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Conic Relaxations with Stable Exactness Conditions for Parametric Robust Convex Polynomial Problems

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Abstract In this paper, we examine stable exact relaxations for classes of parametric robust convex polynomial optimization problems under affinely parameterized data uncertainty in the constraints. We first show that a parametric robust *convex polynomial* problem with convex compact uncertainty sets enjoys stable exact *conic relaxations* under the validation of a characteristic cone constraint qualification. We then show that such stable exact conic relaxations become stable exact *semidefinite programming relaxations* for a parametric robust *SOS-convex polynomial* problem, where the uncertainty sets are assumed to be bounded spectrahedra. In addition, under the corresponding constraint qualification, we derive stable exact *second-order cone programming relaxations* for some classes of parametric robust convex quadratic programs under ellipsoidal uncertainty sets.

Keywords Robust optimization · convex polynomial · stable exact relaxation · spectrahedral uncertainty set · conic relaxation

Mathematics Subject Classification (2010) 49K99 · 65K10 · 90C29 · 90C46

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1 Introduction

Due to noise and lack of information in real world optimization problems, it is important to identify and find optimal solutions that are immunized against data uncertainty. Robust optimization has emerged as a useful and efficient deterministic approach to treat optimization problems under data uncertainty [3, 4, 7, 8]. The cornerstone of robust optimization is that the so-called robust counterpart should be computationally tractable. In that sense, the exact semidefinite programming (SDP) relaxation of a given robust optimization problem is a highly desirable feature because SDP problems can be efficiently solved (e.g., using interior point methods) [2, 9, 34, 38].

If we restrict ourselves to convex polynomial programs, then the notion of SOS-convexity (sum-of-squares-convexity) [1, 20] becomes essential since it has been proposed as a tractable sufficient condition for convexity based on semidefinite programming. The SOS-convex polynomials cover many commonly used convex polynomials such as convex quadratic functions and convex separable polynomials. The benefit of an SOS-convex polynomial is that deciding whether a given polynomial is SOS-convex or not can be equivalently rewritten as an SDP problem (and thus, can be efficiently checked). Furthermore, for an SOS-convex optimization problem, its optimal value and optimal solution can be found by solving an SDP problem [27], that is, an exact SDP relaxation holds for the special class of SOS-convex problems.

Regarding robust optimization problems with SOS-convex polynomials, exact SDP relaxations have been shown to hold for robust convex quadratic optimization problems under ellipsoidal data uncertainty [3], and more generally, for some classes of robust SOS-convex programs [25], including robust quadratically constrained convex optimization problems and robust separable convex polynomial optimization problems. With the help of the Slater condition, the authors of [25] characterized robust solutions and exact SDP relaxations of robust SOS-convex polynomial optimization problems under polytopic and ellipsoidal uncertainty. In addition to that, tight SDP relaxations for a class of robust SOS-convex polynomial problems without the Slater condition were obtained in [15]. This approach has been widely employed in the literature (see, e.g., [10, 13, 14, 23, 30]).

Besides the exact SDP relaxation approach, another trend in robust optimization is to identify classes of robust problems having relaxations which can be efficiently solved by means of a second-order cone programming (SOCP) problem [31, 34]. As a matter of example, it has been shown in [18] that a robust convex quadratic optimization problem under restricted ellipsoidal data uncertainty can be equivalently reformulated as an SOCP problem. Concerning nonconvex polynomial optimization problems, a convergent bounded degree hierarchy of SOCP relaxations was recently proposed in [16], whereas an exact SOCP relaxation for minimax nonconvex separable quadratic problems was established in [24].

In this work, we consider a parametric robust *convex polynomial* optimization problem under general convex compact uncertainty sets, and then we focus on a parametric robust *SOS-convex polynomial* optimization problem where the constraint data are affinely parameterized and the uncertainty sets are assumed to be bounded spectrahedra. The main purpose of this paper is to provide conditions that guarantee exact conic relaxations for classes of robust convex/SOS-convex polynomial optimization problems. More precisely, we make the following main contributions:

(i) We show that a parametric robust *convex polynomial* problem with convex compact uncertainty sets enjoys stable exact *conic relaxations* under the validation of a characteristic cone constraint qualification.

(ii) We also show that the stable exact conic relaxations obtained in (i) become stable exact *semidefinite programming (SDP) relaxations* for a parametric robust *SOS-convex polynomial* problem with spectrahedral uncertainty sets. In this sense, the characteristic cone constraint qualification can be considered as the *weakest* regularity condition that guarantees the validation of exact SDP relaxations for robust SOS-convex polynomial problems.

