. According to [4], this issue can be tackled by constructing an uncertain measurement model (UMM) based on the thresholds of the binary sensors. In this paper, following the line of [4], we adopt the UMM to extract the system state information from binary signals.

Consider the time instant k^* at which the binary signal received from sensor i switches between 0 and 1, i.e., $|\breve{y}_{i,k^*} - \breve{y}_{i,k^*-1}| = 1$. It is clear that the threshold τ_i must lie within the interval between y_{i,k^*-1} and y_{i,k^*}. Hence, there exists a number $\beta \in [0, 1]$ such that

$$(1 - \beta)y_{i,k^*} + \beta y_{i,k^*-1} = \tau_i. \tag{4}$$

Here, β refers to an uncertainty term, whose existence is ensured while its exact value remains unknown due to the lack of measurement information about system outputs. Substituting (2) into (4), we have

$$C_i x_{k^*} + \eta_{i,k^*} + \vartheta_{i,k^*} = \tau_i, \tag{5}$$

where

$$\eta_{i,k^*} = \beta C_i(x_{k^*-1} - x_{k^*}), \quad \vartheta_{i,k^*} = (1 - \beta)v_{i,k^*} + \beta v_{i,k^*-1}.$$

The UMM (5) reveals the relationship between the system states and the thresholds of binary sensors, based on which we are able to deal with the state estimation problem with binary sensors. Note that (5) holds only for sensors switching from time $k^* - 1$ to k^*. Hence, to select these sensors, we introduce the following matrix

$$M_k = \mathrm{diag}\{\theta_{1,k}, \theta_{2,k}, \cdots, \theta_{m,k}\}, \tag{6}$$

where $\theta_{i,k} = |\breve{y}_{i,k} - \breve{y}_{i,k-1}|$.

Based on the UMM and the preceding discussion, the state estimator is constructed in the following form

$$\hat{x}_{k|k-1} = f(\hat{x}_{k-1|k-1}), \tag{7}$$
$$\hat{x}_{k|k} = \hat{x}_{k|k-1} + K_k M_k(\tau - C\hat{x}_{k|k-1}). \tag{8}$$

where $\hat{x}_{k|k-1}$ is the one-step prediction at time k, $\hat{x}_{k|k}$ is the state estimate of x_k at time k with initial value $\hat{x}_{0|0}$, $K_k \in \mathbb{R}^{n\times m}$ is the estimator gain matrix to be determined, and

$$\tau \triangleq [\tau_1, \tau_2, \cdots, \tau_m]^{\mathrm{T}}, \qquad C \triangleq [C_1^{\mathrm{T}}, C_2^{\mathrm{T}}, \cdots, C_m^{\mathrm{T}}]^{\mathrm{T}}. \tag{9}$$

The one-step prediction error $e_{k|k-1}$, the estimation error $e_{k|k}$, and their covariances is defined by

$$e_{k|k-1} \triangleq x_k - \hat{x}_{k|k-1}, \qquad e_{k|k} \triangleq x_k - \hat{x}_{k|k},$$
$$P_{k|k-1} \triangleq \mathbb{E}\{e_{k|k-1}e_{k|k-1}^{\mathrm{T}}\}, \qquad P_{k|k} \triangleq \mathbb{E}\{e_{k|k}e_{k|k}^{\mathrm{T}}\}.$$

Note that $P_{k|k-1}$ and $P_{k|k}$ provide quantitative measures of the covariances associated with prediction and estimation errors, respectively.

From (1), (7) and (8), the dynamics of the one-step prediction error and the estimation error can be obtained as

$$e_{k|k-1} = f(x_{k-1}) - f(\hat{x}_{k-1|k-1}) + w_{k-1}, \tag{10}$$
$$e_{k|k} = e_{k|k-1} - K_k M_k(\tau - C\hat{x}_{k|k-1}). \tag{11}$$

On the other hand, by using the Taylor series expansion around $\hat{x}_{k|k}$, the nonlinear function $f(x_k)$ is rewritten by

$$f(x_k) = f(\hat{x}_{k|k}) + A_k(x_k - \hat{x}_{k|k}) + E\Delta_k(x_k - \hat{x}_{k|k}), \tag{12}$$

where A_k is the Jacobian matrix of f at $\hat{x}_{k|k}$, E is a problem-dependent scaling matrix, and Δ_k is a matrix satisfying $\Delta_k\Delta_k^{\mathrm{T}} \le I$. Specifically, Δ_k is an uncertainty term and there is no specific value assigned to it, nor a probability distribution associated with it, although the existence of Δ_{k-1} can always be ensured. A rigorous justification for the existence of Δ_k will be given in Lemma 2 later. Then, it follows from (10)–(12) that

$$e_{k|k-1} = (A_{k-1} + E\Delta_{k-1})e_{k-1|k-1} + w_{k-1}, \tag{13}$$
$$e_{k|k} = e_{k|k-1} - K_k M_k(\tau - C\hat{x}_{k|k-1}). \tag{14}$$

Now, we are in a position to present the main objectives of this paper. Firstly, considering that it is literally impossible to acquire the exact estimation error covariance due to the existence of binary mapping, we aim to calculate an upper bound of the estimation error covariance $P_{k|k}$. Secondly, we will design an estimator gain K_k to minimize such an upper bound at each time instant k. Finally, we will establish a sufficient condition to verify the boundedness of the prediction error in the mean square sense.

Remark 2: The proposed state estimator consists of two stages: prediction (7) and update (8). In the prediction stage, the system model is utilized to predict the state forward from one time instant to the next. In the update stage, according to UMM (5), we substitute the thresholds τ_i for the system output $y_{i,k}$, thereby addressing the difficulty of obtaining $y_{i,k}$ due to binary mapping. Since UMM holds only for the switching sensors at each time instant, the matrix M_k is employed to select those sensors. This facilitates the construction of the state estimator, as illustrated by (7)–(8).

Remark 3: It should be noted that, by definition, system (1)–(3) is unobservable, since, even if the system is noise-free, the knowledge of $\breve{y}_{i,k}$ over a finite time interval cannot uniquely determine x_0.

Before ending this section, the following lemmas are presented for convenience of the subsequent analysis.

Lemma 1: [25] Let Y and Z be matrices with appropriate dimensions. The inequality $\mathrm{He}(YZ^{\mathrm{T}}) \le \varepsilon YY^{\mathrm{T}} + \varepsilon^{-1}ZZ^{\mathrm{T}}$ holds for all $\varepsilon > 0$.

Lemma 2: Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a vector-valued function satisfying the Lipschitz condition and $\hat{x} \in \mathbb{R}^n$ be a given vector. Then, there exists a scaling matrix E such that for any $x \in \mathbb{R}^n$, there always exists a matrix Δ with the property $\Delta\Delta^{\mathrm{T}} \le I$ such that $f(x) = f(\hat{x}) + (A + E\Delta)(x - \hat{x})$, where A is the Jacobian matrix of f at $\hat{x}$.

Proof: See Appendix A. ■

Lemma 3: Let ε_0 be a given scalar. Define $X_k \triangleq \mathbb{E}\{x_k x_k^{\mathrm{T}}\}$, where x_k is the state of system (1). The sequence of matrices $\{X_k\}_{k=0}^{\infty}$ is bounded by the solutions of the following recursive equation:

$$\bar{X}_k = (1+\varepsilon_0)\bar{A}\bar{X}_{k-1}\bar{A}^{\mathrm{T}} + (1+\varepsilon_0^{-1})\rho(\bar{X}_{k-1})EE^{\mathrm{T}} + Q \tag{15}$$

with initial condition $\bar{X}_0 = X_0$ and $\bar{A} = \left.\frac{\partial f(x)}{\partial x}\right|_{x=0}$.

Proof: See Appendix B. ■

Lemma 4: For matrices $\beta = \mathrm{diag}\{\tilde{\beta}_1, \tilde{\beta}_2, \ldots, \tilde{\beta}_n\}$ and $R = \mathrm{diag}\{R_1, R_2, \ldots, R_n\}$ with $\tilde{\beta}_i \in [0,1]$ and $R_i \ge 0$, $(i = 1, \ldots, n)$, inequality $(I-\beta)R(I-\beta) + \beta R\beta \le R$ holds.

