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# Trading multiple mean reversion

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How should one construct a portfolio from multiple mean-reverting assets? Should one add an asset to a portfolio even if the asset has zero mean reversion? We consider a position management problem for an agent trading multiple mean-reverting assets. We solve an optimal control problem for an agent with power utility, and present an explicit solution for several important special cases and a semi-explicit solution for the general case. The near-explicit nature of the solution allows us to study the effects of parameter mis-specification, and derive a number of properties of the optimal solution.

*Keywords*: portfolio optimization; dynamic portfolio optimization; optimal trading; statistical arbitrage; relative value trading; tactical asset allocation; Ornstein-Uhlenbeck process; mean-reversion; hedging.

## 1. Introduction

One of the basic patterns of statistical arbitrage is mean reversion trading. Typically, one constructs a synthetic asset from one or several traded assets in such a way that its price dynamics is mean-reverting. We shall be calling this mean-reverting synthetic asset a *spread*. Generally, trading a mean reverting asset consists of buying the spread when it is below its mean level and selling when it is above. The main question is how the position should be optimally managed with the movement of the spread, the trader's risk aversion, and the time horizon. When there are several mean-reverting assets available, the trader should additionally solve a dynamic

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portfolio optimization problem in order to decide the best way to combine positions in these assets.

A number of papers addressed this problem by specifying a stochastic differential equation (SDE) for spread dynamics and finding the optimal strategy that optimizes the expected utility over the terminal wealth. The simplest example of mean-reverting dynamics in continuous time is the Ornstein–Uhlenbeck process, the continuous version of the AR(1) discrete process. For a single spread utility maximization trading strategy see Boguslavskaya & Boguslavsky (2004). Another approach based on maximization of Sharpe ratio is described in Lipton & Prado (2020). For a more complicated example of mean-reverting dynamics we refer to paper Altay *et al.* (2018), where the spread is modelled by a Markov modulated Ornstein–Uhlenbeck process, and to papers Fouque & Hu (2019a,b) where the authors consider fractional stochastic processes. The models with uncertainty in the mean reversion level were discussed in Lee & Papanicolaou (2016). For alternative spread models, see Liu & Longstaff (2003) with Brownian bridge models and Zervos & Johnson (2013) for CER/CIR processes. A comprehensive review of mean reversion trading can be found in Leung & Lin (2015). For the methodology of statistical arbitrage we refer to Avellaneda & Lee (2010). In Li & Papanicolaou (2019), the authors assume different mean-reversion dynamics for multiple spread processes. They solve a portfolio optimization problem for several geometric Brownian motions with multiple co-integration terms in drifts.

Usually a portfolio allocator has access to multiple investing opportunities. Optimal sizing and timing of positions in each of these opportunities may be affected by positions in other assets and performance of those assets. To develop intuition about optimal dynamic allocation strategy, we generalize Boguslavskaya & Boguslavsky (2004) to the case of multiple correlated Ornstein-Uhlenbeck and Brownian Motion processes. We solve the problem of the maximization of power utility over terminal wealth for a finite horizon agent. Power utilities are a sufficiently broad family of utility functions, containing log-utility as a special case and linear utility as a limit case.

For the general problem, the optimal strategy is found in a quasi–analytical form as a solution to a matrix Riccati ordinary differential equation. For several important special cases it is possible to solve this equation explicitly. We also propose an efficient approach to analyse effects of parameter mis-specification. Although the proposed model is very simple, one can observe non-trivial qualitative properties of the optimal strategy. The availability of a quasi–analytical solution allows us to study how the trading strategy is affected by the correlation between spreads, and demonstrate the trade-offs between "harvesting" each spread separately and hedging positions in correlated spreads.

The rest of this paper is organized as follows: in Section 2 we give a brief overview of optimal strategy properties. In Section 3 we specify our formal asset and trading model and formulate a stochastic optimal control problem. Section 4 contains explicit formulas for the optimal control and the value function. Section 5 reminds main insights for the one-dimensional case. Optimal solution analysis is presented in Section 6. In Section 7, we present an ODE based framework to analyse the effect of parameter mis-specification and calculate the moments of the terminal wealth's distribution. We then apply this framework to analyse the sensitivity of the optimal strategy and of the value function to reversion rates mis-specification.

Implementation source code in Python and numerical implementation hints are available at https://github.com/DmitryMuravey/TradingMultipleMeanReversion.

## 2. Main results

The main contributions of this paper to the portfolio allocation field are the derivation of a quasi-analytical solution to the problem of portfolio allocation between multiple mean-reverting and Brownian motion assets, an explicit solution to this problem for a number of important special cases, and a number of qualitative observations on the behaviour of the quasi-analytical solution in the general case. Some of these observations may be contrary to the conventional wisdom of portfolio allocators.

### 2.1. *Myopic and hedging demands*

It is well known that in multiple asset portfolio allocation problems the position in each asset is driven by the myopic demand for the asset and by the hedging component, see Merton (1990). It is not surprising that we observe this in our particular problem as well, with the optimal strategy using positions in assets with slower mean reversion to hedge positions in faster mean reverting assets. However, the strength of the hedging demand in our problems leads to several less-intuitive solution properties.

### 2.2. *Low asset correlations are not beneficial*

While asset managers are often trying to benefit from diversification by composing their portfolios from assets with low pairwise correlations, in our model, with all other parameters fixed, higher absolute values of correlations between mean reverting asset driving processes are preferable to lower absolute values, as long as they stay strictly below 1. See Section 6.5 for more details.

### 2.3. *Stronger mean-reversion is not always beneficial*

One could expect that a higher reversion speed is beneficial to the trader. While this is true in the case of a single mean-reverting asset, this is not always so in the multi-dimensional case. An asset with a lower reversion rate and a non-zero correlation with higher reversion rate assets, may be used primarily as a hedge for positions in these assets. Hedge efficiency may be declining with the increases in the lower reversion rate. Assets with zero reversion and zero expected returns can play an important role in portfolio construction as sources of diversification. See Section 6.4 for more details.

### 2.4. *Parameter mis-specification cost is asymmetric*

The optimal strategy has a strong dependence on assumed reversion rates. It turns out that this dependence is quite asymmetric. We find that it is safer to underestimate reversion rates than to overestimate them. The value function is more sensitive to errors in reversion rate ratios between assets than to joint correlated errors in rate estimates. See Section 7.

## 3. The model

### 3.1. *The assets*

We are assuming that we have multiple tradable assets available with futures price processes $X_t^i, i = 1, ..., n$. Each of these processes can take positive or negative values, so they are best thought of as futures contracts on spreads between security prices or long-short portfolios of futures contracts. We are not modeling margin requirements, so positions are limited only by the risk aversion of the trading agents. The agent possesses initial wealth $W_0$. This wealth is assumed to be deposited on a margin account for the duration of trading and is earning no interest.

### 3.2. *Price processes*

Assume the canonical multivariate filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ with filtration $(\mathcal{F}_t)_{t\geq 0}$ to satisfy the usual conditions; see e.g. Karatzas & Shreve (1991). The simplest dynamic for a single mean-reverting tradable asset is a one-dimensional Ornstein-Uhlenbeck process $dX_t = -kX_t dt + \sigma dB_t$. Similarly, a collection $\left[X_t^1, X_t^2, \ldots, X_t^n\right]^\top$ of $n$ Ornstein-Uhlenbeck processes can be defined over this space as a multidimensional Ornstein–Uhlenbeck process:

$$d\mathbf{X}_t = -\boldsymbol{\kappa}\mathbf{X}_t dt + \boldsymbol{\sigma} d\mathbf{B}_t \tag{3.1}$$

Here $\mathbf{B}_t = \left[B_t^1, B_t^2, \ldots, B_t^n\right]^\top$ is an $n$-dimensional Wiener process with correlation matrix $\mathbf{\Theta} \in \mathbb{R}^{n\times n}$ (i.e. $d\mathbf{B}_t d\mathbf{B}_t^\top = \mathbf{\Theta} dt$), and $\boldsymbol{\kappa} \in \mathbb{R}_+^{n\times n}$ and $\boldsymbol{\sigma} \in \mathbb{R}_+^{n\times n}$ are diagonal matrices with non-negative elements that contain reversion rates and driving process volatilities for each asset:

$$\begin{aligned}\boldsymbol{\kappa} &= diag(\kappa_1, \kappa_2, \ldots, \kappa_n), \\ \boldsymbol{\sigma} &= diag(\sigma_1, \sigma_2, \ldots, \sigma_n),\end{aligned} \quad \mathbf{\Theta} = \begin{bmatrix} 1 & \rho_{12} & \ldots & \rho_{1n} \\ \rho_{21} & 1 & \ldots & \rho_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{n1} & \rho_{n2} & \ldots & 1 \end{bmatrix}. \tag{3.2}$$

The diagonality of matrices $\boldsymbol{\kappa}$ and $\boldsymbol{\sigma}$ means that all dependency between the assets comes from the correlations between the driving Brownian motions. We are assuming that all elements of the diagonal matrix $\boldsymbol{\kappa}$ are non-negative. Note that all our results hold for the larger class of matrices $\boldsymbol{\kappa}$ with non-negative eigenvalues, but this is outside of this paper scope. Positive values of $\boldsymbol{\kappa}_i$ correspond to mean-reverting processes, and $\boldsymbol{\kappa}_i = 0$ correspond to assets exhibiting zero mean reversion and simply following correlated Brownian motions. However, we assume that elements of vector $\boldsymbol{\kappa}$ are not all zero to avoid a trivial problem. Correlation matrix $\boldsymbol{\Theta}$ should be symmetric and positive semi-definite with unit diagonal elements, $\rho_{ii} = 1$, $\rho_{ij} = \rho_{ji}$. We shall assume that $\boldsymbol{\Theta}$ has full rank to avoid obvious arbitrages.

Without loss of generality, we can also assume that long-term means of each spread process are equal to zero. The general case can be reduced to equation (3.1) by the substitution $[\mathbf{X}_t - \boldsymbol{\theta}] \to \mathbf{X}_t$, where $\boldsymbol{\theta}$ is a vector of long term means. Equation 3.1 can be solved explicitly in terms of Itô integral:

$$\mathbf{X}_t = \mathrm{e}^{-\boldsymbol{\kappa} t}\mathbf{X}_0 + \int_0^t \mathrm{e}^{-\boldsymbol{\kappa}(t-s)}\boldsymbol{\sigma} d\mathbf{B}_s. \tag{3.3}$$

Here $\mathrm{e}^{\boldsymbol{A}}$ is a matrix exponential:

$$\mathrm{e}^{\boldsymbol{A}} = \sum_{k=0}^{\infty} \frac{1}{k!}\boldsymbol{A}^k, \quad \boldsymbol{A}^0 = \boldsymbol{I}. \tag{3.4}$$

### 3.3. *Wealth process*

The problem can be treated in the general Merton portfolio optimisation framework, see Merton (1990). We are assuming that the agent starts with wealth $W_0$, fully deposited to the margin account at inception at an interest rate of zero. As the agent trades and makes profits/losses, the profits and losses are realized continuously and deposited to the same account at the same zero interest rate. We are not modelling margin requirements and are assuming that $W_t$ is always sufficient to cover them.

Let vector $\boldsymbol{\alpha}_t$

$$\boldsymbol{\alpha}_t = \left[\alpha_t^1, \alpha_t^2, \dots, \alpha_t^n\right]^\top \tag{3.5}$$

be a trader's position at time $t$, i.e. the number of units of each asset held. This is the control in our optimization problem. Assuming no transaction costs, for a given control process $\boldsymbol{\alpha}_t$, the wealth process $W_t^{\boldsymbol{\alpha}}$ is given by

$$dW_t^{\boldsymbol{\alpha}} = \boldsymbol{\alpha}_t^\top d\mathbf{X}_t = \sum_{i=1}^n \alpha_t^i dX_t^i \tag{3.6}$$

or in integral form

$$W_T^{\boldsymbol{\alpha}} = W_t^{\boldsymbol{\alpha}} + \int_t^T \boldsymbol{\alpha}_u^\top d\boldsymbol{X}_u = W_t^{\boldsymbol{\alpha}} + \sum_{i=1}^n \int_t^T \alpha_u^i dX_u^i. \tag{3.7}$$

### 3.4. *Normalization*

Without loss of generality, we can normalize all price processes to unit noises: $\boldsymbol{\sigma} = \boldsymbol{I}$. For the general case, the following parameterization should be used:

$$\mathbf{X}_t \to \boldsymbol{\sigma}^{-1}\mathbf{X}_t, \quad \boldsymbol{\alpha}_t \to \boldsymbol{\sigma}\boldsymbol{\alpha}_t. \tag{3.8}$$

### 3.5. *Value function*

The value function $J(W_t^{\boldsymbol{\alpha}}, \mathbf{X}_t, t) : \mathbb{R}^+ \times \mathbb{R}^n \times [0, T] \to \mathbb{R}$ is the supremum over all admissible controls of the expectation of the terminal utility conditional on the information available at time $t$,

$$J(w, \boldsymbol{x}, t) = \sup_{\boldsymbol{\alpha}_t \in \mathcal{A}} \mathbb{E}\left[U(W_T^{\boldsymbol{\alpha}}) | W_t^{\boldsymbol{\alpha}} = w, \, \mathbf{X}_t = \boldsymbol{x}\right], \tag{3.9}$$

where the set of admissible controls $\mathcal{A}$ is defined as

$$\mathcal{A} = \left\{ \boldsymbol{\alpha} : [0, T] \times \Omega \to \mathbb{R}^n \,|\, \boldsymbol{\alpha}_t \in \mathcal{F}_t, \, \int_0^\top (W_t^{\boldsymbol{\alpha}})^2 \sum_{i=1}^n \left(\boldsymbol{\alpha}_t^i \mathbf{X}_t^i\right)^2 dt < \infty, \, a.s \right\} \tag{3.10}$$

We consider a power utility function with the parameter $\gamma < 1$

$$U = U(W_T^{\boldsymbol{\alpha}}) = \frac{1}{\gamma} \left(W_T^{\boldsymbol{\alpha}}\right)^\gamma. \tag{3.11}$$

The relative risk aversion is measured by $1 - \gamma$. It is convenient to use another measure $\delta$, which is also known as the distortion rate (see Zariphopoulou (2001))

$$\delta = \frac{1}{1 - \gamma}, \quad 0 < \delta < \infty \tag{3.12}$$

so the smaller $\delta$ is, the less risk averse the agent. The case $\gamma = 0$ corresponds to the logarithmic utility function and the investor with $\gamma \to 1$ is a risk seeking investor.

Note that while we assumed that the margin account is earning no interest, our results do not require a zero internal discount rate for the trading agent. Our utility function depends only on terminal wealth $W_T$ and not on wealth at any intermediate moments $0 < t < T$. If the agent is discounting future wealth at a fixed interest rate $r$, utility $U$ is multiplied by a constant $\mathrm{e}^{-rT\gamma}$ and so is the value function $J$. Multiplication of the value function by a constant has no impact on the optimal strategy. Thus, we can assume that $r = 0$ for the rest of this paper.