(iii) Under the corresponding constraint qualification, we derive stable exact *second-order cone programming (SOCP) relaxations* for some classes of parametric robust convex quadratic programs involving ellipsoidal uncertainty sets.

Numerical examples are also given to illustrate the necessity of the assumptions and the significance of the obtained results.

The outline of the paper is as follows. In Section 2, we first present a characterization of stable exact conic relaxations for the class of robust convex polynomial problems, and then derive corresponding results for the classes of robust SOS-convex polynomial/convex quadratic problems. Section 3 is devoted to presenting characterizations of stable exact second-order cone programming relaxations for some classes of robust convex quadratic problems.

2 Stable Exact Relaxations for Classes of Parametric Robust Convex Polynomial Programs

We start this section by presenting some definitions and preliminaries that will be used in the paper. The notation $\mathbb{R}^n$ stands for the n-dimensional Euclidean space whose norm is denoted by $\|\cdot\|_2$ for each $n \in \mathbb{N} := \{1, 2, \ldots\}$. The inner product in $\mathbb{R}^n$ is defined by $\langle x, y\rangle := x^\top y$ for all $x, y \in \mathbb{R}^n$. The origin of any space is denoted by 0 but we may use 0_n for the origin of $\mathbb{R}^n$ in situations where some confusion might be possible. For a nonempty set $\Gamma \subset \mathbb{R}^n$, $\operatorname{conv} \Gamma$ denotes the convex hull of Γ, while $\operatorname{cone} \Gamma := \mathbb{R}_+ \operatorname{conv} \Gamma$ stands for the convex conical hull of Γ, where $\mathbb{R}_+ := [0, +\infty) \subset \mathbb{R}$. As usual, the symbol I_n stands for the identity $(n \times n)$ matrix. A symmetric $(n \times n)$ matrix M is said to be positive semi-definite, denoted by $M \succeq 0$, whenever $x^\top Mx \geq 0$ for all $x \in \mathbb{R}^n$. If $x^\top Mx > 0$ for all $x \in \mathbb{R}^n \setminus \{0_n\}$, then M is called positive definite, denoted by $M \succ 0$.

The space of all real polynomials on $\mathbb{R}^n$ is denoted by $\mathbb{R}[x]$ and the set of all $n \times r$ matrix polynomials is denoted by $\mathbb{R}[x]^{n\times r}$. We say that $f \in \mathbb{R}[x]$ is sum-of-squares (see, e.g., [2, 28, 29]) if there exist $f_j \in \mathbb{R}[x]$, $j = 1, \ldots, r$, such that $f = \sum_{j=1}^r f_j^2$. The set consisting of all sum-of-squares polynomials is denoted by Σ^2, which is a subset of the set of all nonnegative polynomials, denoted by $\mathcal{P}$. Moreover, the set consisting of all sum-of-squares (respectively, nonnegative) polynomials with degree at most d is denoted by Σ^2_d (respectively, $\mathcal{P}_d$). We say that $F \in \mathbb{R}[x]^{n\times n}$ is an SOS matrix polynomial if $F(x) = H(x)H(x)^\top$, where $H(x) \in \mathbb{R}[x]^{n\times r}$ is a matrix polynomial for some $r \in \mathbb{N}$. A real polynomial f on $\mathbb{R}^n$ is called *SOS-convex* if the Hessian matrix function $F : x \mapsto \nabla^2 f(x)$ is an SOS matrix polynomial [20]. Clearly, an SOS-convex polynomial is convex. However, the converse is not true; that is, there exists a convex polynomial that is not SOS-convex [1]. It is known that any convex quadratic function and any convex separable polynomial is SOS-convex. Moreover, an SOS-convex polynomial can be non-quadratic and non-separable. For instance, $f(x) := x_1^8 + x_1^2 + x_1x_2 + x_2^2$ is an SOS-convex polynomial (see [20]) that is non-quadratic and non-separable.

As in [9, 37], for an extended real-valued function $\varphi : \mathbb{R}^n \to \overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$, we set

$$\operatorname{dom} \varphi := \{x \in \mathbb{R}^n \mid \varphi(x) < +\infty\}, \quad \operatorname{epi} \varphi := \{(x, \mu) \in \mathbb{R}^n \times \mathbb{R} \mid \varphi(x) \leq \mu\}.$$

The *conjugate function* of φ, $\varphi^* : \mathbb{R}^n \to \overline{\mathbb{R}}$, is defined by

$$\varphi^*(w) = \sup\{\langle w, x\rangle - \varphi(x) \mid x \in \operatorname{dom} \varphi\}, \quad w \in \mathbb{R}^n.$$