Proof: Evidently. ■

III. Main Results

In this section, we first calculate upper bounds of one-step prediction error covariance and the estimation error covariance. Then, an estimator gain matrix is designed to minimize the obtained upper bound of the estimation error covariance. Moreover, an algorithm is presented to show the proposed state estimator design scheme. Finally, a sufficient condition is presented to verify the boundedness of the mean-square error.

A. Upper bounds of error covariances

In light of (13) and (14), it is easy to obtain the one-step prediction error covariance and estimation error covariance, which are provided in the following lemma.

Lemma 5: The one-step prediction error covariance $P_{k|k-1}$ and estimation error covariance $P_{k|k}$ are of the following recursion forms:

$$P_{k|k-1} = (A_{k-1} + E\Delta_{k-1})P_{k-1|k-1}(A_{k-1} + E\Delta_{k-1})^{\mathrm{T}} + Q, \tag{16}$$
$$\begin{aligned} P_{k|k} &= (I - K_kM_kC)P_{k|k-1}(I - K_kM_kC)^{\mathrm{T}} + \mathfrak{E}_k + \mathfrak{E}_k^{\mathrm{T}}, \\ &\quad + K_k\mathbb{E}\{M_k(Cx_k - \tau)(Cx_k - \tau)^{\mathrm{T}}M_k\}K_k^{\mathrm{T}} \end{aligned} \tag{17}$$

where A_{k-1} is the Jacobian matrix of f at $\hat{x}_{k-1|k-1}$ and $\mathfrak{E}_k = (I - K_kM_kC)\mathbb{E}\{e_{k|k-1}(Cx_k - \tau)^{\mathrm{T}}M_k\}K_k^{\mathrm{T}}$.

Proof: Equation (16) is directly derived from (13), the definition of one-step prediction error covariance, and the mutual independence of $e_{k-1|k-1}$ and w_{k-1}. By adding the zero-value term $K_kM_kCx_k - K_kM_kCx_k$ on the right-hand side of (11), we have $e_{k|k} = (I - K_kM_kC)e_{k|k-1} + K_kM_k(Cx_k - \tau)$. Based on the preceding equation and the definition of estimation error covariance, we can obtain (17) immediately. ■

Remark 4: So far, we have acquired the expressions of the one-step prediction error covariance and the estimation error covariance. Nevertheless, it is worth mentioning that, there exist some unknown terms in (16) and (17), i.e., Δ_{k-1}, $\mathbb{E}\{M_k(Cx_k - \tau)(Cx_k - \tau)^{\mathrm{T}}M_k\}$, and $\mathfrak{E}_k$, which render it difficult to obtain the exact values of the covariances. Inspired by the variance-constrained state estimation scheme given in [16], [17], we turn to seek an upper bound of the estimation error covariance and minimize such an upper bound by designing an appropriate estimator gain.

Theorem 1: Let $\varepsilon_i > 0$ $(i = 0, 1, 2, 3)$ be given scalars. Given system (1) and scaling matrix E, define the operator $\mathcal{G} : \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times n} \times \mathbb{R}^+ \to \mathbb{R}^{n\times n}$ by

$$\mathcal{G}(\mathcal{P}, \mathcal{A}, \varepsilon) = (1+\varepsilon)\mathcal{A}\mathcal{P}\mathcal{A}^{\mathrm{T}} + (1+\varepsilon^{-1})\rho(\mathcal{P})EE^{\mathrm{T}} + Q. \tag{18}$$

Consider the following difference equations

$$\Xi_{k|k-1} = \mathcal{G}(\Xi_{k-1|k-1}, A_{k-1}, \varepsilon_1), \tag{19}$$
$$\Xi_{k|k} = \delta(\mathrm{tr}(M_k))\Xi_{k|k-1} + (1 - \delta(\mathrm{tr}(M_k)))$$

This article has been accepted for publication in a future issue of this journal, but has not been fully edited. Content may change prior to final publication. Citation information: DOI10.1109/TAC.2024.3378776, IEEE Transactions on Automatic Control

$$\times \{(1+\varepsilon_2)(I-K_kM_kC)\Xi_{k|k-1}(I-K_kM_kC)^{\mathrm{T}}$$
$$+(1+\varepsilon_2^{-1})K_kM_k$$
$$\times [\rho(C\mathcal{G}(\bar{X}_{k-1},\bar{A}-I,\varepsilon_3)C^{\mathrm{T}})I+R]M_kK_k^{\mathrm{T}}\} \quad (20)$$

with $\Xi_{0|0}=P_{0|0}$ and $\bar{X}_{k-1}$ being calculated recursively by (15). The solutions of (19) and (20) are, respectively, upper bounds of the one-step prediction error covariance and estimation error covariance, namely, $\Xi_{k|k-1}\geq P_{k|k-1}$ and $\Xi_{k|k}\geq P_{k|k}$, $\forall k\in\mathbb{N}$.

Proof: The mathematical induction is utilized to prove this theorem. First, it is obvious that $\Xi_{0|0}=P_{0|0}$. Then, assume that $\Xi_{k-1|k-1}\geq P_{k-1|k-1}$ for $k\geq 1$. It follows from (16), Lemma 1, and $\Delta_{k-1}\Delta_{k-1}^{\mathrm{T}}\leq I$ that $P_{k|k-1}\leq \mathcal{G}(P_{k-1|k-1},A_{k-1},\varepsilon_1)$. Therefore, one easily verify that $\Xi_{k-1|k-1}\geq P_{k-1|k-1}$ implies

$$\Xi_{k|k-1}\geq P_{k|k-1}. \quad (21)$$

Next, we show $\Xi_{k|k}\geq P_{k|k}$. It is concluded from (5) that, for any sensor that switches from instant $k-1$ to k (i.e., $\theta_{i,k}=1$), there exists $\tilde{\beta}_{i,k}\in[0,1]$ such that

$$\theta_{i,k}(C_ix_k-\tau_i)$$
$$=\theta_{i,k}\left[\tilde{\beta}_{i,k}C_i(x_k-x_{k-1})-(1-\tilde{\beta}_{i,k})v_{i,k}-\tilde{\beta}_{i,k}v_{i,k-1}\right]. \quad (22)$$

On the other hand, if the ith sensor does not switch, i.e., $\theta_{i,k}=0$, the equation (22) still holds, as both sides equal 0. Then, it follows from (6) and (22) that, at any time instant k, there always exists a matrix $\beta_k=\mathrm{diag}\{\tilde{\beta}_{1,k},\tilde{\beta}_{2,k},\cdots,\tilde{\beta}_{m,k}\}$ (where $\tilde{\beta}_{i,k}\in[0,1]$ for all $i=1,2,\ldots,m$) such that

$$M_k(Cx_k-\tau)=M_k[\beta_kC(x_k-x_{k-1})-(I-\beta_k)v_k-\beta_kv_{k-1}]. \quad (23)$$

If $M_k=0$, it follows from (11) that

$$P_{k|k}=P_{k|k-1}\leq\Xi_{k|k-1}=\Xi_{k|k}. \quad (24)$$

If $M_k\neq 0$, according to (17), (23), Lemma 1 and Lemma 4, we have

$$P_{k|k}\leq(1+\varepsilon_2)(I-K_kM_kC)P_{k|k-1}(I-K_kM_kC)^{\mathrm{T}}$$
$$+(1+\varepsilon_2^{-1})K_kM_k[\beta_kC\mathbb{E}\{(x_k-x_{k-1})(x_k-x_{k-1})^{\mathrm{T}}\}C^{\mathrm{T}}\beta_k$$
$$+R+\mathrm{He}(\mathfrak{F}_1+\mathfrak{F}_2+\mathfrak{F}_3)]M_kK_k^{\mathrm{T}}, \quad (25)$$

where $\mathfrak{F}_1\triangleq\beta_k\mathbb{E}\{C(x_k-x_{k-1})v_k^{\mathrm{T}}\}(I-\beta_k)$, $\mathfrak{F}_2\triangleq\beta_k\mathbb{E}\{C(x_k-x_{k-1})v_{k-1}^{\mathrm{T}}\}\beta_k$, and $\mathfrak{F}_3\triangleq(I-\beta_k)\mathbb{E}\{v_kv_{k-1}^{\mathrm{T}}\}\beta_k$ are all equal to 0 since (x_k-x_{k-1}), v_{k-1} and v_k are mutually independent.