## 4. Main result

### 4.1. *The Hamilton–Jacobi–Belman equation*

Our aim is to find the optimal control $\boldsymbol{\alpha}^*(W_t^{\boldsymbol{\alpha}}, \mathbf{X}_t, t)$ and the value function $J(W_t^{\boldsymbol{\alpha}}, \mathbf{X}_t, t)$ as the functions of wealth $W_t^{\boldsymbol{\alpha}}$, prices $\mathbf{X}_t$ and time $t$. The Hamilton–Jacobi–Bellman equation is

$$\sup_{\boldsymbol{\alpha}} \left( (\partial/\partial t + \mathcal{L}) \, J \right) = 0. \tag{4.1}$$

Here $\mathcal{L}$ is the infinitesimal generator of the wealth process $W_t^{\boldsymbol{\alpha}}$:

$$\begin{aligned} \mathcal{L} = & \frac{\boldsymbol{\alpha}^\top \boldsymbol{\Theta} \boldsymbol{\alpha}}{2} \frac{\partial^2}{\partial w^2} + \boldsymbol{\alpha}^\top \boldsymbol{\Theta} \boldsymbol{\nabla} \frac{\partial}{\partial w} - \boldsymbol{\alpha}^\top \boldsymbol{\kappa} \boldsymbol{x} \frac{\partial}{\partial w} \\ & + \frac{\boldsymbol{\nabla}^\top \boldsymbol{\Theta} \boldsymbol{\nabla}}{2} - \boldsymbol{x}^\top \boldsymbol{\kappa} \boldsymbol{\nabla} \end{aligned} \tag{4.2}$$

and the first order optimality condition on the control $\boldsymbol{\alpha}^*$ is

$$\boldsymbol{\alpha}^*(w,\boldsymbol{x},t) = \frac{J_w}{J_{ww}}\boldsymbol{\Theta}^{-1}\boldsymbol{\kappa x} - \frac{\boldsymbol{\nabla} J_w}{J_{ww}}. \tag{4.3}$$

The operator $\boldsymbol{\nabla}$ denotes a vector differential operator

$$\boldsymbol{\nabla} = \left[\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_n}\right]^\top \tag{4.4}$$

for which we define the following operations for any vectors $\mathbf{a} \in \mathbb{R}^{1\times n}$ and matrices $\mathbf{A} \in \mathbb{R}^{n\times n}$:

$$\mathbf{a}^\top \boldsymbol{\nabla} = \sum_{i=1}^{n} \mathbf{a}_i \frac{\partial}{\partial x_i}, \quad \boldsymbol{\nabla}^\top \mathbf{A} \boldsymbol{\nabla} = \sum_{i=1}^{n}\sum_{j=1}^{n} \mathbf{A}_{ij}\frac{\partial^2}{\partial x_i \partial x_j}. \tag{4.5}$$

Note that the first summand in the right-hand side of 4.3 is the myopic demand term corresponding to a static optimization problem while the second term hedges from changes in the investment opportunity set. For a log utility investor ($\gamma = 0$ or, equivalently, $\delta = 1$) the second term vanishes (see Merton (1990).)

Substituting this condition into Eq. 4.1 for the value function, we obtain a non-linear PDE, which can be linearized by the distortion transformation (see Zariphopoulou (2001)):

$$J(w,\boldsymbol{x},t) = \frac{w^\gamma}{\gamma} f^{1/\delta}(\boldsymbol{x},t). \tag{4.6}$$

Here the function $f(\boldsymbol{x},t)$ is a solution to the Cauchy problem for the parabolic PDE:

$$\begin{aligned} &\frac{1}{2}\boldsymbol{\nabla\Theta\nabla} f - \boldsymbol{x}^\top \boldsymbol{\kappa \nabla} f - \frac{\delta-1}{2}\boldsymbol{x}^\top\boldsymbol{\kappa\nabla} f - \frac{\delta-1}{2}\boldsymbol{\nabla}^\top f \boldsymbol{\kappa x} \\ &\qquad + \frac{\delta(\delta-1)}{2}\boldsymbol{x}^\top \boldsymbol{\kappa\Theta}^{-1}\boldsymbol{\kappa x} f + \frac{\partial f}{\partial t} = 0. \\ &f(\boldsymbol{x},T) = 1. \end{aligned} \tag{4.7}$$

The main equation 4.7 can be reduced to the matrix Riccati ODE. The value function $J$ and the optimal control $\boldsymbol{\alpha}^*$ have quasi-analytic representations via solutions to this ODE.

**Theorem 4.1.** *The value function* (3.9) *admits the following representation*

$$J(w,\boldsymbol{x},t) = \frac{w^\gamma}{\gamma}\cdot\exp\left\{\int_0^{T-t}\frac{\boldsymbol{Tr}(\boldsymbol{A}(u)\boldsymbol{\Theta})}{\delta}du + \frac{\boldsymbol{x}^\top \boldsymbol{A}(T-t)\boldsymbol{x}}{\delta}\right\} \tag{4.8}$$

*where* $\boldsymbol{Tr}$ *denotes trace operator and the function* $\boldsymbol{A} : \mathbb{R}^+ \to \mathbb{R}^{n\times n}\times\mathbb{R}^+$ *is a matrix function of inverse time* $\tau = T - t$*:*

$$\boldsymbol{A}(\tau) = \begin{vmatrix} A_{11}(\tau) & A_{12}(\tau) & \ldots & A_{1n}(\tau) \\ A_{21}(\tau) & A_{22}(\tau) & \ldots & A_{2n}(\tau) \\ \vdots & \vdots & \ddots & \vdots \\ A_{n1}(\tau) & A_{n2}(\tau) & \ldots & A_{nn}(\tau) \end{vmatrix} \tag{4.9}$$

*which is defined as a solution to the following matrix Ricatti equation:*

$$\begin{aligned} \boldsymbol{A}'(\tau) &= \mathfrak{R}_{\boldsymbol{\Theta},\boldsymbol{\kappa},\delta}\boldsymbol{A}, \\ \boldsymbol{A}(0) &= \boldsymbol{0}. \end{aligned} \tag{4.10}$$

*with* $\mathfrak{R}_{\boldsymbol{\Theta},\boldsymbol{\kappa},\delta}$ *denoting the nonlinear operator*

$$\begin{aligned} \mathfrak{R}_{\boldsymbol{\Theta},\boldsymbol{\kappa},\delta}\boldsymbol{A} = &\frac{\left(\boldsymbol{A}^\top + \boldsymbol{A}\right)\boldsymbol{\Theta}\left(\boldsymbol{A}^\top + \boldsymbol{A}\right)}{2} - \left[\frac{\delta-1}{2}\boldsymbol{\kappa} + \boldsymbol{\kappa}\right]\left(\boldsymbol{A}^\top + \boldsymbol{A}\right) \\ &- \frac{\delta-1}{2}\left(\boldsymbol{A}^\top + \boldsymbol{A}\right)\boldsymbol{\kappa} + \frac{\delta(\delta-1)}{2}\boldsymbol{\kappa}\boldsymbol{\Theta}^{-1}\boldsymbol{\kappa}. \end{aligned} \tag{4.11}$$

**Proof.** Using an ansatz similar to Brendle (2006) and Li & Papanicolaou (2019), we obtain the representation (4.8). □

The optimal strategy $\boldsymbol{\alpha}^*$ has the following representation:

$$\boldsymbol{\alpha}^*(w, \boldsymbol{x}, t) = w\left[-\delta\boldsymbol{\Theta}^{-1}\boldsymbol{\kappa} + \boldsymbol{A} + \boldsymbol{A}^\top\right]\boldsymbol{x}. \tag{4.12}$$

Introducing a new matrix $\boldsymbol{D}$ as

$$\boldsymbol{D}(\tau) = \delta\boldsymbol{\Theta}^{-1}\boldsymbol{\kappa} - \left(\boldsymbol{A}(\tau) + \boldsymbol{A}^\top(\tau)\right)$$

we get the following formula for the optimal strategy $\boldsymbol{\alpha}^*$:

$$\boldsymbol{\alpha}^*(w, \boldsymbol{x}, t) = -w\boldsymbol{D}(\tau)\boldsymbol{x}. \tag{4.13}$$

Matrix $\boldsymbol{D}$ can be found directly from another Riccati ODE (see Appendix B for details):

$$\begin{aligned} \boldsymbol{D}'(\tau) &= -\boldsymbol{D}^\top\boldsymbol{\Theta}\boldsymbol{D} + \delta\boldsymbol{\kappa}\boldsymbol{\Theta}^{-1}\boldsymbol{\kappa}, \\ \boldsymbol{D}(0) &= \delta\boldsymbol{\Theta}^{-1}\boldsymbol{\kappa}. \end{aligned} \tag{4.14}$$

If one needs to find only the optimal control, it is sufficient to solve Eq. (4.14). To find the value function we need to solve a more complex system (4.10).

The optimality of the candidate control $\boldsymbol{\alpha}^*$ can be verified using the same arguments as in Li & Papanicolaou (2019); see also Davis & Lleo (2008, 2014).

## 5. Analysis. Review of the one-dimensional case

### 5.1. *The problem*

Before we analyse the multidimensional case, let us present a short review of the one-dimensional case, for more details see Boguslavskaya & Boguslavsky (2004). It is obtained from our problem by setting $n = 1$ in all formulas from Section (3.2). To be more precise, we consider mean-reverting asset $X_t$ which follows an Ornstein–Uhlenbeck process with zero mean and unit variance:

$$dX_t = -\kappa X_t dt + dB_t \tag{5.1}$$

and the wealth process $W_t^{\boldsymbol{\alpha}}$ generated by the trading strategy $\alpha$:

$$dW_t^{\boldsymbol{\alpha}} = \alpha_t dX_t. \tag{5.2}$$

We are looking for the maximizer $\alpha^*$ of the expected utility over the terminal wealth $W_T^{\boldsymbol{\alpha}}$:

$$\alpha^* = \underset{\alpha}{\operatorname{argmax}} \left[\mathbb{E}_t \left[U(W_T^{\boldsymbol{\alpha}})\right]\right]. \tag{5.3}$$

### 5.2. *The structure of the optimal strategy*

The optimal control $\alpha^*$ can be expressed as

$$\alpha^*(w, x, t) = -wD_\kappa(T-t)x, \tag{5.4}$$

where the function $D_\kappa(\tau)$ is a solution to the following Riccati equation:

$$\begin{aligned} D_\kappa' = &\quad -D_\kappa^2 + \delta k^2, \\ D_\kappa(0) = &\quad \delta\kappa. \end{aligned} \tag{5.5}$$

This one-dimensional problem (5.5) can be solved explicitly via the substitution $\tau(D_\kappa) = D_\kappa^{-1}$. The function $D_\kappa(\tau)$ is a shifted and scaled sigmoid function of the inverse time $\tau = T - t$ :

$$D_\kappa(\tau) = \kappa\sqrt{\delta}\frac{\sqrt{\delta}\cosh\kappa\sqrt{\delta}\tau + \sinh\kappa\sqrt{\delta}\tau}{\sqrt{\delta}\sinh\kappa\sqrt{\delta}\tau + \cosh\kappa\sqrt{\delta}\tau}. \tag{5.6}$$

It is worth to mention that for $\gamma < 0$ the function $D_\kappa$ can be represented as

$$D_\kappa(\tau) = \kappa\sqrt{\delta}\tanh\left(\kappa\sqrt{\delta}\tau + \varphi\right), \quad \tanh\varphi = \sqrt{\delta}. \tag{5.7}$$

The behavior of the function $D_\kappa(T-t)$ depends on the value of risk aversion $\gamma$: a trading agent with a negative gamma (less risk averse than a log-utility agent) becomes less aggressive as time approaches terminal time, while traders with positive gamma become more aggressive (see Figure 1). For the log-utility agent ($\gamma = 0$, red line on Figure 1), the optimal strategy is static, i.e. $D_\kappa(\tau) \equiv const$.

### 5.3. *Value function structure*

The value function $J(w, x, t)$ can be split into three multiplicative terms:

$$J(w,x,t) = \underbrace{\frac{w^\gamma}{\gamma}}_{\mathbf{a}} \cdot \underbrace{\exp\left\{-\int_0^{T-t}\frac{D(u)-\delta\kappa}{2\delta}du\right\}}_{\mathbf{b}} \cdot \underbrace{\exp\left\{-\frac{x^2(D(T-t)-\delta\kappa)}{2\delta}\right\}}_{\mathbf{c}} \tag{5.8}$$

which can be interpreted as follows:

- **a** present wealth utility. This is the only term that depends on present wealth $w$. Value function is proportional to present wealth utility.

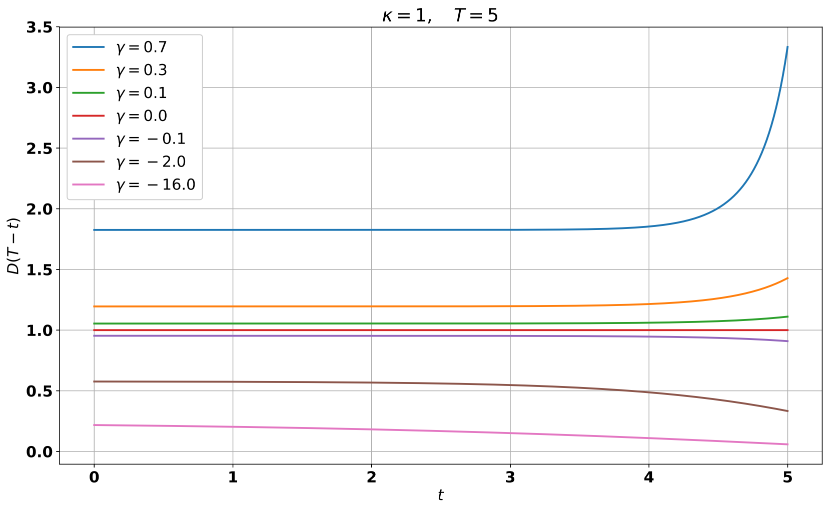



Fig. 1: Position size multiplier $D(T-t)$ for different values of risk aversion. A trading agent with a negative gamma (less risk averse than a log-utility agent) becomes less aggressive as time approaches the time horizon, while trading agents with positive gamma are not just more aggressive at all times but also become more aggressive as the time horizon approaches.

- **b** time value (utility of future expected opportunities). This term is an integral of a time function over the remaining time period; it depends neither on current wealth, nor on current asset price.
- **c** intrinsic value (utility of the immediate investment opportunity set.) This term's logarithm is proportional to the squared current asset price and, in particular, vanishes for current price at 0.

### 5.4. *Wealth process structure*

The stochastic process $W_t^{\boldsymbol{\alpha}}$ generated by the optimal strategy $\boldsymbol{\alpha}^*$ can be represented as (for more details see Appendix C)

$$\log\left(\frac{W_t^{\boldsymbol{\alpha}}}{W_s^{\boldsymbol{\alpha}}}\right) = \underbrace{\int_s^t \frac{D_\kappa(T-u) - \delta\kappa^2 X_u^2}{2} du}_{\mathbf{a}} + \underbrace{\frac{X_s^2 D_\kappa(T-s) - X_t^2 D_\kappa(T-t)}{2}}_{\mathbf{b}}. \quad (5.9)$$

So the log return of wealth between times $s$ and $t$ is the sum of

- **a** profit/loss from dynamic trading in the time period $[s,t]$; this term does not depend on the current price $X_s$
- **b** profit/loss between $s$ and $t$ on position open at time $s$.

### 5.5. *Monte Carlo analysis of parameter mis-specifications*

The higher mean reversion speed $\kappa$ makes the trader more aggressive. Authors of Boguslavskaya & Boguslavsky (2004) also make the following observations based on Monte Carlo simulations:

> The influence of mean reversion coefficient mis-specification is asymmetric. Trading with a conservatively estimated $\kappa$ greatly reduces utility uncertainty. The overestimation of $\kappa$ leads to excessively aggressive positions. It is much safer to underestimate $\kappa$ than to overestimate it.

## 6. Analysis. Multidimensional case.

The main difference between multidimensional and one dimensional cases is that changes in some spreads may affect positions in other spreads via changes in risk exposures. Generally, one might expect two possible motivations to take a position in each of the assets: to extract value from its reversion or to hedge positions in other assets.

In the multidimensional case, the time decay function $\boldsymbol{D}$ is a matrix. The main difficulty is that there are no known techniques to explicitly solve generic matrix Riccati equations. However, we were able to obtain explicit solutions for a number of important special cases, including the case of multiple assets with identical reversion rates and hedging a mean reverting asset with multiple correlated Brownian motions. We also make a number of observations for the general case and discuss the structure of the optimal strategy and the impact of correlation on value functions.

To simplify the notation, we shall be assuming below that the price process is at its long term mean $\mathbf{X}_0 \equiv \boldsymbol{\theta}$.