For a closed convex subset $\Gamma \subset \mathbb{R}^n$, its indicator function $\delta_\Gamma : \mathbb{R}^n \to \overline{\mathbb{R}}$ is defined as $\delta_\Gamma(x) := 0$ if $x \in \Gamma$ and $\delta_\Gamma(x) := +\infty$ if $x \notin \Gamma$. If one of the functions $f_1,\ f_2$ is continuous, then we have

$$\operatorname{epi}(f_1 + f_2)^* = \operatorname{epi} f_1^* + \operatorname{epi} f_2^*. \tag{1}$$

Parametric robust convex polynomial problems. Let us first consider a *parametric* robust *convex* polynomial problem that is defined as follows: For a convex polynomial $f : \mathbb{R}^n \to \mathbb{R}$ (regarded as a *parameter*), one has the following robust convex polynomial program

$$\inf_{x\in\mathbb{R}^n} \{f(x) \mid g_j^0(x) + \sum_{i=1}^{q_j} v_j^i g_j^i(x) \le 0,\ \forall\, v_j := (v_j^1, \ldots, v_j^{q_j}) \in \mathcal{V}_j, j = 1, \ldots, m\}, \tag{P_f}$$

where $v_j, j = 1, \ldots, m$ are uncertain parameters, $\mathcal{V}_j,\ j = 1, \ldots, m$ are the uncertainty sets that are assumed to be nonempty, convex and compact, and $g_j^i : \mathbb{R}^n \to \mathbb{R},\ i = 0, 1, \ldots, q_j, j = 1, \ldots, m$ are given polynomials such that, for each fixed $v_j \in \mathcal{V}_j$, the function $g_j(\cdot, v_j)$ given by

$$g_j(x, v_j) := g_j^0(x) + \sum_{i=1}^{q_j} v_j^i g_j^i(x),\ x \in \mathbb{R}^n, \tag{2}$$

is a convex polynomial on $\mathbb{R}^n$ for $j = 1, \ldots, m$. In what follows, we use the characteristic cone of the constraints given by

$$\widetilde{C} := \operatorname{cone}\{(0_n, 1) \cup \operatorname{epi} g_j^*(\cdot, v_j) \mid v_j \in \mathcal{V}_j, j = 1, \ldots, m\}. \tag{3}$$

The first theorem in this section characterizes stable exact *conic* relaxations in terms of the characteristic cone for the family of robust convex polynomial problems defined by (P_f) when f varies in the class of *convex* polynomials.

Theorem 2.1 (Characterization of stable exact conic relaxations) *Let $F := \{x \in \mathbb{R}^n \mid g_j(x, v_j) \le 0,\ \forall v_j \in \mathcal{V}_j, j = 1, \ldots, m\} \neq \emptyset$, where $g_j, j = 1, \ldots, m$, are given as in* (2)*. Then, the following statements are equivalent:*

(i) *The characteristic cone $\widetilde{C}$ in* (3) *is closed.*

(ii) *For any convex polynomial f on $\mathbb{R}^n$ with $\inf\{f(x) \mid x \in F\} > -\infty$, one has*

$$\inf\{f(x) \mid x \in F\} = \max_{(t, w_j^0, w_j^i)} \big\{t \mid f + \sum_{j=1}^{m}\sum_{i=0}^{q_j} w_j^i g_j^i - t \in \mathcal{P}_d,\ t \in \mathbb{R}, \tag{4}$$
$$(w_j^0, w_j^1, \ldots, w_j^{q_j}) \in \operatorname{cone}(\{1\} \times \mathcal{V}_j), j = 1, \ldots, m\big\},$$

where d is the smallest even number satisfying $d \ge \max\{\deg f, \deg g_j^i, i = 0, 1, \ldots, q_j, j = 1, \ldots, m\}$.

Proof [(i) ⇒ (ii)] Suppose that (i) holds. Let f be a convex polynomial on $\mathbb{R}^n$ such that $\inf\{f(x) \mid x \in F\} > -\infty$. Setting

$$\alpha^* := \sup_{(t,w_j^0,w_j^i)} \Big\{t \mid f + \sum_{j=1}^{m}\sum_{i=0}^{q_j} w_j^i g_j^i - t \in \mathcal{P}_d,\ t \in \mathbb{R},\quad (w_j^0, w_j^1, \ldots, w_j^{q_j}) \in \operatorname{cone}(\{1\}\times\mathcal{V}_j), j = 1,\ldots,m\Big\}, \tag{5}$$