Next, it follows from Lemma 2 and $f(0)=0$ that, there exists a sequence of matrices $\{\bar{\Delta}_i\}_{i=0}^{\infty}$ with $\bar{\Delta}_i\bar{\Delta}_i^{\mathrm{T}}\leq I$, $\forall i\in\mathbb{N}$ such that $f(x_{k-1})=(\bar{A}+E\bar{\Delta}_{k-1})x_{k-1}$, where $\bar{A}$ is deterministic as it is the Jacobian matrix of f at point 0. Then, according to (1), one has $x_k-x_{k-1}=(\bar{A}-I+E\bar{\Delta}_{k-1})x_{k-1}+w_{k-1}$. Since x_{k-1} and w_{k-1} are mutually independent, it follows from Lemma 1 that $\mathbb{E}\{(x_k-x_{k-1})(x_k-x_{k-1})^{\mathrm{T}}\}\leq\mathcal{G}(\mathbb{E}\{x_{k-1}x_{k-1}^{\mathrm{T}}\},\bar{A}-I,\varepsilon_3)$. Because $\mathbb{E}\{x_{k-1}x_{k-1}^{\mathrm{T}}\}\leq\bar{X}_{k-1}$ by Lemma 3, we have

$$\mathbb{E}\{(x_k-x_{k-1})(x_k-x_{k-1})^{\mathrm{T}}\}\leq\mathcal{G}(\bar{X}_{k-1},\bar{A}-I,\varepsilon_3). \quad (26)$$

Note that $\beta_k\beta_k\leq I$. Therefore, it holds that

$$\beta_kY\beta_k<\rho(Y), \qquad \forall\, Y\in\mathbb{S}_m^+. \quad (27)$$

From (25)–(27), we conclude that, if $M_k\neq 0$, then

$$P_{k|k}\leq\Xi_{k|k}, \quad (28)$$

Finally, combining (24) and (28) will complete the proof. ■

Remark 5: In Theorem 1, $\Xi_{k|k-1}$ and $\Xi_{k|k}$ are parameterized by means of scalars ε_i, $(i=0,1,2,3)$. Ideally, the optimal value of ε_i, $(i=0,1,2,3)$ could be chosen by solving the following optimization problem at each time instant:

$$\min_{\varepsilon_0,\varepsilon_1,\varepsilon_2,\varepsilon_3}\mathrm{tr}(\Xi_{k|k}), \qquad \text{s.t. } \varepsilon_i>0,\ i=0,1,2,3.$$

Unfortunately, this problem is non-convex, making it rather challenging to derive analytical solutions. An alternative way is to utilize the evolutionary computation algorithms (e.g. the genetic algorithm or particle swarm optimization algorithm) to search for heuristically optimal solutions.

B. Design of the filter gain matrix and an algorithm

In what follows, we shall proceed to design the estimator gain to minimize the derived upper bound of the estimation error covariance. In addition, an algorithm will be presented to show the proposed state estimator design scheme.

Theorem 2: Let $\Xi_{k|k-1}$ and $\Xi_{k|k}$ be, respectively, upper bounds of one-step prediction error covariance and estimation error covariance given in Theorem 1. Upper bound $\Xi_{k|k}$ can be minimized by using the following estimator parameter

$$K_k^*=(1+\varepsilon_2)\Xi_{k|k-1}C^{\mathrm{T}}\tilde{M}_k^{\mathrm{T}}(\tilde{M}_k\mathcal{F}_k\tilde{M}_k^{\mathrm{T}})^{-1}\tilde{M}_k, \quad (29)$$

where $\mathcal{F}_k\triangleq(1+\varepsilon_2)C\Xi_{k|k-1}C^{\mathrm{T}}+(1+\varepsilon_2^{-1})[\rho(C\mathcal{G}(\bar{X}_{k-1},\bar{A}-I,\varepsilon_3)C^{\mathrm{T}})I+R]$ and $\tilde{M}_k$ is obtained by removing all the zero-rows in M_k. Moreover, $\Xi_{k|k-1}$ and $\Xi_{k|k}$ can be computed recursively by

$$\Xi_{k|k-1}=\mathcal{G}(A_{k-1},\Xi_{k-1|k-1},\varepsilon_1), \quad (30)$$
$$\Xi_{k|k}=\delta(\mathrm{tr}(M_k))\Xi_{k|k-1}+(1-\delta(\mathrm{tr}(M_k)))[(1+\varepsilon_2)\Xi_{k|k-1}$$
$$-(1+\varepsilon_2)^2\Xi_{k|k-1}C^{\mathrm{T}}\tilde{M}_k^{\mathrm{T}}(\tilde{M}_k\mathcal{F}_k\tilde{M}_k^{\mathrm{T}})^{-1}\tilde{M}_kC\Xi_{k|k-1}] \quad (31)$$

with the initial condition $\Xi_{0|0}=P_{0|0}$.

Proof: First, it follows from Theorem 1 that, if $M_k=0$, then $\Xi_{k|k}=\Xi_{k|k-1}$. If $M_k\neq 0$, we can rewrite (20) as $\Xi_{k|k}=(1+\varepsilon_2)[\Xi_{k|k-1}-\mathrm{He}(K_kM_kC\Xi_{k|k-1})]+K_kM_k\mathcal{F}_kM_kK_k^{\mathrm{T}}$. It can be observed easily that $M_k=\tilde{M}_k^{\mathrm{T}}\tilde{M}_k$ and $I=\tilde{M}_k\tilde{M}_k^{\mathrm{T}}$. Therefore, it holds that $\Xi_{k|k}=(1+\varepsilon_2)[\Xi_{k|k-1}-\mathrm{He}(K_k\tilde{M}_k^{\mathrm{T}}\tilde{M}_kC\Xi_{k|k-1})]+K_k\tilde{M}_k^{\mathrm{T}}\tilde{M}_k\mathcal{F}_k\tilde{M}_k^{\mathrm{T}}\tilde{M}_kK_k^{\mathrm{T}}$. The trace of $\Xi_{k|k}$ is minimized when its derivative with respect to the gain matrix K_k is zero. Thus, by letting $\frac{\partial\mathrm{tr}(\Xi_{k|k})}{\partial K_k}=0$, we derive the gain matrix as in (29). In addition, (29) further leads to the minimized $\Xi_{k|k}$ as: $\Xi_{k|k}=(1+\varepsilon_2)\Xi_{k|k-1}-(1+\varepsilon_2)^2\Xi_{k|k-1}C^{\mathrm{T}}\tilde{M}_k^{\mathrm{T}}(\tilde{M}_k\mathcal{F}_k\tilde{M}_k^{\mathrm{T}})^{-1}\tilde{M}_kC\Xi_{k|k-1}$. In summary, the minimized $\Xi_{k|k}$ can be written as (31), as desired. ■

After the calculation of estimator gain using Theorem 2, we can summarize the state estimation scheme as in Algorithm 1.

Algorithm 1 Recursive state estimation algorithm for systems with binary sensors

Initialization: Initialize the parameters $\hat{x}_{0|0}$, $\Xi_{0|0}$, $\breve{y}_{i,0}$ $(i=1,2,...,m)$ and ε_i $(i=0,1,2,3)$. Choose appropriate thresholds τ_i, $(i=1,2,3,...,m)$. Set time instant $k=0$.
Step 1: Measurement update
Collect all the sensor signals $\breve{y}_{i,k}$.
Step 2: Gain calculation
Generate matrix M_k by (6).
Compute $\Xi_{k|k-1}$ by (30).
Calculate $\bar{X}_{k-1}$ by (15).
Calculate the filter gain K_k^* by (29).
Step 3: Estimation update
Compute $\Xi_{k|k}$ by (31).
Compute the updated state estimate $\hat{x}_{k|k}$ by (7) and (8).
Return $\hat{x}_{k|k}$.
Step 4: Set $k=k+1$, then repeat *Step 1-Step 3*.