### 6.1. *Explicitly solvable cases.*

#### 6.1.1. *Non-correlated assets*

Assume that the asset processes are driven by non-correlated Wiener processes, $\boldsymbol{\Theta} = \boldsymbol{I}$. We can expect that the optimal strategy is simply a vector of one dimensional optimal strategies for each asset. That is, a candidate optimal control is

$$\boldsymbol{\alpha}^* = -w\boldsymbol{D}(\tau)\boldsymbol{x}, \quad \boldsymbol{D}(\tau) = diag\left(D_{\kappa_1}(\tau), D_{\kappa_2}(\tau)\ldots, D_{\kappa_n}(\tau)\right), \quad \tau = T - t. \tag{6.1}$$

For the definition of $D_\kappa$ see section 5. One can directly confirm that this control is indeed optimal by checking that it solves the system (4.14).

In this case, there are no interactions between the assets. The position in the $i$-th asset depends only on time $t$, current wealth and $i$-th asset parameters.

#### 6.1.2. *Common reversion rate*

Another case that allows an explicit solution is when the correlations are non-trivial, but the reversion rate $\kappa$ is the same for all assets $\boldsymbol{\kappa} = \kappa\boldsymbol{I}$. Recall SDE for the price process

$$d\mathbf{X}_t = -\kappa \mathbf{X}_t dt + d\mathbf{B}_t, \quad d\mathbf{B}_t d\mathbf{B}_t^\top = \Theta dt. \tag{6.2}$$

We show that for this case the explicit solution can also be constructed.

Indeed, with a single common reversion rate, any non-zero linear combination $\mathbf{Y}_t = \boldsymbol{L}^{-1}\mathbf{X}_t$ of Ornstein–Uhlenbeck processes is also an Ornstein–Uhlenbeck process:

$$d\mathbf{Y}_t = -\kappa \mathbf{Y}_t dt + d\tilde{\mathbf{B}}_t, d\tilde{\mathbf{B}}_t d\tilde{\mathbf{B}}_t^\top = \boldsymbol{L}^{-1}\Theta(\boldsymbol{L}^{-1})^\top dt. \tag{6.3}$$

Here $\tilde{\mathbf{B}}_t$ is an $n$-dimensional Wiener process with correlation matrix

$$\boldsymbol{L}^{-1}\Theta(\boldsymbol{L}^{-1})^\top. \tag{6.4}$$

Assuming the invertibility of $\boldsymbol{L}$, one can find an optimal control $\boldsymbol{\alpha}_Y$ for this new process $\mathbf{Y}_t$ and then transform it to an optimal control for $\mathbf{X}_t$. The transformation is based on the following equality

$$dW_t^{\boldsymbol{\alpha}} = \boldsymbol{\alpha}_Y^\top d\mathbf{Y}_t = \boldsymbol{\alpha}_X^\top d\mathbf{X}_t, \quad \boldsymbol{\alpha}_X(W_t^{\boldsymbol{\alpha}}, \mathbf{X}_t, t) = (\boldsymbol{L}^{-1})^\top \boldsymbol{\alpha}_Y(W_t^{\boldsymbol{\alpha}}, \boldsymbol{L}^{-1}\mathbf{X}_t, t). \tag{6.5}$$

The transformation matrix $\boldsymbol{L}$ is constructed as a Cholesky decomposition of correlation matrix $\boldsymbol{\Theta}$:

$$\boldsymbol{L}^\top \boldsymbol{L} = \boldsymbol{L}\boldsymbol{L}^\top = \boldsymbol{\Theta}, \quad (\boldsymbol{L}^{-1})^\top \boldsymbol{L}^{-1} = \boldsymbol{L}^\top (\boldsymbol{L}^{-1})^\top = \boldsymbol{\Theta}^{-1}. \tag{6.6}$$

Applying this transformation, we obtain the following equation for the optimal control:

$$\boldsymbol{\alpha}^* = -w D_\kappa(T-t)\boldsymbol{\Theta}^{-1}\boldsymbol{x}. \tag{6.7}$$

Thus, the optimal trading rule can be interpreted as the construction of linearly independent factor portfolios, and then trading them as in the case of non-correlated assets. This is similar to the portfolio signal construction approach of Kelly *et al.* (2020).

In this case, there are also no interactions between the assets. The value function $J(w, \mathbf{0}, t)$ does not depend on asset correlations:

$$J(w, \mathbf{0}, t) = \frac{w^\gamma}{\gamma} \exp\left\{ n \int_0^{T-t} \frac{\delta\kappa - D_\kappa(u)}{2\delta} du \right\}. \tag{6.8}$$

6.1.3. *Hedging a mean reverting asset via correlated Brownian Motions*

Let us consider a case where the set of tradable assets consists of a single mean-reverting asset and one or several correlated Brownian motions. We can also consider this case as the limiting case for sets of tradable assets, where one asset's mean reversion rate $\kappa$ is very large in relation to all other assets' reversion rates.

**Theorem 6.1.** *Consider the following matrix of reversion rates:*

$$\boldsymbol{\kappa} = diag(\kappa, 0, 0, \ldots, 0). \tag{6.9}$$

*One can check by a direct calculation that the solution to the Riccati equation (4.14) has the following form:*

$$\boldsymbol{D}(t) = \begin{vmatrix} \boldsymbol{D}_{11} & 0 & \dots & 0 \\ \boldsymbol{D}_{21} & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{D}_{n1} & 0 & \dots & 0 \end{vmatrix} \tag{6.10}$$

$\boldsymbol{D}_{j1} = \delta\kappa \left(\boldsymbol{\Theta}^{-1}\right)_{j1}$. *The term* $\boldsymbol{D}_{11}(\tau)$ *can be derived from the following Riccati ODE:*

$$\begin{aligned} &\boldsymbol{D}'_{11}(\tau) = -\boldsymbol{D}_{11}^2 + 2\delta(\zeta - 1)\kappa\boldsymbol{D}_{11} + \kappa^2\delta\zeta\left(\delta(1-\zeta) + 1\right), \\ &\boldsymbol{D}'_{11}(0) = \delta\zeta\kappa. \end{aligned} \tag{6.11}$$

*This ODE can be solved explicitly to yield the following formula for* $\boldsymbol{D}$*:*

$$\boldsymbol{D}_{11}(\tau) = \begin{cases} \kappa\lambda\frac{\delta\cosh\lambda\kappa\tau + \lambda\sinh\lambda\kappa\tau}{\delta\sinh\lambda\kappa\tau + \lambda\cosh\lambda\kappa\tau} + \delta\kappa(\zeta - 1), & \gamma < 1/\zeta, \\ \frac{\kappa\delta}{1+\kappa\delta\tau} + \kappa\delta(\zeta - 1), & \gamma = 1/\zeta, \\ \kappa\lambda\frac{\delta\cos\lambda\kappa\tau - \lambda\sin\lambda\kappa\tau}{\delta\sin\lambda\kappa\tau + \lambda\cos\lambda\kappa\tau} + \delta\kappa(\zeta - 1), & 1/\zeta < \gamma < 1. \end{cases} \tag{6.12}$$

*Here* $\zeta = \left(\boldsymbol{\Theta}^{-1}\right)_{11}$, $\lambda = \sqrt{|\delta(\delta-1)\zeta - \delta^2|}$.

Thus, in this case we trade the mean-reverting asset and hedge it via correlated Brownian motions. Both the mean-reverting asset position and the hedging positions are larger for large correlations. Availability of correlated hedging assets allow us to take larger positions for given risk aversion and wealth.

### 6.2. *The structure of the optimal strategy*

To illustrate the structure of the optimal strategy, we expand the product $\boldsymbol{D}(\tau)\boldsymbol{x}$ in formula (4.13) for optimal control $\boldsymbol{\alpha}^*$:

$$\begin{vmatrix} \boldsymbol{\alpha}_1^* \\ \boldsymbol{\alpha}_2^* \\ \vdots \\ \boldsymbol{\alpha}_n^* \end{vmatrix} = -w \begin{vmatrix} \boldsymbol{D}_{11}(\tau)\boldsymbol{x}_1 + \boldsymbol{D}_{12}(\tau)\boldsymbol{x}_2 + \dots \boldsymbol{D}_{1n}(\tau)\boldsymbol{x}_n \\ \boldsymbol{D}_{21}(\tau)\boldsymbol{x}_1 + \boldsymbol{D}_{22}(\tau)\boldsymbol{x}_2 + \dots \boldsymbol{D}_{2n}(\tau)\boldsymbol{x}_n \\ \vdots \\ \boldsymbol{D}_{n1}(\tau)\boldsymbol{x}_1 + \boldsymbol{D}_{n2}(\tau)\boldsymbol{x}_2 + \dots \boldsymbol{D}_{nn}(\tau)\boldsymbol{x}_n \end{vmatrix}. \tag{6.13}$$

The summand $\boldsymbol{D}_{ii}x_i$ is a position size multiplier for the mean reversion trading of the $i-th$ asset, while $\boldsymbol{D}_{ij}x_j$ is a quantity of the $i-th$ asset required to hedge the position in the $j-th$ asset. In the case of non-correlated assets, each $\boldsymbol{D}_{ij} = 0$, for $i \neq j$. The quantities $\boldsymbol{D}_{ij}$ and $\boldsymbol{D}_{ji}$ satisfy the following relations :

$$\boldsymbol{D}_{ij} + \delta\boldsymbol{\Theta}_{ij}^{-1}\kappa_j = \boldsymbol{D}_{ji} + \delta\boldsymbol{\Theta}_{ij}^{-1}\kappa_i. \tag{6.14}$$

or, in the matrix form

$$\boldsymbol{D} - \boldsymbol{D}^\top = \delta\left[\boldsymbol{\kappa}, \boldsymbol{\Theta}^{-1}\right] = \delta\left(\boldsymbol{\kappa}\boldsymbol{\Theta}^{-1} - \boldsymbol{\Theta}^{-1}\boldsymbol{\kappa}\right), \tag{6.15}$$

here $[\cdot,\cdot]$ denotes a commutator. Note that the difference between $\boldsymbol{D}_{ij}$ and $\boldsymbol{D}_{ji}$ does not depend on time $t$.

### 6.3. *Wealth dynamics*

Similarly to the one-dimensional case, the wealth process $W_t^{\boldsymbol{\alpha}}$ can be found in the explicit form

**Theorem 6.2.** *The wealth process $W_t^{\boldsymbol{\alpha}}$ associated with the optimal control $\boldsymbol{\alpha}_t^*$ is given by the following formulas*

$$\log\left(\frac{W_t^{\boldsymbol{\alpha}}}{W_s^{\boldsymbol{\alpha}}}\right) = \overbrace{\int_s^t \frac{\boldsymbol{Tr}\,\boldsymbol{\Theta}\boldsymbol{D}(T-u) - \delta \boldsymbol{X}_u^\top \boldsymbol{\kappa}\boldsymbol{\Theta}^{-1}\boldsymbol{\kappa}\boldsymbol{X}_u}{2} du}^{\boldsymbol{a}} + \\ + \underbrace{\frac{\boldsymbol{X}_s^\top \boldsymbol{D}(T-s)\boldsymbol{X}_s - \boldsymbol{X}_t^\top \boldsymbol{D}(T-t)\boldsymbol{X}_t}{2}}_{\boldsymbol{b}} + \underbrace{\frac{1}{2}\int_s^t \boldsymbol{X}_u^\top \left[\boldsymbol{D} - \boldsymbol{D}^\top\right] d\boldsymbol{X}_u}_{\boldsymbol{c}}. \tag{6.16}$$

One term of equation (6.16) that is missing in the one-dimensional case is **c**. This summand corresponds to hedging efficiency. It is easy to see that for cases $\boldsymbol{\Theta} = \boldsymbol{I}$ or $\boldsymbol{\kappa} = \kappa\boldsymbol{I}$ this term vanishes. As we mentioned before, the case $\boldsymbol{\kappa} = \kappa\boldsymbol{I}$ can be reduced to the case $\boldsymbol{\Theta} = \boldsymbol{I}$.

### 6.4. *Example. 2-dimensional model.*

To illustrate interactions between reversion speed and correlation, let us consider a two-dimensional example in more details. We shall use the following parameters for this illustration: the numbers of assets shall be $n = 2$, noise magnitude $\boldsymbol{\sigma} = \boldsymbol{I}$, long term mean and initial point $\boldsymbol{\theta} = \mathbf{X}_0 = 0$, risk aversion $\gamma = -4$ and time horizon $T = 3$. We consider an optimal strategy for a portfolio of two correlated Ornstein–Uhlenbeck processes with $\kappa_1 = 1$ and different values of $\kappa_2$ and correlation $\rho$:

$$n = 2, \quad \gamma = -4, \quad \boldsymbol{\sigma} = \boldsymbol{I}, \quad \boldsymbol{\kappa} = diag(1, \kappa_2), \quad \boldsymbol{\theta} = \mathbf{X}_0 = \mathbf{0}, \quad \boldsymbol{\Theta} = \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}. \tag{6.17}$$

Figure 2 shows the value function $J$ as a function of $\log(\kappa_2/\kappa_1)$ ($\kappa_1 = 1$) for several different values of $\rho$. Here, we are varying the lower of two asset mean-reversion rates. It turns out that for sufficiently high correlation $\rho$, the value function has a proper minimum as a function of $\kappa_2$ and it becomes decreasing in $\kappa_2$ as the correlation gets closer to 1. This means that in these cases, one would prefer to have a lower value for the second asset's mean-reversion rate to a slightly higher value (but not to a much higher value $\kappa_2 >> \kappa_1$.) Therefore, with more that one asset, a higher reversion rate is not always good for extracting value from trading, quite unlike the one-dimensional case.

### 6.5. *Impact of correlation*

We have seen in the previous section that the value function can be non-monotonic in mean-reversion rates. Let us show that it is always increasing with the correlation, if all other parameters are fixed.

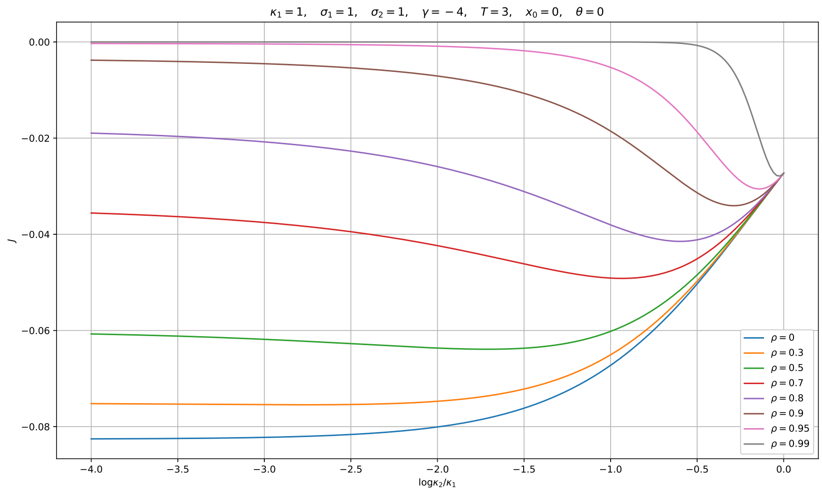



Fig. 2: 2D example. Value function for a range of values for $\kappa_2$ and correlation $\rho$. The horizontal axis is the log-ratio of two mean reversion speeds. When these speeds are equal, the value function does not depend on the correlation. For a sufficiently high absolute value of correlation, the value function has a proper minimum in the log-ratio, so the trader would prefer either a lower or a higher mean-reversion rate for the slower reverting asset, to an intermediate rate. For high correlations, the trader prefers very low or no mean reversion at all in the slower reverting asset to the two equal mean reversion rates.

Suppose now that we start our trading process with no immediate trading opportunities (i.e. $\boldsymbol{x} = \mathbf{0}$). We consider $J(w, \mathbf{0}, t)$ as the function of correlation coefficients $\rho_{mn}$. In the standard Markowitz portfolio optimization problem, one can construct more profitable portfolios when correlations are lower. In our setting, we can prove that the value function has a local minimum at zero correlations $\boldsymbol{\Theta} = \boldsymbol{I}$. Correlations between driving processes enable cross-hedging between positions in different assets and these increase the value function. We have already seen a similar beneficial effect of higher correlations in section 6.1.3 for a special case of a single mean-reverting asset hedged with Brownian motions and the following theorem demonstrates that this effect holds in the general case as well.