we first prove that

$$\inf\{f(x) \mid x \in F\} \geq \alpha^*. \tag{6}$$

If the feasible set of the problem in the right-hand side of (5) is empty, then $\alpha^* = -\infty$, and in this case, (6) holds trivially. Now, let (t, w_j^0, w_j^i) be a feasible point of this problem. This means that $t \in \mathbb{R}$, and there exist $\alpha_{jk} \geq 0, v_{jk} := (v_{jk}^1, \ldots, v_{jk}^{q_j}) \in \mathcal{V}_j, k = 1,\ldots,s_j, j = 1,\ldots,m$ such that

$$w_j^0 = \sum_{k=1}^{s_j} \alpha_{jk}, \quad (w_j^1, \ldots, w_j^{q_j}) = \sum_{k=1}^{s_j} \alpha_{jk} v_{jk},\ j = 1,\ldots,m, \tag{7}$$

$$f + \sum_{j=1}^{m}\sum_{i=0}^{q_j} w_j^i g_j^i - t \in \mathcal{P}_d. \tag{8}$$

Let $j \in \{1,\ldots,m\}$ be arbitrary. We get by (7) that if $w_j^0 = 0$, then $w_j^i = 0$ for all $i = 1,\ldots,q_j$. Now, choose any $\widehat{v}_j := (\widehat{v}_j^1, \ldots, \widehat{v}_j^{q_j}) \in \mathcal{V}_j$ and set $\widetilde{v}_j := (\widetilde{v}_j^1, \ldots, \widetilde{v}_j^{q_j})$ with

$$\widetilde{v}_j^i := \begin{cases} \widehat{v}_j^i & \text{if } w_j^0 = 0, \\ \frac{w_j^i}{w_j^0} & \text{if } w_j^0 \neq 0, \end{cases} \qquad i = 1,\ldots,q_j.$$

Clearly, if $w_j^0 = 0$, then $\widetilde{v}_j = \widehat{v}_j \in \mathcal{V}_j$. Otherwise, $w_j^0 \neq 0$, then, by the convexity of $\mathcal{V}_j$, it holds that

$$\widetilde{v}_j = \left(\sum_{k=1}^{s_j} \frac{\alpha_{jk}}{w_j^0} v_{jk}^1, \ldots, \sum_{k=1}^{s_j} \frac{\alpha_{jk}}{w_j^0} v_{jk}^{q_j}\right) = \sum_{k=1}^{s_j} \frac{\alpha_{jk}}{w_j^0} v_{jk} \in \mathcal{V}_j.$$

Consequently, $\widetilde{v}_j \in \mathcal{V}_j$. Therefore, for any $x \in \mathbb{R}^n$,

$$\begin{aligned}\sum_{i=0}^{q_j} w_j^i g_j^i(x) = w_j^0 g_j^0(x) + \sum_{i=1}^{q_j} w_j^i g_j^i(x) &= w_j^0 g_j^0(x) + \sum_{i=1}^{q_j} (w_j^0 \widetilde{v}_j^i) g_j^i(x) \\ &= w_j^0 \big[g_j^0(x) + \sum_{i=1}^{q_j} \widetilde{v}_j^i g_j^i(x)\big] = w_j^0 g_j(x, \widetilde{v}_j),\end{aligned}$$

where we note that if $w_j^0 = 0$, then $w_j^i = 0$ for all $i = 1,\ldots,q_j$ as said above. So, due to (8), we find $\sigma_0 \in \mathcal{P}_d$ such that

$$f(x) = \sigma_0(x) - \sum_{j=1}^{m} w_j^0 g_j(x, \widetilde{v}_j) + t, \quad \forall x \in \mathbb{R}^n. \tag{9}$$

Then, for any $\widehat{x} \in F$, it follows that $\sum_{j=1}^{m} w_j^0 g_j(\widehat{x}, \widetilde{v}_j) \le 0$ due to $w_j^0 \ge 0$ and $g_j(\widehat{x}, \widetilde{v}_j) \le 0$ for all $j = 1, \ldots, m$. Keeping in mind the non-negativity of σ_0, evaluating (9) at $\widehat{x}$, we arrive at $f(\widehat{x}) \ge t$. This justifies that (6) holds.