C. Performance analysis

In this subsection, we aim to conduct the performance analysis of the proposed state estimation algorithm. Before proceeding further,

This article has been accepted for publication in a future issue of this journal, but has not been fully edited. Content may change prior to final publication. Citation information: DOI10.1109/TAC.2024.3378776, IEEE Transactions on Automatic Control

let us first present the definition of exponential boundedness in the mean square sense.

Definition 1: [31] A stochastic process ϑ_k, $k \in \mathbb{N}$, is said to be exponentially bounded in mean square if there exist constants $c > 0$, $\varsigma > 0$, and $0 < \gamma < 1$ such that $\mathbb{E}\{\|\vartheta_k\|^2\} \leq c\gamma^k + \varsigma$ holds for all $k \in \mathbb{N}$.

Based on Definition 1, a sufficient condition that ensures the exponential boundedness of the prediction error in the mean square sense is given in the following theorem.

Theorem 3: Consider the discrete-time nonlinear systems (1)–(3) with the designed estimator (7)–(8). Let $\boldsymbol{\mathcal{M}} = \{\mathrm{diag}\{\theta_1, ..., \theta_m\} : \theta_i = 0 \text{ or } 1,\ i = 1, ..., m\}$. Define operators $\mathcal{T}^r, \mathcal{T}^p : \mathbb{S}_n^+ \to \mathbb{S}_n^+$ and $\mathcal{U} : \mathbb{S}_n^+ \times \mathbb{R}^{n\times m} \times \mathbb{R}^{m\times m} \to \mathbb{S}_n^+$ by

$$\mathcal{T}^r(\mathcal{X}) = \mathcal{G}(\mathcal{X}, \bar{A}, \varepsilon_0) - Q, \qquad \mathcal{T}^p(\mathcal{P}) = \mathcal{G}(\mathcal{P}, \hat{A}, \varepsilon_1) - Q,$$
$$\mathcal{U}(\mathcal{P}, \mathcal{K}, \mathcal{M}) = (1 + \varepsilon_2)(I - \mathcal{K}\mathcal{M}C)\mathcal{P}(I - \mathcal{K}\mathcal{M}C)^{\mathrm{T}}, \tag{32}$$

where $\mathcal{G}$ is defined in (18), and $\hat{A} \in \mathbb{R}^{n\times n}$ is a matrix satisfying $A_k\mathcal{X}A_k^{\mathrm{T}} \leq \hat{A}\mathcal{X}\hat{A}^{\mathrm{T}}$ for any given $\mathcal{X} > 0$. If there exist $P, X \in \mathbb{S}_n^+$ and $K \in \mathbb{R}^{n\times m}$ such that

$$X > \mathcal{T}^r(X), \qquad P > \mathcal{T}^p(\mathcal{U}(P, K, \mathcal{M})), \qquad \forall \mathcal{M} \in \boldsymbol{\mathcal{M}} \tag{33}$$

then the prediction error is exponentially bounded in mean square.

Proof: First, we define operators $\tilde{\mathcal{T}}_k^p(\mathcal{P}) = \mathcal{G}(\mathcal{P}, A_k, \varepsilon_1) - Q$ and $\mathcal{T}^c(\mathcal{P}, \mathcal{X}, \mathcal{K}, \mathcal{M}) = \mathcal{U}(\mathcal{P}, \mathcal{K}, \mathcal{M}) + \mathcal{V}(\mathcal{X}, \mathcal{K}, \mathcal{M})$, where

$$\mathcal{V}(\mathcal{X}, \mathcal{K}, \mathcal{M}) = (1 + \varepsilon_2^{-1})\mathcal{K}\mathcal{M}[\rho(C\mathcal{G}(\mathcal{X}, \bar{A} - I, \varepsilon_3)C^{\mathrm{T}})I + R]\mathcal{M}\mathcal{K}^{\mathrm{T}}. \tag{34}$$

By Theorems 1 and 2, $\Xi_{k+1|k}$ can be rearranged as

$$\Xi_{k+1|k} = \begin{cases} \tilde{\mathcal{T}}_k^p(\Xi_{k|k-1}) + Q, & M_k = 0, \\ \tilde{\mathcal{T}}_k^p(\mathcal{T}^c(\Xi_{k|k-1}, \bar{X}_{k-1}, K_k^*, M_k)) + Q, & M_k \neq 0, \end{cases}$$

Since $A_k\mathcal{X}A_k^{\mathrm{T}} \leq \hat{A}\mathcal{X}\hat{A}^{\mathrm{T}}$, we can state $\tilde{\mathcal{T}}_k^p(\mathcal{P}) \leq \mathcal{T}^p(\mathcal{P})$ by (18). Moreover, it can be observed that $\mathcal{T}^p(\mathcal{P}) \leq \mathcal{T}^p(\mathcal{P}')$ and $\mathcal{T}^c(\mathcal{P}, \mathcal{X}, \mathcal{K}, \mathcal{M}) \leq \mathcal{T}^c(\mathcal{P}', \mathcal{X}, \mathcal{K}, \mathcal{M})$ for any $\mathcal{P} < \mathcal{P}'$. This leads to

$$\Xi_{k+1|k} \leq \begin{cases} \mathcal{T}^p(\Xi_{k|k-1}) + Q, & M_k = 0, \\ \mathcal{T}^p(\mathcal{T}^c(\Xi_{k|k-1}, \bar{X}_{k-1}, K_k^*, M_k)) + Q, & M_k \neq 0. \end{cases} \tag{35}$$

From (18), (32) and (34), one can verify that the operators $\mathcal{T}^p$, $\mathcal{T}^r$, $\mathcal{U}$ and $\mathcal{V}$ satisfy the following properties:

- homogeneity (w.r.t. the first variable): $\forall\ \mu \in \mathbb{R}^+$,
$$\mathcal{T}^p(\mu\mathcal{P}) = \mu\mathcal{T}^p(\mathcal{P}), \qquad \mathcal{T}^r(\mu\mathcal{X}) = \mu\mathcal{T}^r(\mathcal{X}),$$
$$\mathcal{U}(\mu\mathcal{P}, \mathcal{K}, \mathcal{M}) = \mu\mathcal{U}(\mathcal{P}, \mathcal{K}, \mathcal{M}); \tag{36}$$
- monotonicity (w.r.t. the first variable): given positive definite matrices $\mathcal{P} \leq \mathcal{P}'$, $\mathcal{X} \leq \mathcal{X}'$,
$$\mathcal{T}^p(\mathcal{P}) \leq \mathcal{T}^p(\mathcal{P}'), \qquad \mathcal{U}(\mathcal{P}, \mathcal{K}, \mathcal{M}) \leq \mathcal{U}(\mathcal{P}', \mathcal{K}, \mathcal{M}),$$
$$\mathcal{T}^r(\mathcal{X}) \leq \mathcal{T}^r(\mathcal{X}'), \qquad \mathcal{V}(\mathcal{X}, \mathcal{K}, \mathcal{M}) \leq \mathcal{V}(\mathcal{X}', \mathcal{K}, \mathcal{M}); \tag{37}$$
- summation inequality: $\forall\ \mathcal{P}_1, \mathcal{P}_2 > 0$,
$$\mathcal{T}^p(\mathcal{P}_1 + \mathcal{P}_2) \leq \mathcal{T}^p(\mathcal{P}_1) + \mathcal{T}^p(\mathcal{P}_2). \tag{38}$$