**Theorem 6.3.** *In the absence of immediate trading opportunities ($\boldsymbol{x} = \boldsymbol{0}$) the value function $J(w, \boldsymbol{0}, t)$ as a function of pairwise correlation coefficients $\rho_{mn}$ has a local minimum at $\boldsymbol{\Theta} = \boldsymbol{I}$.*

**Proof.** Recall the representation of the value function:

$$J(w,\mathbf{0},t) = \frac{w^\gamma}{\gamma}\exp\left\{\frac{1}{\delta}\int_0^{T-t}\mathbf{Tr}\left(\boldsymbol{F}(u)\right)du\right\} \tag{6.18}$$

where matrix $\boldsymbol{F}$ is equal

$$\boldsymbol{F} = \frac{1}{2}(\boldsymbol{A}+\boldsymbol{A}^\top)\boldsymbol{\Theta}. \tag{6.19}$$

Let us define a new matrix $\boldsymbol{\Gamma}$ :

$$\boldsymbol{\Gamma} = \boldsymbol{\Theta}^{-1}\boldsymbol{\kappa}\boldsymbol{\Theta} \tag{6.20}$$

Note that $\boldsymbol{\Gamma}$ is a result of a similarity transformation of the matrix $\boldsymbol{\kappa}$ and $\lim_{\boldsymbol{\Theta}\to\boldsymbol{I}}\boldsymbol{\Gamma} = \boldsymbol{\kappa}$. For the matrix $\boldsymbol{F}$ we have the following ODE:

$$\begin{aligned}\boldsymbol{F}' &= 2\boldsymbol{F}^2 - \delta\left(\boldsymbol{\kappa}\boldsymbol{F}+\boldsymbol{F}\boldsymbol{\Gamma}\right) + \frac{\delta(\delta-1)}{2}\boldsymbol{\kappa}\boldsymbol{\Gamma},\\ \boldsymbol{F}(0) &= \mathbf{0}.\end{aligned} \tag{6.21}$$

Let $\rho_{mn}$ be an arbitrary correlation coefficient at the position $mn$ (i.e. $mn = (ij)$, $\boldsymbol{\Theta}_{ij} = \boldsymbol{\Theta}_{ji} = \rho_{mn}$) and let us consider the following partial derivatives:

$$\frac{\partial J(w,\mathbf{0},t)}{\partial\rho_{mn}} = \frac{J(w,\mathbf{0},t)}{\delta}\int_0^{T-t}\mathbf{Tr}\left(\frac{\partial\boldsymbol{F}(u)}{\partial\rho_{mn}}\right)du, \tag{6.22}$$

$$\frac{\partial^2 J(w,\mathbf{0},t)}{\partial\rho_{mn}\partial\rho_{pq}} = \frac{J(w,\mathbf{0},t)}{\delta}\int_0^{T-t}\mathbf{Tr}\left(\frac{\partial^2\boldsymbol{F}(u)}{\partial\rho_{mn}\partial\rho_{pq}}\right)du, \tag{6.23}$$

$$\frac{\partial^2 J(w,\mathbf{0},t)}{\partial\rho_{mn}^2} = \frac{J(w,\mathbf{0},t)}{\delta}\int_0^{T-t}\mathbf{Tr}\left(\frac{\partial^2\boldsymbol{F}(u)}{\partial\rho_{mn}^2}\right)du. \tag{6.24}$$

We shall prove the following properties for any $mn$ and $pq$:

$$\lim_{\boldsymbol{\Theta}\to\boldsymbol{I}}\frac{\partial J(w,\mathbf{0},t)}{\partial\rho_{mn}} = 0, \tag{6.25}$$

$$\lim_{\boldsymbol{\Theta}\to\boldsymbol{I}}\frac{\partial^2 J(w,\mathbf{0},t)}{\partial\rho_{mn}\partial\rho_{pq}} = 0, \tag{6.26}$$

$$sign\lim_{\boldsymbol{\Theta}\to\boldsymbol{I}}\frac{\partial^2 J(w,\mathbf{0},t)}{\partial\rho_{mn}^2} = sign\gamma, \quad (\kappa_i\neq\kappa_j), \tag{6.27}$$

$$\lim_{\boldsymbol{\Theta}\to\boldsymbol{I}}\frac{\partial^2 J(w,\mathbf{0},t)}{\partial\rho_{mn}^2} = 0, \quad (\kappa_i=\kappa_j). \tag{6.28}$$

From equation (6.25), the point $\boldsymbol{\Theta} = \boldsymbol{I}$ is an extrema point. Equation (6.26) implies that the Gessian matrix at $\boldsymbol{\Theta} = \boldsymbol{I}$ is a diagonal matrix. Using Silvester's criterion we prove that the Gessian matrix is a positive definite at the point $\boldsymbol{\Theta} = \boldsymbol{I}$, for more details see Appendix D. □

## 7. Wealth distribution moments and analysis of parameter mis-specification

### 7.1. *Closed form formulas*

In practice, one does not know the true values for model parameters, so it is important to understand value function sensitivities to errors in parameters estimation. In this section, we present an ODE based framework for the analysis of parameter mis-specification sensitivity. We provide semi-explicit formulas for the value function corresponding to misspecified parameters. Let $\hat{\boldsymbol{\kappa}}, \hat{\boldsymbol{\sigma}}, \hat{\boldsymbol{\Theta}}$ be estimates of reversion rates, volatility and correlation. We consider the control $\hat{\boldsymbol{\alpha}}$ as a function of these estimates

$$\hat{\boldsymbol{\alpha}} = w\hat{\boldsymbol{\sigma}}^{-1}\left[-\delta\hat{\boldsymbol{\Theta}}^{-1}\hat{\boldsymbol{\kappa}} + \left(\hat{\boldsymbol{A}}^{\top} + \hat{\boldsymbol{A}}\right)\right]\hat{\boldsymbol{\sigma}}^{-1}\boldsymbol{x}. \tag{7.1}$$

Here the matrix $\hat{\boldsymbol{A}}$ is a solution to the following ODE

$$\begin{aligned} \hat{\boldsymbol{A}}'(\tau) &= \mathfrak{R}_{\hat{\boldsymbol{\Theta}},\hat{\boldsymbol{\kappa}},\delta}\hat{\boldsymbol{A}}, \\ \hat{\boldsymbol{A}}(0) &= \mathbf{0}, \end{aligned} \tag{7.2}$$

where the differential operator $\mathfrak{R}$ is defined in (4.11). The wealth process $\hat{W}_t$ generated by the strategy $\hat{\boldsymbol{\alpha}}$ is a solution to the following SDE

$$d\hat{W}_t = \hat{\boldsymbol{\alpha}}_t^{\top} d\mathbf{X}_t. \tag{7.3}$$

**Theorem 7.1.** *Let $P_\epsilon(w, \boldsymbol{x}, t)$ be the following expectation of a function of terminal wealth $\hat{W}_T$ defined by (7.3):*

$$P_\epsilon(w, \boldsymbol{x}, t) = \mathbb{E}\left[\frac{\hat{W}_T^{\epsilon}}{\epsilon}\,\Big|\,\hat{W}_t = w,\ \boldsymbol{X}_t = \boldsymbol{x}\right]. \tag{7.4}$$

*The expectation $P_\epsilon(w, \boldsymbol{x}, t)$ can be explicitly found in the following form*

$$P_\epsilon(w, \boldsymbol{x}, t) = \frac{w^\epsilon}{\epsilon}\exp\left\{\int_0^{T-t}\boldsymbol{Tr}\left(\boldsymbol{\Theta}\boldsymbol{Q}(u)\right)du + \boldsymbol{x}^{\top}\boldsymbol{\sigma}^{-1}\boldsymbol{Q}(T-t)\boldsymbol{\sigma}^{-1}\boldsymbol{x}\right\}, \tag{7.5}$$

*where the matrix $\boldsymbol{Q}$ is a solution to Riccati equation*

$$\begin{aligned} \boldsymbol{Q}' &= \mathfrak{B}\boldsymbol{Q}, \\ \boldsymbol{Q}(0) &= \boldsymbol{0}. \end{aligned} \tag{7.6}$$

*The nonlinear operator $\mathfrak{B}$ is given by*

$$\mathfrak{B}\boldsymbol{Q} = \frac{\left(\boldsymbol{Q}+\boldsymbol{Q}^{\top}\right)\boldsymbol{\Theta}\left(\boldsymbol{Q}+\boldsymbol{Q}^{\top}\right)}{2} + \\ + \left(\epsilon\boldsymbol{\beta}^{\top}\boldsymbol{\Theta} - \boldsymbol{\kappa}\right)\left(\boldsymbol{Q}+\boldsymbol{Q}^{\top}\right) + \frac{\epsilon(\epsilon-1)}{2}\boldsymbol{\beta}^{\top}\boldsymbol{\Theta}\boldsymbol{\beta} - \epsilon\boldsymbol{\beta}^{\top}\boldsymbol{\kappa} \tag{7.7}$$

*and the matrix $\boldsymbol{\beta}$ is defined as*

$$\boldsymbol{\beta} = \boldsymbol{\sigma}\hat{\boldsymbol{\sigma}}^{-1}\left[-\delta\hat{\boldsymbol{\Theta}}^{-1}\hat{\boldsymbol{\kappa}} + \left(\hat{\boldsymbol{A}} + \hat{\boldsymbol{A}}^{\top}\right)\right]\hat{\boldsymbol{\sigma}}^{-1}\boldsymbol{\sigma}, \tag{7.8}$$

*here the matrix $\hat{\boldsymbol{A}}$ is a solution to the equation (7.2).*

When $\epsilon = \gamma$, we obtain the expected utility corresponding to the misspecified parameters. The values $\epsilon = 1$ or $\epsilon = 2$ correspond to the first two moments of $W_T$, so we can calculate the Sharpe ratio

$$Sh[\hat{\boldsymbol{\alpha}}] = \frac{P_1(w, \boldsymbol{x}, t)}{\sqrt{2P_2(w, \boldsymbol{x}, t) - P_1^2(w, \boldsymbol{x}, t)}}. \tag{7.9}$$

It is worth to mention that the effects on a misspecified long term mean level $\boldsymbol{\theta}$ can be also analyzed in the same way. In this case, we have to add the extra term $\exp\left\{\boldsymbol{x}^\top \mathbf{V}\right\}$ to the equation (7.5). Here $\mathbf{V}$ is an $n \times 1$ vector function of inverse time $T - t$.

As an alternative, one can analyse the effect of parameter mis-specification by using Monte-Carlo methods. However, from our point of view, the proposed ODE approach is computationally much more efficient than Monte-Carlo simulations.

### 7.2. *Impact of misspecified parameters*

We illustrate the method presented above with the analysis of misspecified reversion rates $\boldsymbol{\kappa}$ and correlations $\boldsymbol{\Theta}$. For simplicity, we consider the case of just two assets. The results are presented on figures 3 and 4. We measure the effect of misspecification by the difference between the value functions corresponding to true and misspecified parameters (color and value of z-axis respectively.)

#### 7.2.1. *Misspecified reversion rates*

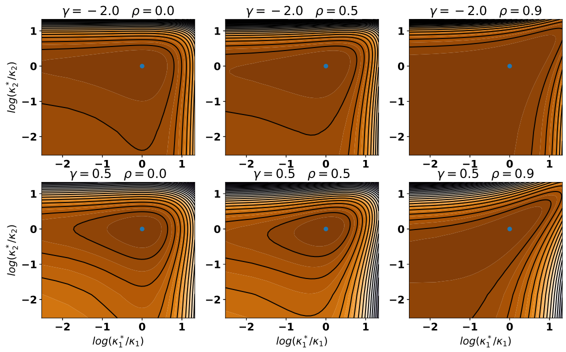
Similarly to the one-dimensional case, the influence of mean reversion coefficient mis-specification is asymmetric. Depending on the value of the correlation, it is more important to correctly estimate the ratio between reversion rates than to estimate the exact value of each mean-reversion rate. It is not surprising given that the optimal strategy hedges the faster mean-reverting asset with the slower one and the hedging accuracy depends on the ratio between reversion speeds.

#### 7.2.2. *Misspecified correlation*

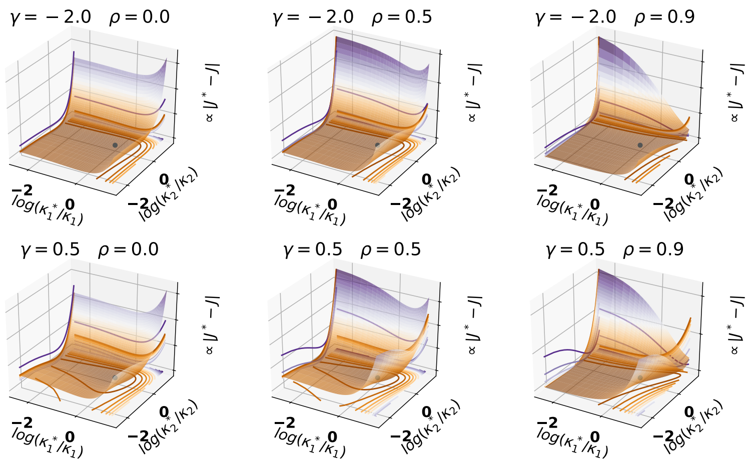
According to the numerical results, the sensitivity to errors in estimation $\rho^*$ of correlation coefficient is increasing for the large absolute values of true correlation coefficient $\rho$. The influence of correlation coefficient mis-specification is symmetric, i.e. the performance depends on the absolute value of the difference $|\rho - \rho^*|$.

## 8. Conclusion

We have obtained quasi-analytical solutions for the problem of optimal trading in multiple correlated Ornstein-Uhlenbeck and Brownian processes. In a general case, the problem boils down to a Ricatti equation. We were able to solve that equation




(a) Heatmap plot.

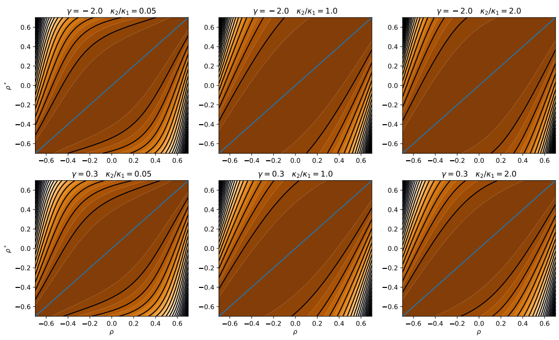



(b) 3D plot.

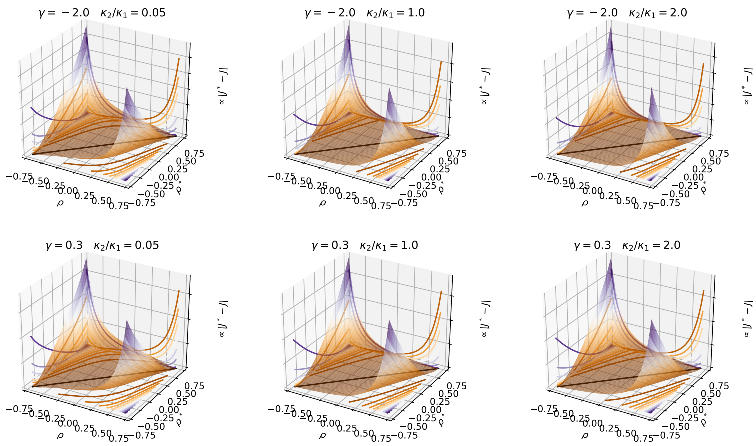
Fig. 3: Misspecified reversion rates. Heatmap plot and 3D plot. The influence of mean reversion coefficient mis-specification is asymmetric. Depending on the value of correlation, it is more important to correctly estimate the ratio between reversion rates than to estimate the exact value of each mean-reversion rate.

for several special cases, including the case of non-correlated assets, the case of correlated assets with identical mean-reversion speeds, and the case of a single mean reverting asset in addition to multiple Brownian motions.

While our model is quite simple, it is sufficient to demonstrate that the optimal trading strategy has a number of non-trivial properties, with the value function



(a) Heatmap plot.



(b) 3D plot.