Now, letting $\alpha := \inf\{f(x) \mid x \in F\}$, it holds that $\alpha \in \mathbb{R}$ and so

$$f(x) + \delta_F(x) \ge \alpha \ \text{ for all } x \in \mathbb{R}^n.$$

This together with (1) implies that

$$(0_n, -\alpha) \in \operatorname{epi}(f + \delta_F)^* = \operatorname{epi} f^* + \operatorname{epi} \delta_F^*$$

due to the continuity of f on $\mathbb{R}^n$. Moreover, we have (cf. [21, Page 951]) that $\operatorname{epi} \delta_F^* = \operatorname{cl} \widetilde{C} = \widetilde{C}$, where the last equality holds as the cone $\widetilde{C}$ is assumed to be closed. Consequently,

$$(0_n, -\alpha) \in \operatorname{epi} f^* + \widetilde{C},$$

which means that there exist $(u, \beta) \in \operatorname{epi} f^*$ and $\overline{v}_{jk} := (\overline{v}_{jk}^1, \ldots, \overline{v}_{jk}^{q_j}) \in \mathcal{V}_j$, $(\overline{u}_{jk}, \overline{\beta}_{jk}) \in \operatorname{epi} g_j^*(\cdot, \overline{v}_{jk})$, $\overline{\alpha}_{jk} \ge 0$, $k = 1, \ldots, s_j$, $j = 1, \ldots, m$ and $\overline{\alpha} \ge 0$ such that

$$0_n = u + \sum_{j=1}^{m} \sum_{k=1}^{s_j} \overline{\alpha}_{jk} \overline{u}_{jk}, \quad -\alpha = \overline{\alpha} + \beta + \sum_{j=1}^{m} \sum_{k=1}^{s_j} \overline{\alpha}_{jk} \overline{\beta}_{jk}. \tag{10}$$

Observe by $(\overline{u}_{jk}, \overline{\beta}_{jk}) \in \operatorname{epi} g_j^*(\cdot, \overline{v}_{jk})$, $k = 1, \ldots, s_j$, $j = 1, \ldots, m$ that, for each $x \in \mathbb{R}^n$, one has $\overline{\beta}_{jk} \ge \overline{u}_{jk}^\top x - g_j(x, \overline{v}_{jk})$. This together with (10) and the relation $(u, \beta) \in \operatorname{epi} f^*$ implies that, for each $x \in \mathbb{R}^n$,

$$f(x) \ge u^\top x - \beta \ge -\sum_{j=1}^{m} \sum_{k=1}^{s_j} \overline{\alpha}_{jk} g_j(x, \overline{v}_{jk}) + \alpha. \tag{11}$$

It shows that

$$f(x) + \sum_{j=1}^{m} \sum_{k=1}^{s_j} \overline{\alpha}_{jk} g_j(x, \overline{v}_{jk}) - \alpha \ge 0, \quad \forall x \in \mathbb{R}^n.$$

Hence, by setting $w_j^0 := \sum_{k=1}^{s_j} \overline{\alpha}_{jk} \in \mathbb{R}_+$ and $w_j^i := \sum_{k=1}^{s_j} \overline{\alpha}_{jk} \overline{v}_{jk}^i$ for all $i = 1, \ldots, q_j$, $j = 1, \ldots, m$ and taking into account (2), we have

$$f + \sum_{j=1}^{m} \sum_{k=1}^{s_j} \overline{\alpha}_{jk} \big(g_j^0 + \sum_{i=1}^{q_j} \overline{v}_{jk}^i g_j^i \big) - \alpha = f + \sum_{j=1}^{m} \sum_{i=0}^{q_j} w_j^i g_j^i - \alpha \in \mathcal{P}_d.$$

In addition, we observe that

$$(w_j^0, w_j^1, \ldots, w_j^{q_j}) = \sum_{k=1}^{s_j} \overline{\alpha}_{jk} \big(1, \overline{v}_{jk}^1, \ldots, \overline{v}_{jk}^{q_j}\big) \in \operatorname{cone}(\{1\} \times \mathcal{V}_j), \ j = 1, \ldots, m,$$

as $\overline{\alpha}_{jk} \geq 0$ and $\overline{v}_{jk} \in \mathcal{V}_j$ for all $k = 1, \ldots, s_j$, $j = 1, \ldots, m$. Therefore, (α, w_j^0, w_j^i) is a feasible point of the problem in the right-hand side of (5). So, $\alpha \leq \alpha^*$, which together with (6) proves that (4) is valid, i.e., we obtain (ii).