Note that $\mathcal{T}^r(X) < X$ indicates that there exist $\mu \in (0, 1)$ such that $\mathcal{T}^r(X) < \mu X$. We now consider Lemma 3. For any initial condition X_0, we can choose a sufficient large σ such that both $X_0 \leq \sigma X$ and $Q \leq \sigma X$ hold. Then, it can be derived from (15), (36) and (37) that $\bar{X}_1 = \mathcal{T}^r(X_0) + Q \leq \mathcal{T}^r(\sigma X) + \sigma X < (\mu + 1)\sigma X$. By induction, equation (15), and the homogeneity of $\mathcal{T}^r$, we can derive that $\bar{X}_k < \sigma\sum_{i=0}^k \mu^i X < \frac{\sigma}{1-\mu}X$, which, together with (37), indicates that $\mathcal{V}(\bar{X}_k, K, \mathcal{M})$ is uniformly bounded. That is, $\mathcal{V}(\bar{X}_k, K, \mathcal{M}) \leq \mathcal{V}((\sigma/(1-\mu))X, K, \mathcal{M})$, $\forall\ k \in \mathbb{N}$. Moreover, there must exist a matrix $\bar{\mathcal{V}} = \max_{\mathcal{M}\in\boldsymbol{\mathcal{M}}} \mathcal{V}((\sigma/(1-\mu))X, K, \mathcal{M})$, such that

$$\mathcal{V}(\bar{X}_k, K, \mathcal{M}) \leq \bar{\mathcal{V}}, \qquad \forall\ k \in \mathbb{N},\ \mathcal{M} \in \boldsymbol{\mathcal{M}}, \tag{39}$$

since $\boldsymbol{\mathcal{M}}$ is a finite set.

Now, we consider the evolution of $\Xi_{k+1|k}$. Given $\Xi_{1|0}$, we can choose a sufficient large $\sigma > 0$ such that $\Xi_{1|0} < \sigma P$ and $Q < \sigma P$. If $M_1 = 0$, then it follows from (32) that $\mathcal{U}(\mathcal{P}, K, M_1) = (1 + \varepsilon_2)\mathcal{P} > \mathcal{P}$. Hence, (33) implies $P > \mathcal{T}^p((1 + \varepsilon_2)P) > \mathcal{T}^p(P)$. Therefore, according to (35)–(37), there is a $\mu_1 \in (0, 1)$ such that

$$\Xi_{2|1} \leq \mathcal{T}^p(\Xi_{1|0}) + Q < \mathcal{T}^p(\sigma P) + \sigma P < \mu_1\sigma P + \sigma P. \tag{40}$$

If $M_1 \neq 0$, then it follows from Theorem 2 that

$$\mathcal{T}^c(\Xi_{1|0}, \bar{X}_0, K_1^*, M_1) \leq \mathcal{T}^c(\Xi_{1|0}, \bar{X}_0, K, M_1) \tag{41}$$

for any $K \neq K_1^*$. By (35)–(39) and (41), we can choose a sufficient large $\sigma > 0$ such that $\Xi_{1|0} < \sigma P$ and conclude that there is a $\mu_2 \in (0, 1)$ such that

$$\begin{aligned} \Xi_{2|1} &\leq \mathcal{T}^p(\mathcal{T}^c(\Xi_{1|0}, \bar{X}_0, K_1^*, M_1)) + Q \\ &\leq \mathcal{T}^p(\mathcal{T}^c(\Xi_{1|0}, \bar{X}_0, K, M_1)) + Q \\ &\leq \mathcal{T}^p(\mathcal{U}(\Xi_{1|0}, K, M_1)) + \mathcal{T}^p(\mathcal{V}(\bar{X}_0, K, M_1)) + Q \\ &\leq \mathcal{T}^p(\mathcal{U}(\sigma P, K, M_1)) + \mathcal{T}^p(\bar{\mathcal{V}}) + Q \\ &< \mu_2\sigma P + \mathcal{T}^p(\bar{\mathcal{V}}) + Q. \end{aligned} \tag{42}$$

By letting $\bar{\mu} = \max\{\mu_1, \mu_2\}$ and choosing a $\bar{\sigma} \geq \sigma$ such that $\mathcal{T}^p(\bar{\mathcal{V}}) + Q < \bar{\sigma}P$, we can derive from (40) and (42) that, $\Xi_{2|1} < (\bar{\mu}\sigma + \bar{\sigma})P$ for any $M_1 \in \boldsymbol{\mathcal{M}}$. By induction and the homogeneity property, we have $\Xi_{k+1|k} < \left(\bar{\mu}^k\sigma + \sum_{i=0}^{k-1}\bar{\mu}^i\bar{\sigma}\right)P < \left(\bar{\mu}^k\sigma + \frac{\bar{\sigma}}{1-\bar{\mu}}\right)P$, which leads us to

$$\mathbb{E}\{\|e_{k+1|k}\|^2\} = \mathrm{tr}(P_{k+1|k}) \leq \mathrm{tr}(\Xi_{k+1|k}) < \mathrm{tr}(P)\sigma\bar{\mu}^k + \frac{\mathrm{tr}(P)\bar{\sigma}}{1-\bar{\mu}}. \tag{43}$$

The proof is complete. ■

Remark 6: It is worth noting that $\mathcal{T}^p(\mathcal{P}) > (A + E\Delta)\mathcal{P}(A + E\Delta)^{\mathrm{T}}$ and $\mathcal{U}(\mathcal{P}, \mathcal{K}, \mathcal{M}) > \mathcal{P}$ if $\mathcal{M} = 0$. Therefore, if there exist $P \in \mathbb{S}_n^+$ and $K \in \mathbb{R}^{n\times m}$ such that $P > \mathcal{T}^p(\mathcal{U}(P, K, \mathcal{M}))$ for any $\mathcal{M}$, then it must hold that $P > (A + E\Delta)P(A + E\Delta)^{\mathrm{T}}$, implying the quadratic stability[1] of system (1). That is to say, the performance analysis in Theorem 3 is actually a sufficient condition for the convergence of *steady-state* filters.

Remark 7: Up to now, we have addressed the state estimation problems for a class of discrete-time nonlinear systems with binary sensors. A new state estimation algorithm has been proposed and the performance analysis has been conducted to reveal the boundedness behavior of the prediction error in the mean square sense. The proposed algorithm has the following advantages: 1) the developed estimation scheme is cost-effective in the sense that only binary measurements are transmitted and utilized, which is preferable in the network with limited communication bandwidth; 2) the proposed state estimation algorithm is of a recursive form which is suitable for real-time implementation; and 3) the proposed recursive state estimation algorithm processes the received data sequentially rather than a batch. As such, it is not necessary to store the complete data set nor to reprocess existing data if a new measurement becomes available, which makes our algorithm more computationally friendly.

IV. Examples

This section presents two examples. In Example 1, we employ an unstable system to illustrate the effectiveness of the proposed state estimation scheme. Then, Example 2 introduces a quadratically stable system to validate Theorem 3.

Example 1. Consider the discrete-time nonlinear system (1)–(3) with

$$f(x_k) = \begin{bmatrix} x_k^{(1)} + 0.5x_k^{(2)} + 0.4\sin x_k^{(1)} + 0.5\sin x_k^{(2)} \\ -0.1x_k^{(1)} + 0.85x_k^{(2)} + 0.4\sin x_k^{(1)} \end{bmatrix},$$

[1]The definition of quadratic stability can be found in [36].

$$C_i = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad i = 1, 2, ..., 9,$$
$$C_i = \begin{bmatrix} 0 & 1 \end{bmatrix}, \quad i = 10, 11, ..., 18.$$

The variances of the process and measurement noise are given by $Q = 4I$ and $R = 9I$, respectively. The initial state x_0 is a Gaussian random vector with zero mean and covariance $\mathbb{E}\{x_0 x_0^{\mathrm{T}}\} = 25I$. The system (1) is monitored by 18 binary sensors, whose thresholds are chosen as

$$\tau_1 = -30, \tau_2 = -20, \tau_3 = -10, \tau_4 = -5, \tau_5 = 0, \tau_6 = 5,$$
$$\tau_7 = 10, \tau_8 = 20, \tau_9 = 30, \tau_{10} = -10, \tau_{11} = -7.5, \tau_{12} = -5,$$
$$\tau_{13} = -2.5, \tau_{14} = 0, \tau_{15} = 2.5, \tau_{16} = 5, \tau_{17} = 7.5, \tau_{18} = 10.$$