Fig. 4: Misspecified correlations. Heatmap plot and 3D plot for $\kappa_1 = 0.1$. The sensitivity to errors in the estimate $\rho^*$ of the correlation coefficient is increasing for the large absolute values of true correlation coefficient $\rho$. The influence of correlation coefficient mis-specification is symmetric.

increasing with cross-asset correlations and also sometimes decreasing with some of the reversion rates. We also show that zero mean reversion assets with zero drifts can be quite valuable sources of portfolio diversification.

We also propose a semi-analytical solution for the effect of parameter mis-specification, demonstrate its properties on several examples, and derive a semi-analytical formula for the optimal strategy Sharpe ratio.

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## Appendix A. Reducing the HJB equation to a linear PDE

### A.1. *Distortion transformation*

The first order optimality condition on the control $\boldsymbol{\alpha}^*$ yields the following linear system of equations in $\boldsymbol{\alpha}^*$:

$$\begin{aligned} \frac{J_{ww}}{2}\left[(\boldsymbol{\alpha}^*)^\top \boldsymbol{\Theta} + \boldsymbol{\Theta}\boldsymbol{\alpha}^*\right] = & \quad \boldsymbol{\kappa x} J_w - \boldsymbol{\Theta}\boldsymbol{\nabla} J_w, \\ J_{ww}\boldsymbol{\Theta}\boldsymbol{\alpha}^* = & \quad \boldsymbol{\kappa x} J_w - \boldsymbol{\Theta}\boldsymbol{\nabla} J_w. \end{aligned} \tag{A.1}$$

The solution to this system is

$$\boldsymbol{\alpha}^* = \frac{1}{J_{ww}}\left[\boldsymbol{\Theta}^{-1} + \boldsymbol{\kappa x} - \boldsymbol{\nabla}\right] J_w. \tag{A.2}$$

Using again the first order optimality condition, we get

$$(\boldsymbol{\alpha}^*)^\top \boldsymbol{\kappa x} J_w - (\boldsymbol{\alpha}^*)^\top \boldsymbol{\Theta}\boldsymbol{\nabla} J_w = (\boldsymbol{\alpha}^*)^\top \boldsymbol{\Theta}\boldsymbol{\alpha}^* J_{ww}. \tag{A.3}$$

Substituting it into the HJB equation we arrive at the following terminal value problem:

$$\begin{aligned} J_t - \frac{1}{2}\left(\boldsymbol{\alpha}^*\right)^\top \boldsymbol{\Theta}\boldsymbol{\alpha}^* J_{ww} \boldsymbol{x}^\top \boldsymbol{\kappa}\boldsymbol{\nabla} J + \frac{1}{2}\boldsymbol{\nabla}^\top \boldsymbol{\Theta}\boldsymbol{\nabla} J = & 0, \\ J(w, \boldsymbol{x}, T) = & \frac{w^\gamma}{\gamma}. \end{aligned} \tag{A.4}$$

Substituting the solution for the optimal control $\boldsymbol{\alpha}^*$ yields a non-linear PDE

$$\begin{aligned} J_t - & \quad \frac{1}{2}\frac{J_w^2}{J_{ww}}\left(\boldsymbol{\kappa x}\right)^\top \boldsymbol{\Theta}^{-1}\boldsymbol{\kappa x} + \frac{1}{2}\frac{J_w}{J_{ww}}\left[\left(\boldsymbol{\kappa x}\right)^\top \boldsymbol{\nabla} J_w + \boldsymbol{\nabla}^\top J_w \left(\boldsymbol{\kappa x}\right)\right] \\ - & \quad \frac{1}{2}\frac{1}{J_{ww}}\boldsymbol{\nabla}^\top J_w \boldsymbol{\Theta}\boldsymbol{\nabla} J_w - \boldsymbol{x}^\top \boldsymbol{\kappa}\boldsymbol{\nabla} J + \frac{1}{2}\boldsymbol{\nabla}^\top \boldsymbol{\Theta}\boldsymbol{\nabla} J = 0. \end{aligned} \tag{A.5}$$

We then apply the so-called distortion transformation

$$J = \frac{w^\gamma}{\gamma} f^{1/\delta}(x, t), \quad \delta = \frac{1}{1-\gamma}. \tag{A.6}$$

The exact formulas for the partial derivatives of the value function $J$ are

$$\begin{aligned} J_t = \frac{1}{\delta}\frac{J}{f}\frac{\partial f}{\partial t}, \quad J_w = \frac{\gamma}{w} J, \quad J_{ww} = \frac{\gamma(\gamma-1)}{w^2} J \\ \boldsymbol{\nabla} J = \frac{1}{\delta}\frac{J}{f}\boldsymbol{\nabla} f, \quad \boldsymbol{\nabla} J_w = \frac{\gamma}{w}\frac{1}{\delta}\frac{J}{f}\boldsymbol{\nabla} f \end{aligned} \tag{A.7}$$

Substituting these expressions into the non-linear HJB PDE we obtain

$$
\begin{aligned}
\frac{1}{2}\boldsymbol{\nabla}^\top \boldsymbol{\Theta} \boldsymbol{\nabla} J &= \frac{1}{2}\frac{1}{\delta}\frac{J}{f}\boldsymbol{\nabla}^\top \boldsymbol{\Theta}\boldsymbol{\nabla} f + \frac{1}{2}\frac{1}{\delta}\left(\frac{1}{\delta}-1\right)\frac{J}{f^2}\boldsymbol{\nabla}^\top f \boldsymbol{\Theta}\boldsymbol{\nabla} f. \\
-\frac{1}{2}\frac{1}{J_{ww}}\boldsymbol{\nabla}^\top J_w \boldsymbol{\Theta}\boldsymbol{\nabla} J_w &= -\frac{1}{2}\frac{\gamma^2}{w^2}\frac{1}{\delta^2}\frac{J^2}{f^2}\frac{w^2}{\gamma(\gamma-1)J}\boldsymbol{\nabla}^\top f\boldsymbol{\Theta}\boldsymbol{\nabla} f \\
&= \frac{1}{2}\frac{\gamma}{\delta}\frac{J}{f^2}\boldsymbol{\nabla}^\top f \boldsymbol{\Theta}\boldsymbol{\nabla} f \qquad \text{(A.8)} \\
&= -\frac{1}{2}\frac{1}{\delta}\left(\frac{1}{\delta}-1\right)\frac{J}{f^2}\boldsymbol{\nabla}^\top f\boldsymbol{\Theta}\boldsymbol{\nabla} f \\
-\frac{1}{2}\frac{J_w^2}{J_{ww}} &= -\frac{1}{2}\frac{\gamma^2}{w^2}J^2\frac{w^2}{\gamma(\gamma-1)J} \qquad \text{(A.9)} \\
&= \frac{1}{2}\frac{\gamma}{1-\gamma}J \\
&= \frac{1}{2}\frac{1}{\delta}\delta(\delta-1)J \\
\frac{1}{2}\frac{J_w}{J_{ww}}\left[(\boldsymbol{\kappa x})^\top \boldsymbol{\nabla} J_w + \boldsymbol{\nabla}^\top J_w (\boldsymbol{\kappa x})\right] &= \frac{1}{2}\frac{\gamma J}{w}\frac{w^2}{\gamma(\gamma-1)J}\left[(\boldsymbol{\kappa x})^\top\left(\frac{\gamma}{w}\frac{1}{\delta}\frac{J}{f}\boldsymbol{\nabla} f\right)\right. \\
&\quad \left. + \left(\frac{\gamma}{w}\frac{1}{\delta}\frac{J}{f}\boldsymbol{\nabla} f\right)^\top (\boldsymbol{\kappa x})\right] \\
&= \frac{1}{2}\frac{1}{\delta}\frac{\gamma}{\gamma-1}\frac{J}{f}\left[\boldsymbol{x}^\top\boldsymbol{\kappa}\boldsymbol{\nabla} f + \boldsymbol{\nabla}^\top f\boldsymbol{\kappa x}\right] \qquad \text{(A.10)} \\
&= \frac{1-\delta}{2}\frac{1}{\delta}\frac{J}{f}\left[\boldsymbol{x}^\top\boldsymbol{\kappa}\boldsymbol{\nabla} f + \boldsymbol{\nabla}^\top f\boldsymbol{\kappa x}\right] \\
&= -\frac{\delta-1}{2}\frac{1}{\delta}\frac{J}{f}\left[\boldsymbol{x}^\top\boldsymbol{\kappa}\boldsymbol{\nabla} f + \boldsymbol{\nabla}^\top f\boldsymbol{\kappa x}\right]
\end{aligned}
$$

This yields the following linear equation for the function $f$

$$
\begin{aligned}
\frac{1}{2}\boldsymbol{\nabla}\boldsymbol{\Theta}\boldsymbol{\nabla} f - \boldsymbol{x}^\top\boldsymbol{\kappa}\boldsymbol{\nabla} f - \frac{\delta-1}{2}\boldsymbol{x}^\top\boldsymbol{\kappa}\boldsymbol{\nabla} f - \frac{\delta-1}{2}\boldsymbol{\nabla}^\top f(\boldsymbol{\kappa x}) \\
+\frac{1}{2}\delta(\delta-1)(\boldsymbol{\kappa x})^\top\boldsymbol{\Theta}^{-1}(\boldsymbol{\kappa x}) f + \frac{\partial f}{\partial t} = 0.
\end{aligned} \qquad \text{(A.11)}
$$

The optimal control $\boldsymbol{\alpha}^*$ is then

$$
\boldsymbol{\alpha}^*(w,\boldsymbol{x},t) = w\left[-\delta\boldsymbol{\Theta}^{-1}\boldsymbol{\kappa x} + \frac{\boldsymbol{\nabla} f}{f}\right]. \qquad \text{(A.12)}
$$

## Appendix B. Derivation of the Riccati equation for matrix D

$$\begin{aligned}\left(\boldsymbol{A}+\boldsymbol{A}^\top\right)' &= \left(\boldsymbol{A}^\top+\boldsymbol{A}\right)\boldsymbol{\Theta}\left(\boldsymbol{A}^\top+\boldsymbol{A}\right) - [(\delta-1)\boldsymbol{\kappa}+\boldsymbol{\kappa}]\left(\boldsymbol{A}^\top+\boldsymbol{A}\right)\\ &\quad - \left(\boldsymbol{A}^\top+\boldsymbol{A}\right)((\delta-1)\boldsymbol{\kappa}+\boldsymbol{\kappa}) + \delta(\delta-1)\boldsymbol{\kappa}\boldsymbol{\Theta}^{-1}\boldsymbol{\kappa}\\ &= \left(\boldsymbol{A}^\top+\boldsymbol{A}\right)\boldsymbol{\Theta}\left(\boldsymbol{A}^\top+\boldsymbol{A}\right) - (\delta\boldsymbol{\kappa})\left(\boldsymbol{A}^\top+\boldsymbol{A}\right)\\ &\quad - \left(\boldsymbol{A}^\top+\boldsymbol{A}\right)(\delta\boldsymbol{\kappa}) + \delta(\delta-1)\boldsymbol{\kappa}\boldsymbol{\Theta}^{-1}\boldsymbol{\kappa}\\ &= \left(\delta\boldsymbol{\kappa}\boldsymbol{\Theta}^{-1}-\boldsymbol{D}^\top\right)\boldsymbol{\Theta}\left(\delta\boldsymbol{\Theta}^{-1}\boldsymbol{\kappa}-\boldsymbol{D}\right)\\ &\quad -\delta\boldsymbol{\kappa}\left(\delta\boldsymbol{\Theta}^{-1}\boldsymbol{\kappa}-\boldsymbol{D}\right) - \left(\delta\boldsymbol{\kappa}\boldsymbol{\Theta}^{-1}-\boldsymbol{D}^\top\right)\delta\boldsymbol{\kappa} + \delta(\delta-1)\boldsymbol{\kappa}\boldsymbol{\Theta}^{-1}\boldsymbol{\kappa}\\ &= \boldsymbol{D}^\top\boldsymbol{\Theta}\boldsymbol{D} - \delta\boldsymbol{\kappa}\boldsymbol{\Theta}^{-1}\boldsymbol{\kappa}.\end{aligned} \tag{B.1}$$

## Appendix C. The solution to the wealth SDE

The wealth process under the optimal control is

$$dW_t = -W_t\mathbf{X}_t^\top\boldsymbol{D}^\top d\mathbf{X}_t. \tag{C.1}$$

We can represent the process $W_t$ in a stochastic exponent form:

$$W_t = W_0 e^{\boldsymbol{\lambda}^\top\mathbf{Y}_t}, \quad d\mathbf{Y}_t = u\,dt + \boldsymbol{\eta}\,d\mathbf{X}_t. \tag{C.2}$$

and apply Itô's lemma

$$dW_t = W_t\left[\boldsymbol{\lambda}^\top d\mathbf{Y}_t + \frac{1}{2}\boldsymbol{\lambda}^\top d\mathbf{Y}_t d\mathbf{Y}_t^\top\boldsymbol{\lambda}\right]. \tag{C.3}$$

Note that

$$\boldsymbol{\lambda}^\top u = -\frac{1}{2}\boldsymbol{\lambda}^\top\boldsymbol{\eta}\boldsymbol{\Theta}\boldsymbol{\eta}^\top\boldsymbol{\lambda} \tag{C.4}$$

$$\boldsymbol{\lambda}^\top\boldsymbol{\eta} = -\mathbf{X}_t^\top\boldsymbol{D}^\top \tag{C.5}$$

$$\boldsymbol{\eta}^\top\boldsymbol{\lambda} = -\boldsymbol{D}\mathbf{X}_t \tag{C.6}$$

$$\boldsymbol{\lambda}^\top u = -\frac{1}{2}\mathbf{X}_t^\top\boldsymbol{D}^\top\boldsymbol{\Theta}\boldsymbol{D}\mathbf{X}_t \tag{C.7}$$

$$\boldsymbol{\lambda}^\top d\mathbf{Y}_t = \boldsymbol{\lambda}^\top u\,dt + \boldsymbol{\lambda}\boldsymbol{\eta}\,d\mathbf{Y}_t. \tag{C.8}$$

$$\boldsymbol{\lambda}^\top d\mathbf{Y}_t = -\frac{1}{2}\mathbf{X}_t^\top\boldsymbol{D}^\top\boldsymbol{\Theta}\boldsymbol{D}\mathbf{X}_t dt - \mathbf{X}_t^\top\boldsymbol{D}^\top d\mathbf{X}_t. \tag{C.9}$$

Therefore,

$$\int_0^t\boldsymbol{\lambda}^\top d\mathbf{Y}_s = -\frac{1}{2}\int_0^t\mathbf{X}_s^\top\boldsymbol{D}(T-s)^\top\boldsymbol{\Theta}\boldsymbol{D}(T-s)\mathbf{X}_s ds - \int_0^t\mathbf{X}_s^\top\boldsymbol{D}(T-s)^\top d\mathbf{X}_s. \tag{C.10}$$

Since the matrix $\boldsymbol{D}$ is a solution to the following Riccati ODE

$$-\frac{d\boldsymbol{D}}{dt} = \boldsymbol{D}^\top\boldsymbol{\Theta}\boldsymbol{D} - \delta\boldsymbol{\kappa}\boldsymbol{\Theta}^{-1}\boldsymbol{\kappa}, \tag{C.11}$$

we get

$$W_t = W_0 \exp\left\{-\frac{\delta}{2}\int_0^t \mathbf{X}_s^\top \boldsymbol{\kappa}\boldsymbol{\Theta}^{-1}\boldsymbol{\kappa}\mathbf{X}_s ds - \frac{1}{2}\left[\mathbf{X}_t^\top \boldsymbol{D}(T-t)\mathbf{X}_t - \mathbf{X}_0^\top \boldsymbol{D}(T)\mathbf{X}_0\right]\right\}$$
$$\cdot \exp\left\{\frac{1}{2}\int_0^t \mathbf{Tr}\boldsymbol{\Theta}\boldsymbol{D}(T-s)ds + \frac{1}{2}\int_0^t \mathbf{X}_s^\top\left[\boldsymbol{D}-\boldsymbol{D}^\top\right]d\mathbf{X}_s\right\} \quad \text{(C.12)}$$