[(ii) ⇒ (i)] Suppose that (ii) holds. Observe (cf. [21, Page 951]) that

$$\operatorname{epi} \delta_F^* = \operatorname{cl} \widetilde{C} = \operatorname{cl} \operatorname{cone} \{(0_n, 1) \cup \operatorname{epi} g_j^*(\cdot, v_j) \mid v_j \in \mathcal{V}_j, j = 1, \ldots, m\}. \tag{12}$$

To prove that $\widetilde{C}$ is closed, let $(b, \beta) \in \operatorname{cl} \widetilde{C}$. Then, by (12), one has $b^\top x \leq \beta$ for all $x \in F$, and so, $\inf\{(-b)^\top x \mid x \in F\} \geq -\beta$. By (ii), we find $\alpha_{jk} \geq 0, v_{jk} := (v_{jk}^1, \ldots, v_{jk}^{q_j}) \in \mathcal{V}_j, k = 1, \ldots, s_j, j = 1, \ldots, m$, $\overline{t} \in \mathbb{R}$ and $\overline{\sigma} \in \mathcal{P}_d$ such that

$$\overline{w}_j^0 = \sum_{k=1}^{s_j} \alpha_{jk}, \quad (\overline{w}_j^1, \ldots, \overline{w}_j^{q_j}) = \sum_{k=1}^{s_j} \alpha_{jk} v_{jk}, \; j = 1, \ldots, m,$$
$$- b^\top x + \sum_{j=1}^{m} \sum_{i=0}^{q_j} \overline{w}_j^i g_j^i(x) - \overline{t} = \overline{\sigma}(x), \quad \forall x \in \mathbb{R}^n,$$

and $\overline{t} = \inf\{(-b)^\top x \mid x \in F\}$. Hence, we have

$$-b^\top x + \sum_{j=1}^{m} \sum_{i=0}^{q_j} \overline{w}_j^i g_j^i(x) \geq -\beta, \quad \forall x \in \mathbb{R}^n. \tag{13}$$

Arguing similarly as in the proof of [(i) ⇒ (ii)], we can find $\widetilde{v}_j \in \mathcal{V}_j, j = 1, \ldots, m$, such that

$$\sum_{i=0}^{q_j} \overline{w}_j^i g_j^i(x) = \overline{w}_j^0 g_j(x, \widetilde{v}_j), \quad \forall x \in \mathbb{R}^n.$$

Then, we get from (13) that $\beta \geq b^\top x - \sum_{j=1}^{m} \overline{w}_j^0 g_j(x, \widetilde{v}_j)$ for all $x \in \mathbb{R}^n$. Hence,

$$(b, \beta) \in \operatorname{epi} \left(\sum_{j=1}^{m} \overline{w}_j^0 g_j(\cdot, \widetilde{v}_j) \right)^* \subset \operatorname{cone} \{(0_n, 1) \cup \operatorname{epi} g_j^*(\cdot, v_j) \mid v_j \in \mathcal{V}_j, j = 1, \ldots, m\},$$

which shows that $\widetilde{C}$ is closed, and consequently, (i) holds. The proof of the theorem is complete. □

Remark 2.1 Let us make some remarks regarding the above theorem.

(a) A closer inspection of the proof of [(ii) ⇒ (i)] reveals that the convex polynomials f in the statement of (ii) to be replaced by affine functions is *sufficient* for this implication. That is why the exact relaxation in Theorem 2.1 is called *stable*, since it continues to hold when the objective function of the primal problem is perturbed with any affine function. We refer the interested reader to [17] for characterizations of stable robust duality for a class of general optimization problems under convexity and closedness assumptions.

(b) If the Slater constraint qualification holds, i.e., if there exists $\widehat{x} \in \mathbb{R}^n$ such that $g_j(\widehat{x}, v_j) < 0$ for all $v_j \in \mathcal{V}_j, j = 1, \ldots, m$, then the characteristic cone $\widetilde{C}$ in (3) is closed (see, e.g., [22, Proposition 3.2]).

Parametric robust SOS-convex polynomial problems. Let us now focus on a parametric robust *SOS-convex* polynomial problem that is defined as follows: For an SOS-convex polynomial $f : \mathbb{R}^n \to \mathbb{R}$, one has the robust SOS-convex polynomial program defined as in (P_f), where the polynomials $g_j^i : \mathbb{R}^n \to \mathbb{R}$, $i = 0, 1, \ldots, q_j, j = 1, \ldots, m$ are given such that, for each fixed $v_j \in \mathcal{V}_j$, the function $g_j(\cdot, v_j)$ given by

$$g_j(x, v_j) := g_j^0(x) + \sum_{i=1}^{q_j} v_j^i g_j^i(x),\ x \in \mathbb{R}^n, \tag{14}$$

is an *SOS-convex* polynomial on $\mathbb{R}^n$ for $j = 1, \ldots, m$, and the uncertainty sets $\mathcal{V}_j$, $j = 1, \ldots, m$ are nonempty and bounded sets, which are of the spectrahedral forms (see e.g., [36, 39]) described by

$$\mathcal{V}_j := \{v_j := (v_j^1, \ldots, v_j^{q_j}) \in \mathbb{R}^{q_j} \mid A_j^0 + \sum_{i=1}^{q_j} v_j^i A_j^i \succeq 0\} \tag{15}$$

with A_j^i, $i = 0, 1, \ldots, q_j$, $j = 1, \ldots, m$, symmetric matrices of order $n_j \in \mathbb{N}$.