It is easy to verify that the nonlinear functions $f(\cdot)$ satisfies the Lipschitz conditions with Lipschitz constant $l = 2.3065$. Then, by Lemma 2, the Jacobian A_k and scaling matrix E are calculated as:

$$A_k = \begin{bmatrix} 1 + 0.4\cos \hat{x}_k^{(1)} & 0.5 + 0.5\cos \hat{x}_k^{(2)} \\ -0.1 + 0.4\cos \hat{x}_k^{(1)} & 0.85 \end{bmatrix}, \quad E = \begin{bmatrix} 0.8 & 1 \\ 0.8 & 0 \end{bmatrix}.$$

By employing Theorem 2 and letting $\varepsilon_0 = 0.5$, $\varepsilon_i = 0.15$ $(i = 1, 2, 3)$ $\forall\ k \in \mathbb{N}$, we can recursively obtain $\hat{x}_{k|k-1}$, $\hat{x}_{k|k}$, and the estimator gain K_k^*.

The simulation results are given in Figs. 1–4. In Figs. 1 and 2, the trajectories of the true state, together with the state estimates obtained via Algorithm 1 are displayed. It is clear that the proposed state estimator can provide satisfactory tracking performance. Fig. 3 depicts the sensors that switch at each time instant. By combining the state trajectories (in Figs. 1 and 2) and Fig. 3, it is vividly indicated that the information of system states can be reflected by the thresholds of the binary sensors that switch at each time instant. In Fig. 4, the mean-square errors obtained over 1000 independent trials are given, which further illustrates the feasibility and effectiveness of the developed state estimation scheme.

Example 2. Consider nonlinear system (1)–(3) with

$$f(x_k) = 0.5x_k - 0.1\sin x_k, \qquad C = \begin{bmatrix} C_1 & C_2 \end{bmatrix}^{\mathrm{T}} = \begin{bmatrix} 1 & 0.5 \end{bmatrix}^{\mathrm{T}}.$$

The variances of the process and measurement noise are set to be $Q = 1$, $R = I$. The initial state x_0 is a Gaussian variable with zero mean and covariance $\mathbb{E}\{x_0^2\} = 4$. The thresholds of binary sensors are chosen as $\tau_1 = 1$, $\tau_2 = -1$. Taking the Taylor expansion of f, we can derive the Jacobian matrix $A_k = 0.5 - 0.1\cos \hat{x}_k$ and scaling matrix $E = 0.2$. Furthermore, it can be verified that $\bar{A} = \frac{df}{dx}\big|_{x=0} = 0.4$. Other parameters are set to be $\varepsilon_0 = 1$, $\varepsilon_1 = 1/3$, $\varepsilon_2 = 1/8$ and $\varepsilon_3 = 1$.

Now, we consider Theorem 3. For any given $\mathcal{X} > 0$, it is clear that one can let $\hat{A} = 0.6$ so that $A_k \mathcal{X} A_k^{\mathrm{T}} < \hat{A}\mathcal{X}\hat{A}^{\mathrm{T}}$ for all k. From the definition of $\mathcal{T}^r$, $\mathcal{T}^p$ and $\mathcal{U}$ in (32), we obtain that $\mathcal{T}^r(\mathcal{X}) = 0.4\mathcal{X}$ and $\mathcal{T}^p(\mathcal{U}(\mathcal{P},\mathcal{K},\mathcal{M})) = 0.72(I - \mathcal{K}\mathcal{M}C)\mathcal{P}(I - \mathcal{K}\mathcal{M}C)^{\mathrm{T}}$. Note that the inequality $\mathcal{X} > \mathcal{T}^r(\mathcal{X})$ holds for any $\mathcal{X} > 0$. In addition, we can let $K = [0.05\ 0.1]$ so that $\mathcal{P} > \mathcal{T}^p(\mathcal{U}(\mathcal{P}, K, \mathcal{M}))$ holds for any $\mathcal{P} > 0$ and $\mathcal{M} \in \boldsymbol{\mathcal{M}}$. Therefore, according to Theorem 3, the prediction error is exponential bounded in mean square.

The simulation results are shown in Fig. 5. Same as Example 1, the value of mean square error is obtained through 1000 independent repeated trials since the actual mean-square error versus time cannot be analytically computed. It is noted that the traces of $\Xi_{k+1|k}$ is always larger than the mean-square error $\mathbb{E}\{\|e_{k+1|k}\|^2\}$, which implies that $\Xi_{k+1|k}$ is indeed an upper bound of $P_{k+1|k}$. Moreover, an exponential bound calculated by the right-hand side of (43) is also depicted in Fig. 5. One can observe that both trajectories of $\mathbb{E}\{\|e_{k+1|k}\|^2\}$ and $\mathrm{tr}(\Xi_{k+1|k})$ are restricted by the exponential bound, which indicates the validity of Theorem 3.

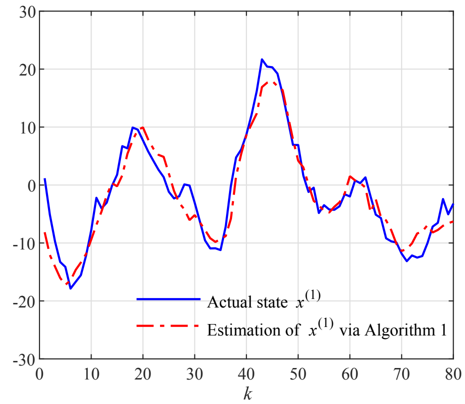

Fig. 1: Trajectory of $x_k^{(1)}$ and its estimate.

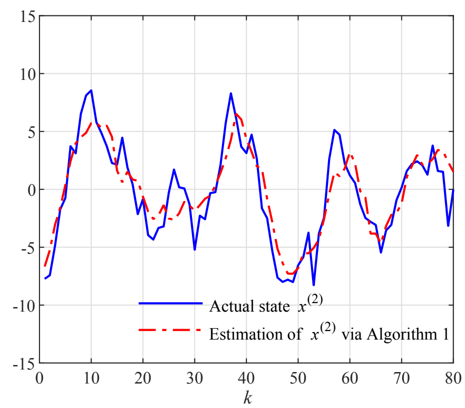

Fig. 2: Trajectory of $x_k^{(2)}$ and its estimate.

V. Conclusions

In this paper, we have addressed the state estimation problems for a class of discrete-time nonlinear systems with binary sensors. A recursive state estimator has been developed and the thresholds of binary sensors have been utilized to compensate for the missing information induced by binary mapping. An optimized estimation scheme has been put forward, in which an upper bound of the estimation error covariance has been obtained and then minimized by designing a proper estimator gain. Moreover, performance analysis has been carried out by investigating the boundedness of the prediction error in the mean square sense. Finally, some numerical simulations have been presented to demonstrate the validity of the proposed state estimation scheme.