## Appendix D. Proof of Theorem 6.3

To prove Theorem 6.3, it is sufficient to demonstrate the following properties of $\boldsymbol{F}$:

$$\lim_{\boldsymbol{\Theta}\to\boldsymbol{I}}\left(\frac{\partial \boldsymbol{F}}{\partial\rho_{mn}}\right)_{ij} = 0, \quad (ij)\notin mn, \quad \text{(D.1)}$$

$$\lim_{\boldsymbol{\Theta}\to\boldsymbol{I}}\mathbf{Tr}\frac{\partial \boldsymbol{F}}{\partial\rho_{mn}} = 0, \quad \text{(D.2)}$$

$$\lim_{\boldsymbol{\Theta}\to\boldsymbol{I}}\mathbf{Tr}\frac{\partial^2 \boldsymbol{F}}{\partial\rho_{mn}\partial\rho_{pq}} \equiv 0, \quad \text{(D.3)}$$

$$\lim_{\boldsymbol{\Theta}\to\boldsymbol{I}}\mathbf{Tr}\frac{\partial^2 \boldsymbol{F}}{\partial\rho_{mn}^2} > 0, \quad \gamma>0, \quad \kappa_i\neq\kappa_j, \quad \text{(D.4)}$$

$$\lim_{\boldsymbol{\Theta}\to\boldsymbol{I}}\mathbf{Tr}\frac{\partial^2 \boldsymbol{F}}{\partial\rho_{mn}^2} < 0, \quad \gamma<0, \quad \kappa_i\neq\kappa_j, \quad \text{(D.5)}$$

$$\lim_{\boldsymbol{\Theta}\to\boldsymbol{I}}\mathbf{Tr}\frac{\partial^2 \boldsymbol{F}}{\partial\rho_{mn}^2} \equiv 0, \quad \gamma=0 \quad or \quad \kappa_i=\kappa_j. \quad \text{(D.6)}$$

### D.1. *Proof of formulas (D.1) and (D.2)*

Consider the partial derivative of $\boldsymbol{F}$ with respect to the correlation $\rho_{mn}$:

$$\left(\frac{\partial \boldsymbol{F}}{\partial\rho_{mn}}\right)' = \frac{\partial}{\partial\rho_{mn}}\left(2\boldsymbol{F}\boldsymbol{F} - \delta\left(\boldsymbol{\kappa}\boldsymbol{F}+\boldsymbol{F}\boldsymbol{\Gamma}\right) + \frac{\delta(\delta-1)}{2}\boldsymbol{\kappa}\boldsymbol{\Gamma}\right)$$
$$= 2\left(\frac{\partial \boldsymbol{F}}{\partial\rho_{mn}}\boldsymbol{F} + \boldsymbol{F}\frac{\partial \boldsymbol{F}}{\partial\rho_{mn}}\right) - \delta\left(\boldsymbol{\kappa}\frac{\partial \boldsymbol{F}}{\partial\rho_{mn}} + \frac{\partial \boldsymbol{F}}{\partial\rho_{mn}}\boldsymbol{\Gamma} + \boldsymbol{F}\frac{\partial \boldsymbol{\Gamma}}{\partial\rho_{mn}}\right) \quad \text{(D.7)}$$
$$+ \frac{\delta(\delta-1)}{2}\boldsymbol{\kappa}\frac{\partial \boldsymbol{\Gamma}}{\partial\rho_{mn}}$$

As $\mathbf{\Theta}$ tends to $\boldsymbol{I}$, we get

$$\begin{aligned}
\boldsymbol{\lambda}' &= 2\left(\boldsymbol{\lambda}\boldsymbol{\Psi} + \boldsymbol{\Psi}\boldsymbol{\lambda}\right) - \delta\left(\boldsymbol{\kappa}\boldsymbol{\lambda} + \boldsymbol{\lambda}\boldsymbol{\kappa} + \boldsymbol{\Psi}\overbrace{\left[\boldsymbol{\kappa}\boldsymbol{I}^{mn} - \boldsymbol{I}^{mn}\boldsymbol{\kappa}\right]}^{under\ Lemma Appendix\ F.1, F.3}\right) \\
&+ \frac{\delta(\delta-1)}{2}\boldsymbol{\kappa}\underbrace{\left[\boldsymbol{\kappa}\boldsymbol{I}^{mn} - \boldsymbol{I}^{mn}\boldsymbol{\kappa}\right]}_{under\ Lemma Appendix\ F.1, F.3} \qquad (D.8) \\
\boldsymbol{\lambda}'_{ij} &= 2\boldsymbol{\lambda}_{ij}\left(\boldsymbol{\Psi}_{ii} + \boldsymbol{\Psi}_{jj} - \delta\left[\kappa_i + \kappa_j\right]\right) - \delta\sum_{s=1}^{n}\sum_{k=1}^{n}\left(\boldsymbol{\Psi}_{is}\boldsymbol{\kappa}_{sk}\boldsymbol{I}^{mn}_{kj} - \boldsymbol{\Psi}_{is}\boldsymbol{I}^{mn}_{sk}\boldsymbol{\kappa}_{kj}\right) \\
&+ \frac{\delta(\delta-1)}{2}\sum_{s=1}^{n}\sum_{k=1}^{n}\left[\boldsymbol{\kappa}_{is}\boldsymbol{\kappa}_{sk}\boldsymbol{I}^{mn}_{kj} - \boldsymbol{\kappa}_{is}\boldsymbol{I}^{mn}_{sk}\boldsymbol{\kappa}_{kj}.\right] \qquad (D.9) \\
\boldsymbol{\lambda}'_{ij} &= \boldsymbol{\lambda}_{ij}\left(2\boldsymbol{\Psi}_{ii} + 2\boldsymbol{\Psi}_{jj} - \delta\left[\kappa_i + \kappa_j\right]\right) - \delta\boldsymbol{\Psi}_{ii}\boldsymbol{I}^{mn}_{ij}\left[\kappa_i - \kappa_j\right] \\
&+ \frac{\delta(\delta-1)}{2}\kappa_i\boldsymbol{I}^{mn}_{ij}\left[\kappa_i - \kappa_j\right]. \qquad (D.10) \\
\boldsymbol{\lambda}'_{ij} &= \boldsymbol{\lambda}_{ij}\left(2\boldsymbol{\Psi}_{ii} + 2\boldsymbol{\Psi}_{jj} - \delta\left[\kappa_i + \kappa_j\right]\right) - \delta\boldsymbol{I}^{mn}_{ij}\left[\kappa_i - \kappa_j\right]\left[\boldsymbol{\Psi}_{ii} + \frac{1-\delta}{2}\kappa_i\right]. \qquad (D.11) \\
\boldsymbol{\lambda}_{ij}(0) &= 0. \qquad (D.12)
\end{aligned}$$

We have $\boldsymbol{I}^{mn}_{ij} = 0$ for $(ij) \neq mn$, hence $\boldsymbol{\lambda}_{ij} \equiv 0$. Moreover, for diagonal elements $(ii) \neq mn$, $\forall i = 1..n$, therefore $\mathbf{Tr}\boldsymbol{\lambda} \equiv 0$.

## D.2. *Proof of formula (D.3)*

$$\begin{aligned}
\left(\frac{\partial^2 \boldsymbol{F}}{\partial\rho_{mn}\partial\rho_{pq}}\right)' &= \frac{\partial}{\partial\rho_{mn}\partial\rho_{pq}}\left(2\boldsymbol{F}\boldsymbol{F} - \delta\left(\boldsymbol{\kappa}\boldsymbol{F} + \boldsymbol{F}\mathbf{\Gamma}\right) + \frac{\delta(\delta-1)}{2}\boldsymbol{\kappa}\mathbf{\Gamma}\right) \qquad (D.13) \\
&= 2\left(\frac{\partial^2 \boldsymbol{F}}{\partial\rho_{mn}\partial\rho_{pq}}\boldsymbol{F} + \frac{\partial \boldsymbol{F}}{\partial\rho_{mn}}\frac{\partial \boldsymbol{F}}{\partial\rho_{pq}} + \frac{\partial \boldsymbol{F}}{\partial\rho_{pq}}\frac{\partial \boldsymbol{F}}{\partial\rho_{mn}} + \boldsymbol{F}\frac{\partial^2 \boldsymbol{F}}{\partial\rho_{mn}\partial\rho_{pq}}\right) \\
&- \delta\left(\boldsymbol{\kappa}\frac{\partial^2 \boldsymbol{F}}{\partial\rho_{mn}\partial\rho_{pq}} + \frac{\partial^2 \boldsymbol{F}}{\partial\rho_{mn}\partial\rho_{pq}}\mathbf{\Gamma} + \frac{\partial \boldsymbol{F}}{\partial\rho_{mn}}\frac{\partial \mathbf{\Gamma}}{\partial\rho_{pq}} + \frac{\partial \boldsymbol{F}}{\partial\rho_{pq}}\frac{\partial \mathbf{\Gamma}}{\partial\rho_{mn}}\right. \\
&\left. + \boldsymbol{F}\frac{\partial^2 \mathbf{\Gamma}}{\partial\rho_{mn}\partial\rho_{pq}}\right) + \frac{\delta(\delta-1)}{2}\boldsymbol{\kappa}\frac{\partial^2 \mathbf{\Gamma}}{\partial\rho_{mn}\partial\rho_{pq}}.
\end{aligned}$$

Let us define

$$\boldsymbol{\eta} = \lim_{\mathbf{\Theta}\to\boldsymbol{I}}\frac{\partial^2 \boldsymbol{F}}{\partial\rho_{mn}\partial\rho_{pq}}, \quad \tilde{\boldsymbol{\lambda}} = \lim_{\mathbf{\Theta}\to\boldsymbol{I}}\frac{\partial \boldsymbol{F}}{\partial\rho_{pq}}, \qquad (D.14)$$

therefore

$$\begin{aligned}
\boldsymbol{\eta}' &= 2\left[\boldsymbol{\eta}\boldsymbol{\Psi} + \boldsymbol{\Psi}\boldsymbol{\eta}\right] - \delta\left[\boldsymbol{\kappa}\boldsymbol{\eta} + \boldsymbol{\eta}\boldsymbol{\kappa} + \boldsymbol{\lambda}\left(\boldsymbol{\kappa}\boldsymbol{I}^{pq} - \boldsymbol{I}^{pq}\boldsymbol{\kappa}\right) + \tilde{\boldsymbol{\lambda}}\left(\boldsymbol{\kappa}\boldsymbol{I}^{mn} - \boldsymbol{I}^{mn}\boldsymbol{\kappa}\right) + \boldsymbol{\Psi}\boldsymbol{Q}\right] \\
&\quad + \frac{\delta(\delta-1)}{2}\boldsymbol{\kappa}\boldsymbol{Q} && \text{(D.15)} \\
\boldsymbol{\eta}'_{ii} &= 4\boldsymbol{\eta}_{ii}\boldsymbol{\Psi}_{ii} - 2\delta\kappa_i\boldsymbol{\eta}_{ii} - \delta\boldsymbol{\Psi}_{ii}\boldsymbol{Q}_{ii} && \text{(D.16)} \\
&\quad - \delta\sum_{s=1}^{n}\sum_{k=1}^{n}\left[\boldsymbol{\lambda}_{is}\boldsymbol{\kappa}_{sk}\boldsymbol{I}^{pq}_{ki} - \boldsymbol{\lambda}_{is}\boldsymbol{I}^{pq}_{sk}\boldsymbol{\kappa}_{ki} + \tilde{\boldsymbol{\lambda}}_{is}\boldsymbol{\kappa}_{sk}\boldsymbol{I}^{mn}_{ki} - \tilde{\boldsymbol{\lambda}}_{is}\boldsymbol{I}^{mn}_{sk}\boldsymbol{\kappa}_{ki}\right] + \frac{\delta(\delta-1)}{2}\kappa_i\boldsymbol{Q}_{ii} \\
\boldsymbol{\eta}'_{ii} &= 2\boldsymbol{\eta}_{ii}\ \left[2\boldsymbol{\Psi}_{ii} - \delta\boldsymbol{\kappa}_{ii}\right] - \delta\sum_{s=1}^{n}\left[\boldsymbol{\lambda}_{is}\kappa_s\boldsymbol{I}^{pq}_{si} - \boldsymbol{\lambda}_{is}\boldsymbol{I}^{pq}_{si}\kappa_i + \tilde{\boldsymbol{\lambda}}_{is}\kappa_s\boldsymbol{I}^{mn}_{si} - \tilde{\boldsymbol{\lambda}}_{is}\boldsymbol{I}^{mn}_{si}\boldsymbol{\kappa}_i\right] && \text{(D.17)} \\
\boldsymbol{\eta}'_{ii} &= 2\boldsymbol{\eta}_{ii}\ \left[2\boldsymbol{\Psi}_{ii} - \delta\boldsymbol{\kappa}_{ii}\right], \quad \boldsymbol{\eta}_{ii}(0) = 0 && \text{(D.18)} \\
\boldsymbol{\eta}_{ii} &\equiv 0 && \text{(D.19)} \\
\mathbf{Tr}\boldsymbol{\eta} &\equiv 0. && \text{(D.20)}
\end{aligned}$$

### D.3. *Proof of formulas (D.4)-(D.6)*

From the definition of $\boldsymbol{\varphi}$, we obtain the following ODE:

$$\begin{aligned}
\boldsymbol{\varphi}' &= 2\left[\boldsymbol{\varphi}\boldsymbol{\Psi} + \boldsymbol{\Psi}\boldsymbol{\varphi}\right] - \delta\left[\boldsymbol{\kappa}\boldsymbol{\varphi} + \boldsymbol{\varphi}\boldsymbol{\kappa} + 2\boldsymbol{\lambda}\left(\boldsymbol{\kappa}\boldsymbol{I}^{mn} - \boldsymbol{I}^{mn}\boldsymbol{\kappa}\right) + \boldsymbol{\Psi}\boldsymbol{P}\right] + \frac{\delta(\delta-1)}{2}\boldsymbol{\kappa}\boldsymbol{P} \\
\boldsymbol{\varphi}(0) &= \mathbf{0}. && \text{(D.21)}
\end{aligned}$$

or in the element-wise notation

$$\begin{aligned}
\boldsymbol{\varphi}'_{ii} &= 2\boldsymbol{\varphi}_{ii}\left[2\boldsymbol{\Psi}_{ii} - \delta\boldsymbol{\kappa}_{ii}\right] - 2\delta\boldsymbol{\lambda}_{ij}\boldsymbol{I}^{mn}_{ij}(\kappa_j - \kappa_i) - \delta\boldsymbol{\Psi}_{ii}\boldsymbol{P}_{ii} + \frac{\delta(\delta-1)}{2}\kappa_i\boldsymbol{P}_{ii} && \text{(D.22)} \\
\boldsymbol{\varphi}'_{ii} &= \boldsymbol{\varphi}_{ii}\left[4\boldsymbol{\Psi}_{ii} - 2\delta\boldsymbol{\kappa}_{ii}\right] + 2\delta\boldsymbol{\lambda}_{ij}\boldsymbol{I}^{mn}_{ij}(\kappa_i - \kappa_j) - \delta\boldsymbol{P}_{ii}\left(\boldsymbol{\Psi}_{ii} + \frac{1-\delta}{2}\kappa_i\right) && \text{(D.23)} \\
\boldsymbol{\varphi}'_{ii} &= \boldsymbol{\varphi}_{ii}\left[4\boldsymbol{\Psi}_{ii} - 2\delta\boldsymbol{\kappa}_{ii}\right] + 2\delta\boldsymbol{I}^{mn}_{ij}(\kappa_i - \kappa_j)\left[\boldsymbol{\lambda}_{ij} - \kappa_i\frac{(1-\sqrt{\delta})}{2}\frac{\mathrm{e}^{2\kappa_i\sqrt{\delta}\tau} + 1}{\mathrm{e}^{2\kappa_i\sqrt{\delta}\tau} + \omega}\right] && \text{(D.24)} \\
\boldsymbol{\varphi}(0) &= 0. && \text{(D.25)}
\end{aligned}$$

It is easy to check that for $\kappa_i = \kappa_j$

$$\boldsymbol{\varphi}_{ii} = \boldsymbol{\varphi}_{jj} = 0. \qquad \text{(D.26)}$$

Eq. D.26 also holds for the special case $\gamma = 0$ ($\delta = 1$). Indeed, for this case $\boldsymbol{\lambda}_{ij} = \boldsymbol{\lambda}_{ji} = 0$. It turns out that the right hand side of the last equation for $\boldsymbol{\varphi}_{ij}$ is equal to zero, so $\boldsymbol{\varphi}_{ii} = \boldsymbol{\varphi}_{jj} = 0$.