It is worth mentioning here that checking whether $g_j(\cdot, v_j), j \in \{1, \ldots, m\}$, in (14) is SOS-convex for each $v_j \in \mathcal{V}_j$ is, in general, NP-hard. For example, consider a quadratic polynomial defined by $g(x, v) := x^\top B^0 x + \sum_{i=1}^{q} v^i (x^\top B^i x), x \in \mathbb{R}^n, v := (v^1, \ldots, v^q) \in \mathcal{V} := [-1, 1]^q$, for given symmetric $(n \times n)$ matrices $B^i, i = 0, 1, \ldots, q$ and assume that we need to check if $g(\cdot, v)$ is SOS-convex (equivalently in this case, convex) for all $v \in \mathcal{V}$. This is the so-called *matrix cube problem* of Ben-Tal and Nemirovski [6], which is NP-hard [33]. However, in some special circumstances, the afforesaid assumption is automatically satisfied. For instance, if $g_j^i, i = 0, 1, \ldots, q_j, j = 1, \ldots, m$ are convex quadratic functions or convex separable polynomials and $\mathcal{V}_j, j = 1, \ldots, m$ are contained in the nonnegative orthant of $\mathbb{R}^{q_j}$, then $g_j(\cdot, v_j), j = 1, \ldots, m$ are SOS-convex polynomials for all $v_j \in \mathcal{V}_j$.

In this setting, we obtain a characterization of stable exact *semidefinite programming* (SDP) relaxations in terms of the characteristic cone for the family of robust SOS-convex polynomial problems defined by (P_f) when f varies in the class of SOS-convex polynomials.

Theorem 2.2 (Characterization of stable exact SDP relaxations) *Let* $F := \{x \in \mathbb{R}^n \mid g_j(x, v_j) \leq 0,\ \forall v_j \in \mathcal{V}_j, j = 1, \ldots, m\} \neq \emptyset$, *where* $g_j, j = 1, \ldots, m$, *are given as in* (14), *and* $\mathcal{V}_j, j = 1, \ldots, m$, *are given as in* (15). *Then, the closedness of the characteristic cone* $\widetilde{C}$ *in* (3) *is equivalent to the following statement: For any SOS-convex polynomial* f *on* $\mathbb{R}^n$ *with* $\inf\{f(x) \mid x \in F\} > -\infty$, *one has*

$$\inf\{f(x) \mid x \in F\} = \max_{(t, w_j^0, w_j^i)} \{t \mid f + \sum_{j=1}^{m} \sum_{i=0}^{q_j} w_j^i g_j^i - t \in \varSigma_d^2,\ w_j^0 A_j^0 + \sum_{i=1}^{q_j} w_j^i A_j^i \succeq 0, \tag{16}$$

$$t \in \mathbb{R}, w_j^0 \in \mathbb{R}_+, w_j^i \in \mathbb{R}, i = 1, \ldots, q_j, j = 1, \ldots, m\},$$

where d is the smallest even number satisfying $d \geq \max\{\deg f, \deg g_j^i, i = 0, 1, \ldots, q_j, j = 1, \ldots, m\}$.

Proof [$\Longrightarrow$] Let the characteristic cone $\widetilde{C}$ in (3) be closed, and let f on $\mathbb{R}^n$ be an SOS-convex polynomial with $\inf\{f(x) \mid x \in F\} > -\infty$. Then, by Theorem 2.1, we have

$$\inf\{f(x) \mid x \in F\} = \max_{(t, w_j^0, w_j^i)} \Big\{t \mid f + \sum_{j=1}^{m} \sum_{i=0}^{q_j} w_j^i g_j^i - t \in \mathcal{P}_d,\ t \in \mathbb{R},$$
$$(w_j^0, w_j^1, \ldots, w_j^{q_j}) \in \operatorname{cone}(\{1\} \times \mathcal{V}_j), j = 1, \ldots, m\Big\},$$

where d is the smallest even number satisfying $d \geq \max\{\deg f, \deg g_j^i, i = 0, 1, \ldots, q_j, j = 1, \ldots, m\}$. To justify (16), it suffices to show that