Appendix A
Proof of Lemma 2

Proof: Consider a nonlinear Lipschitz function $f : \mathbb{R}^n \to \mathbb{R}^n$, denote the ith component of f by f_i, and let $\hat{x} \in \mathbb{R}^n$ be a given point. Taking the first-order Taylor series expansion of f gives $f(x) = f(\hat{x}) + A(x - \hat{x}) + \mathcal{R}(x - \hat{x})$, where A is the Jacobian matrix at $\hat{x}$ and

$$\mathcal{R} = \frac{1}{2}\begin{bmatrix} (x - \hat{x})^{\mathrm{T}} F_1(x(\theta_1)) \\ (x - \hat{x})^{\mathrm{T}} F_2(x(\theta_2)) \\ \vdots \\ (x - \hat{x})^{\mathrm{T}} F_n(x(\theta_n)) \end{bmatrix}$$

is the remainder. In the above equation, $F_i(x(\theta_i))$ is the Hessian matrix at $x(\theta_i)$ and $x(\theta_i) = \theta_i x + (1 - \theta_i)\hat{x}$ for some $\theta_i \in [0, 1]$. Since f is

This article has been accepted for publication in a future issue of this journal, but has not been fully edited. Content may change prior to final publication. Citation information: DOI10.1109/TAC.2024.3378776, IEEE Transactions on Automatic Control

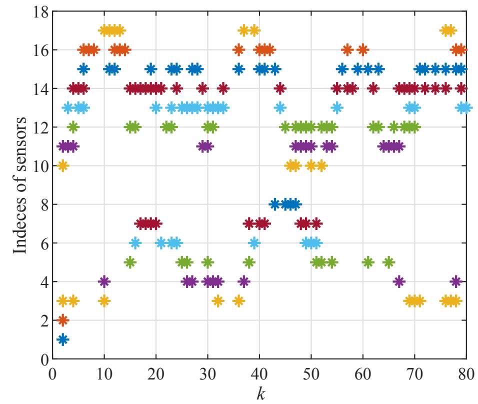

Fig. 3: Switching sensors versus time.

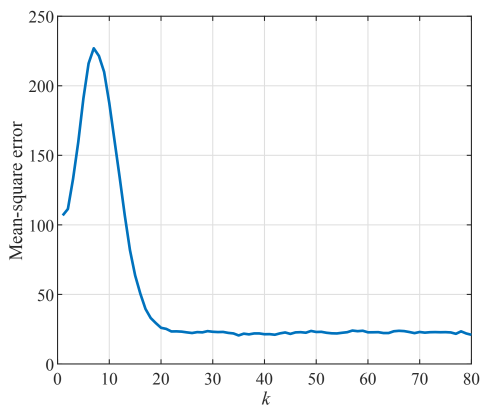

Fig. 4: Mean-square estimation error.

Lipschitz, it holds trivially that $(A+\mathcal{R})^{\mathrm{T}}(A+\mathcal{R}) \leq l^2 I$. Furthermore, we can verify $(A+\mathcal{R})(A+\mathcal{R})^{\mathrm{T}} \leq \operatorname{tr}[(A+\mathcal{R})^{\mathrm{T}}(A+\mathcal{R})]I \leq nl^2 I$. By letting ξ_i and ζ_i be the ith column of A and $\mathcal{R}$, we can rewrite the preceding inequality as $\sum_{i=1}^{n}(\xi_i+\zeta_i)(\xi_i+\zeta_i)^{\mathrm{T}} \leq nl^2 I$. Therefore, for any $i = 1, 2, ..., n$, it yields $\|\xi_i+\zeta_i\|^2 = \operatorname{tr}[(\xi_i+\zeta_i)(\xi_i+\zeta_i)^{\mathrm{T}}] \leq n^2 l^2$, which implies $\|\xi_i+\zeta_i\| \leq nl$. Since f is Lipschitz and ζ_i is the ith column of the Jacobian matrix, one can verify that there exist γ_i $(i = 1, 2, 3, \ldots, n)$ such that $\|\zeta_i\| \leq \gamma_i$. From the triangle inequality, we have $\|\zeta_i\| \leq \|\xi_i+\zeta_i\| + \|-\xi_i\| \leq nl+\gamma_i$. That is, for any $i = 1, 2, \ldots, n$, there always exists a positive number $r_i = nl+\gamma_i$ such that $\|\zeta_i\| \leq r_i$. By letting $\Delta = \frac{1}{\sqrt{n}} \cdot \left[\frac{1}{r_1}\zeta_1, \quad \frac{1}{r_2}\zeta_2, \quad \cdots, \quad \frac{1}{r_n}\zeta_n\right]$ and $E = \sqrt{n} \cdot \operatorname{diag}\{r_1, \quad r_2, \quad \cdots, \quad r_n\}$, the matrix $\mathcal{R}$ can be rewritten as $\mathcal{R} = E\Delta$. Moreover, we can observe that $\Delta\Delta^{\mathrm{T}} \leq I$, as desired. ■

Appendix B
Proof of Lemma 3

Proof: Consider system (1). By taking the Taylor series expansion of f at point 0, it yields that $x_k = (\bar{A} + E\bar{\Delta}_{k-1})x_{k-1} + w_{k-1}$. Applying mathematical induction, we can readily deduce (15) from Lemma 1, $\bar{\Delta}_{k-1}\bar{\Delta}_{k-1}^{\mathrm{T}} < I$ and the independence of x_{k-1} and w_{k-1}. ■

References

[1] M. V. Basin, A. G. Loukianov, and M. Hernandez-Gonzalez, Joint state and parameter estimation for uncertain stochastic nonlinear polynomial systems, *International Journal of Systems Science*, vol. 44, no. 7, pp. 1200–1208, 2013.

[2] G. Battistelli, L. Chisci, N. Forti, and S. Gherardini, MAP moving horizon state estimation with binary measurements, in *Proceedings of the American Control Conference*, Boston, 2016, pp. 5413–5418.

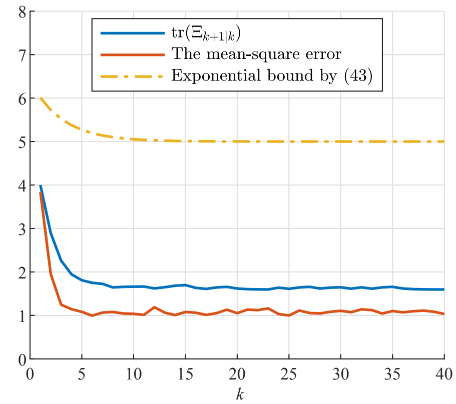

Fig. 5: Comparison of $\operatorname{tr}(\Xi_{k+1|k})$, mean-square error $\mathbb{E}\{\|e_{k+1|k}\|^2\}$ and an exponential bound.

[3] G. Battistelli, L. Chisci, S. Gherardini, Moving horizon state estimation for discrete-time linear systems with binary sensors, in *54th IEEE Conference on Decision and Control (CDC)*, Osaka, Japan, 2015, pp. 2414–2419.

[4] G. Battistelli, L. Chisci, and S. Gherardini, Moving horizon estimation for discrete-time linear systems with binary sensors: Algorithms and stability results, *Automatica*, vol. 85, pp. 374–385, 2017.

[5] D. Bertsekas and I. Rhodes, Recursive state estimation for a set-membership description of uncertainty, *IEEE Transactions on Automatic Control*, vol. 16, no. 2, pp. 117–128, 1971.

[6] F. S. Cattivelli and A. H. Sayed, Diffusion strategies for distributed Kalman filtering and smoothing, *IEEE Transactions on Automatic Control*, vol. 55, no. 9, pp. 2069–2084, 2010.

[7] H.-Y. Chung and W.-J. Chang, Covariance control with variance constraints for continuous perturbed stochastic systems, *Systems & Control Letters*, vol. 19, no. 5, pp. 413–417, 1992.

[8] D. Ciuonzo, A. Aubry, and V. Carotenuto, Rician MIMO channel and jamming-aware decision fusion, *IEEE Transactions on Signal Processing*, vol. 65, no. 15, pp. 3866–3880, 2017.

[9] P. M. Djurić, M. Vemula, and M. F. Bugallo, Signal processing by particle filtering for binary sensor networks, in *Proceedings of 2004 IEEE 11th Digital Signal Processing Workshop and IEEE Signal Processing Education Workshop* , Taos Ski Valley, NM, USA, 2004, pp. 263–267.

[10] P. M. Djurić, M. Vemula, and M. F. Bugallo, Target tracking by particle filtering in binary sensor networks, *IEEE Transactions on Signal Processing*, vol. 56, no. 6, pp. 2229–2238, 2008.

[11] H. Dong, Z. Wang, D. W. C. Ho, and H. Gao, Variance-constrained $\mathcal{H}_\infty$ filtering for a class of nonlinear time-varying systems with multiple missing measurements: The finite-horizon case, *IEEE Transactions on Signal Processing*, vol. 58, no. 5, pp. 2534–2543, 2010.