We proceed with the case $i \notin mn$. Each element $\boldsymbol{P}_{ii}$ equals 0, i.e. $\boldsymbol{\varphi}_{ii}(\tau) \equiv 0$. Therefore, the trace of the matrix $\boldsymbol{\varphi}$ contains only two non-zero terms with multi-index $mn$. For simplicity of notation, we denote it as $i$ and $j$, i.e $mn = (ij)$. The summands $\boldsymbol{\varphi}_{ii}$ and $\boldsymbol{\varphi}_{jj}$ can be found via the following ODEs

$$\boldsymbol{\varphi}'_{ii} - \boldsymbol{\varphi}_{ii}\left[4\boldsymbol{\Psi}_{ii} - 2\delta\kappa_i\right] = 2\delta(\kappa_i - \kappa_j)\left[\boldsymbol{\lambda}_{ij} - \kappa_i \frac{(1-\sqrt{\delta})}{2}\frac{\mathrm{e}^{2\kappa_i\sqrt{\delta}\tau}+1}{\mathrm{e}^{2\kappa_i\sqrt{\delta}\tau}+\omega}\right] \tag{D.27}$$

$$\boldsymbol{\varphi}'_{jj} - \boldsymbol{\varphi}_{jj}\left[4\boldsymbol{\Psi}_{jj} - 2\delta\kappa_j\right] = 2\delta(\kappa_j - \kappa_i)\left[\boldsymbol{\lambda}_{ji} - \kappa_j \frac{(1-\sqrt{\delta})}{2}\frac{\mathrm{e}^{2\kappa_j\sqrt{\delta}\tau}+1}{\mathrm{e}^{2\kappa_j\sqrt{\delta}\tau}+\omega}\right] \tag{D.28}$$

$$\boldsymbol{\varphi}_{ii}(0) = \boldsymbol{\varphi}_{jj}(0) = 0. \tag{D.29}$$

Lemma Appendix E.3 concludes the proof.

## Appendix E. Properties of $\boldsymbol{F}$ in the zero correlation case

This appendix demonstrates several properties of matrix $\boldsymbol{F}$ for the zero correlation case that were used in proofs above. Let us define matrices $\boldsymbol{\Psi}$, $\boldsymbol{\lambda}$ and $\boldsymbol{\varphi}$ as

$$\boldsymbol{\Psi} = \lim_{\boldsymbol{\Theta}\to\boldsymbol{I}} \boldsymbol{F}, \quad \boldsymbol{\lambda} = \lim_{\boldsymbol{\Theta}\to\boldsymbol{I}} \frac{\partial \boldsymbol{F}}{\partial \rho_{mn}}, \quad \boldsymbol{\varphi} = \lim_{\boldsymbol{\Theta}\to\boldsymbol{I}} \frac{\partial^2 \boldsymbol{F}}{\partial \rho_{mn}^2}. \tag{E.1}$$

**Lemma Appendix E.1.** *The matrix $\boldsymbol{\Psi}$ is a diagonal matrix with the following entries*

$$\boldsymbol{\Psi} = diag\left(\Psi(\kappa_1,\tau), \Psi(\kappa_1,\tau), \ldots, \Psi(\kappa_n,\tau)\right). \tag{E.2}$$

*Here the function $\Psi(\kappa,\tau)$ can be defined as a solution to the following one-dimensional Riccati equation*

$$\frac{d\Psi}{d\tau} = 2\Psi^2 - 2\delta\kappa\Psi + \frac{\delta(\delta-1)}{2}\kappa^2, \quad \Psi(0) = 0, \tag{E.3}$$

*which can be solved explicitly:*

$$\Psi(\kappa,\tau) = \frac{\kappa\sqrt{\delta}(\sqrt{\delta}-1)}{2}\frac{\mathrm{e}^{2\kappa\sqrt{\delta}\tau}-1}{\mathrm{e}^{2\kappa\sqrt{\delta}\tau}+\omega}, \quad \omega = \frac{1-\sqrt{\delta}}{1+\sqrt{\delta}}. \tag{E.4}$$

*The function $\Psi$ has the following properties:*

$$\int \Psi(\kappa,\tau)d\tau = \frac{\delta+\sqrt{\delta}}{2}\kappa\tau - \frac{1}{2}\ln\left(\mathrm{e}^{2\kappa\sqrt{\delta}\tau}+\omega\right) + C, \tag{E.5}$$

$$\Psi(\kappa,\tau) + \frac{1-\delta}{2}\kappa = \frac{\kappa(1-\sqrt{\delta})}{2}\frac{\mathrm{e}^{2\kappa\sqrt{\delta}\tau}+1}{\mathrm{e}^{2\kappa\sqrt{\delta}\tau}+\omega}. \tag{E.6}$$

**Proof.**

$$\frac{d\Psi}{d\tau} = 2\Psi^2 - 2\delta\kappa\Psi + \frac{\delta(\delta-1)}{2}\kappa^2, \tag{E.7}$$

$$d\tau = \frac{d\Psi}{2\Psi^2 - 2\delta\kappa\Psi + \delta(\delta-1)\kappa^2/2} \tag{E.8}$$

$$\int d\tau = \int \frac{d\Psi}{2\Psi^2 - 2\delta\kappa\Psi + \delta(\delta-1)\kappa^2/2} \tag{E.9}$$

$$\tau + c = \frac{1}{2\sqrt{\delta}\kappa}\left[\ln\left(\frac{\delta\kappa - 2\Psi}{\sqrt{\delta}\kappa} + 1\right) - \ln\left(1 - \frac{\delta\kappa - 2\Psi}{\sqrt{\delta}\kappa}\right)\right] \tag{E.10}$$

$$\tau + c = \frac{1}{2\sqrt{\delta}\kappa}\ln\left(\frac{\delta\kappa - 2\Psi + \sqrt{\delta}\kappa}{-\delta\kappa + 2\Psi + \sqrt{\delta}\kappa}\right) \tag{E.11}$$

$$2\sqrt{\delta}\kappa\tau + \ln\left(\frac{\delta\kappa + \sqrt{\delta}\kappa}{-\delta\kappa + \sqrt{\delta}\kappa}\right) = \ln\left(\frac{\delta\kappa - 2\Psi + \sqrt{\delta}\kappa}{-\delta\kappa + 2\Psi + \sqrt{\delta}\kappa}\right) \tag{E.12}$$

$$2\sqrt{\delta}\kappa\tau = \ln\left(\frac{\delta\kappa - 2\Psi + \sqrt{\delta}\kappa}{-\delta\kappa + 2\Psi + \sqrt{\delta}\kappa}\right) - \ln\left(\frac{1+\sqrt{\delta}}{1-\sqrt{\delta}}\right) \tag{E.13}$$

$$\mathrm{e}^{2\sqrt{\delta}\kappa\tau} = \frac{\left(\delta\kappa - 2\Psi + \sqrt{\delta}\kappa\right)(1-\sqrt{\delta})}{\left(-\delta\kappa + 2\Psi + \sqrt{\delta}\kappa\right)(1+\sqrt{\delta})} \tag{E.14}$$

$$\mathrm{e}^{2\sqrt{\delta}\kappa\tau} = \frac{2\Psi(\sqrt{\delta}-1) + \sqrt{\delta}\kappa(1-\delta)}{2\Psi(\sqrt{\delta}+1) + \sqrt{\delta}\kappa(1-\delta)} \tag{E.15}$$

Hence for $\Psi$ we have

$$\Psi = \frac{1}{2}\frac{\sqrt{\delta}\kappa(1-\delta)\left(1 - \mathrm{e}^{2\sqrt{\delta}\kappa\tau}\right)}{\mathrm{e}^{2\sqrt{\delta}\kappa\tau}\left(1+\sqrt{\delta}\right) + 1 - \sqrt{\delta}} \tag{E.16}$$

$$\Psi = -\frac{\sqrt{\delta}\kappa}{2}\frac{(1-\sqrt{\delta})\left(1 - \mathrm{e}^{-2\sqrt{\delta}\kappa\tau}\right)}{1 + \frac{1-\sqrt{\delta}}{1+\sqrt{\delta}}\mathrm{e}^{-2\sqrt{\delta}\kappa\tau}} \tag{E.17}$$

$$\Psi = \frac{\kappa\sqrt{\delta}(\sqrt{\delta}-1)}{2}\frac{\mathrm{e}^{2\kappa\sqrt{\delta}\tau} - 1}{\mathrm{e}^{2\kappa\sqrt{\delta}\tau} + \omega}, \quad \omega = \frac{1-\sqrt{\delta}}{1+\sqrt{\delta}}. \tag{E.18}$$

□

**Lemma Appendix E.2.** *Each element* $\boldsymbol{\lambda}_{ij}$ *of the matrix* $\boldsymbol{\lambda}$ *can be represented as*

$$\begin{aligned}\boldsymbol{\lambda}_{ij} = \kappa_i &\frac{\sqrt{\delta}(1-\sqrt{\delta})}{2(\mathrm{e}^{2\kappa_i\sqrt{\delta}\tau} + \omega)(\mathrm{e}^{2\kappa_j\sqrt{\delta}\tau} + \omega)} \times \\ &\times \left[\frac{\kappa_j - \kappa_i}{\kappa_j + \kappa_i}\left(\mathrm{e}^{(\kappa_j+\kappa_i)\sqrt{\delta}\tau} - 1\right)\left(\mathrm{e}^{(\kappa_j+\kappa_i)\sqrt{\delta}\tau} + \omega\right)\right. \\ &\left. + \mathrm{e}^{2\kappa_i\sqrt{\delta}\tau}\left(\mathrm{e}^{(\kappa_j-\kappa_i)\sqrt{\delta}\tau} - 1\right)\left(\mathrm{e}^{(\kappa_j-\kappa_i)\sqrt{\delta}\tau} + \omega\right)\right].\end{aligned} \tag{E.19}$$

***Proof.*** *Differentiating the matrix equation 6.19 with respect to time $t$ and taking the limit $\boldsymbol{\Theta} \to \boldsymbol{I}$, we get the following element-wise ODEs for the $\boldsymbol{\lambda}_{ij}$:*

$$\begin{aligned} \boldsymbol{\lambda}_{ij}' &= \boldsymbol{\lambda}_{ij}\left(2\boldsymbol{\Psi}_{ii} + 2\boldsymbol{\Psi}_{jj} - \delta\left[\kappa_i + \kappa_j\right]\right) - \delta\left[\kappa_i - \kappa_j\right]\left[\boldsymbol{\Psi}_{ii} + \frac{1-\delta}{2}\kappa_i\right] \\ \boldsymbol{\lambda}_{ij}(0) &= 0. \end{aligned} \tag{E.20}$$

*The corresponding homogeneous ODE can be solved explicitly:*

$$\frac{\mathrm{e}^{\kappa_i\sqrt{\delta}\tau + \kappa_j\sqrt{\delta}\tau}}{(\mathrm{e}^{2\kappa_i\sqrt{\delta}\tau} + \omega)(\mathrm{e}^{2\kappa_j\sqrt{\delta}\tau} + \omega)}. \tag{E.21}$$

*Thus, the solution to the non-homogeneous problem is*

$$\begin{aligned} \boldsymbol{\lambda}_{ij} &= -\delta\left[\kappa_i - \kappa_j\right]\frac{\kappa_i(1-\sqrt{\delta})}{2}\frac{\mathrm{e}^{\kappa_i\sqrt{\delta}\tau + \kappa_j\sqrt{\delta}\tau}}{(\mathrm{e}^{2\kappa_i\sqrt{\delta}\tau} + \omega)(\mathrm{e}^{2\kappa_j\sqrt{\delta}\tau} + \omega)} \\ &\times \int_0^\tau \frac{(\mathrm{e}^{2\kappa_i\sqrt{\delta}\zeta} + 1)(\mathrm{e}^{2\kappa_j\sqrt{\delta}\zeta} + \omega)}{\mathrm{e}^{\kappa_i\sqrt{\delta}\zeta + \kappa_j\sqrt{\delta}\zeta}} d\zeta \\ &= -\delta\left[\kappa_i - \kappa_j\right]\frac{\kappa_i(1-\sqrt{\delta})}{2}\frac{\mathrm{e}^{\kappa_i\sqrt{\delta}\tau + \kappa_j\sqrt{\delta}\tau}}{(\mathrm{e}^{2\kappa_i\sqrt{\delta}\tau} + \omega)(\mathrm{e}^{2\kappa_j\sqrt{\delta}\tau} + \omega)} \\ &\times \int_0^\tau \left[\mathrm{e}^{(\kappa_i+\kappa_j)\sqrt{\delta}\zeta} + \omega\mathrm{e}^{(\kappa_i-\kappa_j)\sqrt{\delta}\zeta} + \mathrm{e}^{(\kappa_j-\kappa_i)\sqrt{\delta}\zeta} + \omega\mathrm{e}^{-(\kappa_i+\kappa_j)\sqrt{\delta}\zeta}\right] d\zeta; \\ &= \delta\left[\kappa_j - \kappa_i\right]\frac{\kappa_i(1-\sqrt{\delta})}{2}\frac{\mathrm{e}^{(\kappa_i+\kappa_j)\sqrt{\delta}\tau}}{(\mathrm{e}^{2\kappa_i\sqrt{\delta}\tau} + \omega)(\mathrm{e}^{2\kappa_j\sqrt{\delta}\tau} + \omega)} \\ &\times \left[\frac{\mathrm{e}^{(\kappa_i+\kappa_j)\sqrt{\delta}\tau} - \omega\mathrm{e}^{-(\kappa_i+\kappa_j)\sqrt{\delta}\tau} + \omega - 1}{(\kappa_i+\kappa_j)\sqrt{\delta}} + \frac{\mathrm{e}^{(\kappa_j-\kappa_i)\sqrt{\delta}\tau} - \omega\mathrm{e}^{-(\kappa_j-\kappa_i)\sqrt{\delta}\tau} + \omega - 1}{(\kappa_j-\kappa_i)\sqrt{\delta}}\right] \\ &= \kappa_i\frac{\sqrt{\delta}(1-\sqrt{\delta})}{2(\mathrm{e}^{2\kappa_i\sqrt{\delta}\tau} + \omega)(\mathrm{e}^{2\kappa_j\sqrt{\delta}\tau} + \omega)} \times \\ &\times \left[\frac{\kappa_j - \kappa_i}{\kappa_j + \kappa_i}\left(\mathrm{e}^{(\kappa_j+\kappa_i)\sqrt{\delta}\tau} - 1\right)\left(\mathrm{e}^{(\kappa_j+\kappa_i)\sqrt{\delta}\tau} + \omega\right)\right. \\ &\left. + \mathrm{e}^{2\kappa_i\sqrt{\delta}\tau}\left(\mathrm{e}^{(\kappa_j-\kappa_i)\sqrt{\delta}\tau} - 1\right)\left(\mathrm{e}^{(\kappa_j-\kappa_i)\sqrt{\delta}\tau} + \omega\right)\right] \end{aligned} \tag{E.22}$$