[12] R. Frühwirth, Application of Kalman filtering to track and vertex fitting, *Nuclear Instruments and Methods in Physics Research Section A: Accelerators, Spectrometers, Detectors and Associated Equipment*, vol. 262, no. 2, pp. 444–450, 1987.

[13] H. Gao, J. Lam, L. Xie, and C. Wang, New approach to mixed $\mathcal{H}_2/\mathcal{H}_\infty$ filtering for polytopic discrete-time systems, *IEEE Transactions on Signal Processing*, vol. 53, no. 8, pp. 3183–3192, 2005.

[14] X. Ge and Q.-L. Han, Distributed sampled-data asynchronous H_∞ filtering of Markovian jump linear systems over sensor networks, *Signal Processing*, vol. 127, pp. 86–99, 2016.

[15] J. W. Grizzle, K. L. Dobbins, and J. A. Cook, Individual cylinder air-fuel ratio control with a single EGO sensor, *IEEE Transactions on Vehicular Technology*, vol. 40, no. 1, pp. 280-286.

[16] J. Hu, Z. Wang, G. Liu, and H. Zhang, Variance-constrained recursive state estimation for time-varying complex networks with quantized measurements and uncertain inner coupling, *IEEE Transactions on Neural Networks and Learning Systems*, vol. 31, no. 6, pp. 1955–1967, 2020.

[17] J. Hu, Z. Wang, S. Liu, and H. Gao, A variance-constrained approach to recursive state estimation for time-varying complex networks with missing measurements, *Automatica*, vol. 64, pp. 155–162, 2016.

[18] Y. S. Hung and F. Yang, Robust H_∞ filtering with error variance constraints for discrete time-varying systems with uncertainty, *Automatica*, vol. 39, no. 7, pp. 1185–1194, 2003.

[19] M. R. James and I. R. Petersen, Nonlinear state estimation for uncertain systems with an integral constraint, *IEEE Transactions on Signal Processing*, vol. 46, no. 11, pp. 2926–2937, 1998.
[20] R. E. Kalman, A new approach to linear filtering and prediction problems, *Transactions of the ASME–Journal of Basic Engineering*, vol. 82 (Series D), pp. 35–45, 1960.
[21] X. Kai, C. Wei, and L. Liu, Robust extended Kalman filtering for nonlinear systems with stochastic uncertainties, *IEEE Transactions on Systems, Man, and Cybernetics - Part A: Systems and Humans*, vol. 40, no. 2, pp. 399–405, 2010.
[22] A. G. Kallapur, I. R. Petersen, and S. G. Anavatti, A discrete-time robust extended Kalman filter for uncertain systems with sum quadratic constraints, *IEEE Transactions on Automatic Control*, vol. 54, no. 4, pp. 850–854, 2009.
[23] J. Li, G. Wei, D. Ding, and Y. Li, Set-membership filtering for discrete time-varying nonlinear systems with censored measurements under Round-Robin protocol, *Neurocomputing*, vol. 281, pp. 20–26, 2018.
[24] W. Li, J. Sun, Y. Jia, J. Du, and X. Fu, Variance-constrained state estimation for nonlinear complex networks with uncertain coupling strength, *Digital Signal Processing*, vol. 67, pp. 107–115, 2017.
[25] Q. Liu, Z. Wang, X. He, D. H. Zhou, Event-triggered resilient filtering with measurement quantization and random sensor failures: Monotonicity and convergence, *Automatica*, vol. 94, pp. 458–464, 2018.
[26] W. NaNacara and E. E. Yaz, Recursive estimator for linear and nonlinear systems with uncertain observations, *Signal Processing*, vol. 62, no. 2, pp. 215–228, 1997.
[27] Z.-H. Pang, L.-Z. Fan, K. Liu and G.-P. Liu, Detection of stealthy false data injection attacks against networked control systems via active data modification, *Information Sciences*, vol. 546, pp. 192-205, 2021.
[28] Z.-H. Pang, L.-Z. Fan, Z. Dong, Q.-L. Han and G.-P. Liu, False data injection attacks against partial sensor measurements of networked control systems, *IEEE Transactions on Circuits and Systems II: Express Briefs*, vol. 69, no. 1, pp. 149-153, 2022.
[29] U. Shaked, L. Xie, and Y. C. Soh, New approaches to robust minimum variance filter design, *IEEE Transactions on Signal Processing*, vol. 49, no. 11, pp. 2620–2629, 2001.
[30] W. Song, Z. Wang, J. Wang, F. E. Alsaadi, and J. Shan, Secure particle filtering for cyber-physical systems with binary sensors under multiple attacks, *IEEE Systems Journal*, vol. 16, no. 1, pp. 603-613, March 2022.
[31] T.-J. Tarn and Y. Rasis, Observers for nonlinear stochastic systems, *IEEE Transactions on Automatic Control*, vol. 21, no. 4, pp. 441–448, 1976.
[32] L. Y. Wang, G. G. Yin, and J.-F. Zhang, Joint identification of plant rational models and noise distribution functions using binary-valued observations, *Automatica*, vol. 42, no. 4, pp. 535–547, 2006.
[33] L. Y. Wang, J.-F. Zhang, and G. G. Yin, System identification using binary sensors, *IEEE Transactions on Automatic Control*, vol. 48, no. 11, pp. 1892–1907, 2003.
[34] Z. Wang, D. W. C. Ho, and X. Liu, Variance-constrained filtering for uncertain stochastic systems with missing measurements, *IEEE Transactions on Automatic Control*, vol. 48, no. 7, pp. 1254–1258, 2003.
[35] L. Wu, P. Shi, and H. Gao, State estimation and sliding-mode control of Markovian jump singular systems, *IEEE Transactions on Automatic Control*, vol. 55, no. 5, pp. 1213–1219, 2010.
[36] L. Xie, Y. C. Soh, and C. E. de Souza, Robust Kalman filtering for uncertain discrete-time systems, *IEEE Transactions on Automatic Control*, vol. 39, no. 6, pp 1310–1314, 1994.
[37] Y. Xu, Z. Yao, R. Lu and B. K. Ghosh, A novel fixed-time protocol for first-order consensus tracking with disturbance rejection, *IEEE Transactions on Automatic Control*, DOI: 10.1109/TAC.2021.3131549, 2022.
[38] Y. Xu, W. Lv, W. Lin, R. Lu and D. E. Quevedo, On extended state estimation for nonlinear uncertain systems with round-robin protocol, *Automatica*, vol. 138, Article 110154, 2022.
[39] N. Yang, D. Chen, D. Ji and Z. Wu, Resilient state estimation for nonlinear complex networks with time-delay under stochastic communication protocol, *Neurocomputing*, vol. 346, pp. 38–47, 2019.
[40] D. Yue and Q.-L. Han, Network-based robust H_∞ filtering for uncertain linear systems, *IEEE Transactions on Signal Processing*, vol. 54, no. 11, pp. 4293–4301, 2006.
[41] X.-M. Zhang and Q.-L. Han, Network-based $\mathcal{H}_\infty$ filtering using a logic jumping-like trigger, *Automatica*, vol. 49, no. 5, pp. 1428–1435, 2013.
[42] X.-M. Zhang and Q.-L. Han, Event-based H_∞ filtering for sampled-data systems, *Automatica*, vol. 51, pp. 55–69, 2015.
[43] Y. Zhang, B. Chen, and L. Yu, Fusion estimation under binary sensors, *Automatica*, vol. 115, art. no. 108861, 2020.
[44] Y. Zhang, B. Chen, and L. Yu, Distributed fusion Kalman filtering under binary sensors, *International Journal of Robust and Nonlinear Control*, vol. 30, pp. 2570–2578, 2015.
[45] X. Zheng and H. Fang, Recursive state estimation for discrete-time nonlinear systems with event-triggered data transmission, norm-bounded uncertainties and multiple missing measurements, *International Journal of Robust and Nonlinear Control*, vol. 26, no. 17, pp. 3673–3695, 2016.