□

**Lemma Appendix E.3.** *Any diagonal element $\boldsymbol{\varphi}_{ii}$ of the matrix $\boldsymbol{\varphi}$ is a solution to the following ODE*

$$\begin{aligned} \boldsymbol{\varphi}_{ii}' &= -2\kappa_i\sqrt{\delta}\frac{\mathrm{e}^{2\kappa_i\sqrt{\delta}\tau} - \omega}{\mathrm{e}^{2\kappa_i\sqrt{\delta}\tau} + \omega}\boldsymbol{\varphi}_{ii} + \delta(1-\sqrt{\delta})\kappa_i(\kappa_i - \kappa_j) \times \\ &\times \left[-\frac{\mathrm{e}^{2\kappa_i\sqrt{\delta}\tau} + 1}{\mathrm{e}^{2\kappa_i\sqrt{\delta}\tau} + \omega} + \sqrt{\delta}\frac{\kappa_j - \kappa_i}{\kappa_j + \kappa_i}\frac{\mathrm{e}^{(\kappa_j+\kappa_i)\sqrt{\delta}\tau} - 1}{\mathrm{e}^{2\kappa_i\sqrt{\delta}\tau} + \omega}\frac{\mathrm{e}^{(\kappa_j+\kappa_i)\sqrt{\delta}\tau} + \omega}{\mathrm{e}^{2\kappa_j\sqrt{\delta}\tau} + \omega}\right. \\ &\left. + \sqrt{\delta}\mathrm{e}^{2\kappa_i\sqrt{\delta}\tau}\frac{\mathrm{e}^{(\kappa_j-\kappa_i)\sqrt{\delta}\tau} - 1}{\mathrm{e}^{2\kappa_i\sqrt{\delta}\tau} + \omega}\frac{\mathrm{e}^{(\kappa_j-\kappa_i)\sqrt{\delta}\tau} + \omega}{\mathrm{e}^{2\kappa_j\sqrt{\delta}\tau} + \omega}\right] \\ \boldsymbol{\varphi}_{ii}(0) &= 0 \end{aligned} \tag{E.23}$$

*Moreover, the following inequalities hold for any* $\kappa_i > 0$, $\kappa_j > 0$, $T > 0$ *and* $\delta > 0$*:*

$$\begin{aligned}
&\int_0^T \left[\boldsymbol{\varphi}_{ii}(u) + \boldsymbol{\varphi}_{jj}(u)\right] du > 0, \quad \delta > 1, \quad \kappa_i \neq \kappa_j \\
&\int_0^T \left[\boldsymbol{\varphi}_{ii}(u) + \boldsymbol{\varphi}_{jj}(u)\right] du \equiv 0, \quad \delta = 1 \quad or \quad \kappa_i = \kappa_j \\
&\int_0^T \left[\boldsymbol{\varphi}_{ii}(u) + \boldsymbol{\varphi}_{jj}(u)\right] du < 0, \quad 0 < \delta < 1 \qquad \kappa_i \neq \kappa_j
\end{aligned} \tag{E.24}$$

**Proof.** Can be checked by direct calculations. □

## Appendix F. Properties of correlation matrices

In this section we use two special types of square symmetric matrices, $\boldsymbol{I}^{mn}$ and $\boldsymbol{I}^{uu}$. They are defined as follows: matrix $\boldsymbol{I}^{mn}$ has zero entries, except elements with multiindex $(mn)$; these elements are equal to 1:

$$\boldsymbol{I}^{mn}_{ij} = 0, \forall (ij) \neq (mn), \quad \boldsymbol{I}^{mn}_{ij} = 1, (ij) = (mn), \quad or \quad (ji) = (mn). \tag{F.1}$$

Matrix $\boldsymbol{I}^{mn}$ is a traceless matrix, $\mathbf{Tr}\boldsymbol{I}^{mn} = 0$. The matrix $\boldsymbol{I}^{uu}$ also has zero entries, except only one element on $(u, u)$. This element is equal to 1.

The following lemma describes several properties of the correlation matrix $\boldsymbol{\Theta}$ and of the similarity transform $\boldsymbol{\Gamma} = \boldsymbol{\Theta}^{-1}\boldsymbol{\kappa}\boldsymbol{\Theta}$ of the matrix $\boldsymbol{\kappa}$.

**Lemma Appendix F.1.** *Correlation matrix* $\boldsymbol{\Theta}$ *and its similarity transform* $\boldsymbol{\Gamma}$ *satisfy the following equations:*

$$\frac{\partial \boldsymbol{\Theta}^{-1}}{\partial \rho_{mn}} = -\boldsymbol{\Theta}^{-1}\frac{\partial \boldsymbol{\Theta}}{\partial \rho_{mn}}\boldsymbol{\Theta}^{-1} \tag{F.2}$$

$$\lim_{\boldsymbol{\Theta}\to \boldsymbol{I}} \frac{\partial \boldsymbol{\Gamma}}{\partial \rho_{mn}} = \boldsymbol{\kappa}\boldsymbol{I}^{mn} - \boldsymbol{I}^{mn}\boldsymbol{\kappa}. \tag{F.3}$$

$$\lim_{\boldsymbol{\Theta}\to \boldsymbol{I}} \frac{\partial^2 \boldsymbol{\Gamma}}{\partial \rho_{mn}\partial \rho_{pq}} = \boldsymbol{Q}, \quad \boldsymbol{Q}_{ii} = 0, \forall i = 1..n. \tag{F.4}$$

$$\lim_{\boldsymbol{\Theta}\to \boldsymbol{I}} \frac{\partial^2 \boldsymbol{\Gamma}}{\partial \rho_{mn}^2} = \boldsymbol{P}, \quad \boldsymbol{P}_{ii} = 2\mathcal{I}(ij \in mn)\left[\kappa_i - \kappa_j\right]. \tag{F.5}$$

**F.1. *Proof of formula (F.2).***

$$
\begin{aligned}
\boldsymbol{\Theta}\boldsymbol{\Theta}^{-1} &= \boldsymbol{I} && \text{(F.6)}\\
\frac{\partial}{\partial \rho_{mn}}\left(\boldsymbol{\Theta}\boldsymbol{\Theta}^{-1}\right) &= \frac{\partial \boldsymbol{I}}{\partial \rho_{mn}} && \text{(F.7)}\\
\boldsymbol{\Theta}\frac{\partial \boldsymbol{\Theta}^{-1}}{\partial \rho_{mn}} &= -\frac{\partial \boldsymbol{\Theta}}{\partial \rho_{mn}}\boldsymbol{\Theta}^{-1} && \text{(F.8)}\\
\boldsymbol{\Theta}\frac{\partial \boldsymbol{\Theta}^{-1}}{\partial \rho_{mn}} &= -\frac{\partial \boldsymbol{\Theta}}{\partial \rho_{mn}}\boldsymbol{\Theta}^{-1} && \text{(F.9)}\\
\frac{\partial \boldsymbol{\Theta}^{-1}}{\partial \rho_{mn}} &= -\boldsymbol{\Theta}^{-1}\frac{\partial \boldsymbol{\Theta}}{\partial \rho_{mn}}\boldsymbol{\Theta}^{-1} && \text{(F.10)}
\end{aligned}
$$

**F.2. *Proof of formula (F.3)***

$$
\begin{aligned}
\lim_{\boldsymbol{\Theta}\to\boldsymbol{I}}\frac{\partial \boldsymbol{\Gamma}}{\partial \rho_{mn}} &= \lim_{\boldsymbol{\Theta}\to\boldsymbol{I}}\frac{\partial\left(\boldsymbol{\Theta}^{-1}\boldsymbol{\kappa}\boldsymbol{\Theta}\right)}{\partial \rho_{mn}}\\
&= \lim_{\boldsymbol{\Theta}\to\boldsymbol{I}}\frac{\partial \boldsymbol{\Theta}^{-1}}{\partial \rho_{mn}}\boldsymbol{\kappa}\boldsymbol{\Theta} + \lim_{\boldsymbol{\Theta}\to\boldsymbol{I}}\boldsymbol{\Theta}^{-1}\boldsymbol{\kappa}\frac{\partial \boldsymbol{\Theta}}{\partial \rho_{mn}}\\
&= \lim_{\boldsymbol{\Theta}\to\boldsymbol{I}}\frac{\partial \boldsymbol{\Theta}^{-1}}{\partial \rho_{mn}}\boldsymbol{\kappa}\boldsymbol{I} + \boldsymbol{I}\boldsymbol{\kappa}\lim_{\boldsymbol{\Theta}\to\boldsymbol{I}}\frac{\partial \boldsymbol{\Theta}}{\partial \rho_{mn}} && \text{(F.11)}\\
&= -\lim_{\boldsymbol{\Theta}\to\boldsymbol{I}}\frac{\partial \boldsymbol{\Theta}}{\partial \rho_{mn}}\boldsymbol{\kappa} + \boldsymbol{\kappa}\lim_{\boldsymbol{\Theta}\to\boldsymbol{I}}\frac{\partial \boldsymbol{\Theta}}{\partial \rho_{mn}}\\
&= -\boldsymbol{I}^{mn}\boldsymbol{\kappa} + \boldsymbol{\kappa}\boldsymbol{I}^{mn}\\
&= \boldsymbol{\kappa}\boldsymbol{I}^{mn} - \boldsymbol{I}^{mn}\boldsymbol{\kappa}.
\end{aligned}
$$

### F.3. *Proof of formula (F.4)*

$$\begin{aligned}
\frac{\partial^2 \boldsymbol{\Gamma}}{\partial \rho_{mn}\partial \rho_{pq}} &= \frac{\partial^2}{\partial \rho_{mn}\partial \rho_{pq}} \boldsymbol{\Theta}^{-1}\boldsymbol{\kappa}\boldsymbol{\Theta} \quad \text{(F.12)}\\
&= \frac{\partial^2 \boldsymbol{\Theta}^{-1}}{\partial \rho_{mn}\partial \rho_{pq}}\boldsymbol{\kappa}\boldsymbol{\Theta} + \frac{\partial \boldsymbol{\Theta}^{-1}}{\partial \rho_{mn}}\boldsymbol{\kappa}\frac{\partial \boldsymbol{\Theta}}{\partial \rho_{pq}} + \frac{\partial \boldsymbol{\Theta}^{-1}}{\partial \rho_{pq}}\boldsymbol{\kappa}\frac{\partial \boldsymbol{\Theta}}{\partial \rho_{mn}} + \boldsymbol{\Theta}^{-1}\boldsymbol{\kappa}\frac{\partial^2 \boldsymbol{\Theta}}{\partial \rho_{mn}\partial \rho_{pq}}\\
&= -\frac{\partial}{\partial \rho_{pq}}\left[\boldsymbol{\Theta}^{-1}\frac{\partial \boldsymbol{\Theta}}{\partial \rho_{mn}}\boldsymbol{\Theta}^{-1}\right]\boldsymbol{\kappa}\boldsymbol{\Theta} - \boldsymbol{\Theta}^{-1}\frac{\partial \boldsymbol{\Theta}}{\partial \rho_{mn}}\boldsymbol{\Theta}^{-1}\boldsymbol{\kappa}\frac{\partial \boldsymbol{\Theta}}{\partial \rho_{pq}}\\
&\quad - \boldsymbol{\Theta}^{-1}\frac{\partial \boldsymbol{\Theta}}{\partial \rho_{pq}}\boldsymbol{\Theta}^{-1}\boldsymbol{\kappa}\frac{\partial \boldsymbol{\Theta}}{\partial \rho_{mn}}\\
&= -\frac{\partial \boldsymbol{\Theta}^{-1}}{\partial \rho_{pq}}\frac{\partial \boldsymbol{\Theta}}{\partial \rho_{mn}}\boldsymbol{\Theta}^{-1}\boldsymbol{\kappa}\boldsymbol{\Theta} - \boldsymbol{\Theta}^{-1}\frac{\partial \boldsymbol{\Theta}}{\partial \rho_{mn}}\frac{\partial \boldsymbol{\Theta}^{-1}}{\partial \rho_{pq}}\boldsymbol{\kappa}\boldsymbol{\Theta}\\
&\quad - \boldsymbol{\Theta}^{-1}\frac{\partial \boldsymbol{\Theta}}{\partial \rho_{mn}}\boldsymbol{\Theta}^{-1}\boldsymbol{\kappa}\frac{\partial \boldsymbol{\Theta}}{\partial \rho_{pq}} - \boldsymbol{\Theta}^{-1}\frac{\partial \boldsymbol{\Theta}}{\partial \rho_{pq}}\boldsymbol{\Theta}^{-1}\boldsymbol{\kappa}\frac{\partial \boldsymbol{\Theta}}{\partial \rho_{mn}}\\
&= \boldsymbol{\Theta}^{-1}\frac{\partial \boldsymbol{\Theta}}{\partial \rho_{pq}}\boldsymbol{\Theta}^{-1}\frac{\partial \boldsymbol{\Theta}}{\partial \rho_{mn}}\boldsymbol{\Theta}^{-1}\boldsymbol{\kappa}\boldsymbol{\Theta} + \boldsymbol{\Theta}^{-1}\frac{\partial \boldsymbol{\Theta}}{\partial \rho_{mn}}\boldsymbol{\Theta}^{-1}\frac{\partial \boldsymbol{\Theta}}{\partial \rho_{pq}}\boldsymbol{\Theta}^{-1}\boldsymbol{\kappa}\boldsymbol{\Theta}\\
&\quad - \boldsymbol{\Theta}^{-1}\frac{\partial \boldsymbol{\Theta}}{\partial \rho_{mn}}\boldsymbol{\Theta}^{-1}\boldsymbol{\kappa}\frac{\partial \boldsymbol{\Theta}}{\partial \rho_{pq}} - \boldsymbol{\Theta}^{-1}\frac{\partial \boldsymbol{\Theta}}{\partial \rho_{pq}}\boldsymbol{\Theta}^{-1}\boldsymbol{\kappa}\frac{\partial \boldsymbol{\Theta}}{\partial \rho_{mn}}\\
\boldsymbol{Q} &= \boldsymbol{I}^{pq}\boldsymbol{I}^{mn}\boldsymbol{\kappa} + \boldsymbol{I}^{mn}\boldsymbol{I}^{pq}\boldsymbol{\kappa} - \boldsymbol{I}^{mn}\boldsymbol{\kappa}\boldsymbol{I}^{pq} - \boldsymbol{I}^{pq}\boldsymbol{\kappa}\boldsymbol{I}^{mn} \quad \text{(F.13)}\\
\boldsymbol{Q}_{ii} &= \sum_{s=1}^{n}\sum_{k=1}^{n}\left[\boldsymbol{I}^{pq}_{is}\boldsymbol{I}^{mn}_{sk}\boldsymbol{\kappa}_{si} + \boldsymbol{I}^{mn}_{is}\boldsymbol{I}^{pq}_{sk}\boldsymbol{\kappa}_{si} - \boldsymbol{I}^{mn}_{is}\boldsymbol{\kappa}_{sk}\boldsymbol{I}^{pq}_{ki} - \boldsymbol{I}^{pq}_{is}\boldsymbol{\kappa}_{sk}\boldsymbol{I}^{mn}_{ki}\right]. \quad \text{(F.14)}\\
\boldsymbol{Q}_{ii} &= \sum_{s=1}^{n}\left[\boldsymbol{I}^{pq}_{is}\boldsymbol{I}^{mn}_{si}\boldsymbol{\kappa}_{ii} + \boldsymbol{I}^{mn}_{is}\boldsymbol{I}^{pq}_{si}\boldsymbol{\kappa}_{ii} - \boldsymbol{I}^{mn}_{is}\boldsymbol{\kappa}_{ss}\boldsymbol{I}^{pq}_{si} - \boldsymbol{I}^{pq}_{is}\boldsymbol{\kappa}_{ss}\boldsymbol{I}^{mn}_{si}\right] \quad \text{(F.15)}\\
\boldsymbol{Q}_{ii} &= 0. \quad \text{(F.16)}
\end{aligned}$$

Here we used $\boldsymbol{I}^{mn}_{is} = 0$ if $\boldsymbol{I}^{pq}_{si} = 1$ for each $s = 1..n$ and vice versa.

### F.4. *Proof of formula (F.5)*

$$\boldsymbol{P}_{ii} = 2\sum_{s=1}^{n}\left[\boldsymbol{I}^{mn}_{is}\boldsymbol{I}^{mn}_{si}\boldsymbol{\kappa}_{ii} - \boldsymbol{I}^{mn}_{is}\boldsymbol{\kappa}_{ss}\boldsymbol{I}^{mn}_{si}\right] \quad \text{(F.17)}$$

$$\boldsymbol{P}_{ii} = 2\mathcal{I}(ij \in mn)\left[\kappa_i - \kappa_j\right] \quad \text{(F.18)}$$