$$\max_{(t, w_j^0, w_j^i)} \{t \mid f + \sum_{j=1}^{m} \sum_{i=0}^{q_j} w_j^i g_j^i - t \in \Sigma_d^2,\ w_j^0 A_j^0 + \sum_{i=1}^{q_j} w_j^i A_j^i \succeq 0,$$
$$t \in \mathbb{R}, w_j^0 \in \mathbb{R}_+, w_j^i \in \mathbb{R}, i = 1, \ldots, q_j, j = 1, \ldots, m\}$$
$$= \max_{(t, w_j^0, w_j^i)} \Big\{t \mid f + \sum_{j=1}^{m} \sum_{i=0}^{q_j} w_j^i g_j^i - t \in \mathcal{P}_d,\ t \in \mathbb{R},$$
$$(w_j^0, w_j^1, \ldots, w_j^{q_j}) \in \operatorname{cone}(\{1\} \times \mathcal{V}_j), j = 1, \ldots, m\Big\}. \tag{17}$$

To see this, let $t \in \mathbb{R}$, $w_j^0 \geq 0$, $w_j^i \in \mathbb{R}$, $i = 1, \ldots, q_j$, $j = 1, \ldots, m$. In this setting, we can verify that $(w_j^0, w_j^1, \ldots, w_j^{q_j}) \in \operatorname{cone}(\{1\} \times \mathcal{V}_j), j = 1, \ldots, m$ amount to $w_j^0 A_j^0 + \sum_{i=1}^{q_j} w_j^i A_j^i \succeq 0,\ j = 1, \ldots, m$, and that $f + \sum_{j=1}^{m} \sum_{i=0}^{q_j} w_j^i g_j^i - t \in \mathcal{P}_d$ is equivalent to $f + \sum_{j=1}^{m} \sum_{i=0}^{q_j} w_j^i g_j^i - t \in \Sigma_d^2$ (see e.g., [26, Cor. 2.1]). So, the equality in (17) is valid.

[$\Longleftarrow$] Assume the statement in (16) holds. In this setting, it holds that

$$\inf\{f(x) \mid x \in F\} = \max_{(t, w_j^0, w_j^i)} \Big\{t \mid f + \sum_{j=1}^{m} \sum_{i=0}^{q_j} w_j^i g_j^i - t \in \mathcal{P}_d,\ t \in \mathbb{R},$$
$$(w_j^0, w_j^1, \ldots, w_j^{q_j}) \in \operatorname{cone}(\{1\} \times \mathcal{V}_j), j = 1, \ldots, m\Big\}$$

for any SOS-convex polynomial f on $\mathbb{R}^n$ with $\inf\{f(x) \mid x \in F\} > -\infty$. To prove that $\widetilde{C}$ is closed, we just follow similar arguments as in the proof of Theorem 2.1. □

Remark 2.2 Let us make some remarks regarding the above theorem.

(a) As the sum-of-squares constraint of each relaxation problem in Theorem 2.2 can be equivalently written as a linear matrix inequality (see, e.g., [28, Proposition 2.1]), the corresponding relaxation is an SDP problem.

(b) Theorem 2.2 continues to hold if the sets $\mathcal{V}_j$ in (15) are assumed to be compact projections of spectrahedra (compact SDP-representable sets [20]), which cover bounded spectrahedra (bounded LMI-representable sets). So, this fact constitutes an extension of the obtained result since projections do not necessarily preserve spectrahedrality. However, for the sake of simplicity in the notation, we just show the result for the case of bounded spectrahedra as above.

It is well known that the family of spectrahedra covers a range of commonly used uncertainty sets in the literature such as polytopes, balls, ellipsoids and intersection of ellipsoids. Next, we point out how Theorem 2.2 can be used to derive a characterization of stable exact SDP relaxations for the family of robust SOS-convex polynomial problems defined by (P_f), where the uncertainty sets $\mathcal{V}_j \subset \mathbb{R}^{q_j}, j = 1, \ldots, m$, are replaced by the following intersection of ellipsoids (see e.g., [15])

$$\mathcal{V}_j := \left\{ v_j \in \mathbb{R}^{q_j} \mid v_j^\top E_j^l v_j + b_j^{l\top} v_j + \beta_j^l \leq 0, l = 1, \ldots, s_j \right\} \tag{18}$$

with positive semidefinite symmetric $(q_j \times q_j)$ matrices $E_j^l, l = 1, \ldots, s_j$ and $b_j^l := (b_j^{l,1}, \ldots, b_j^{l,q_j}) \in \mathbb{R}^{q_j}, \beta_j^l \in \mathbb{R}, l = 1, \ldots, s_j$. In what follows, for each $j \in \{1, \ldots, m\}$ and $l \in \{1, \ldots, s_j\}$, we use the notation $L_j^l := (L_j^{l,1}, \ldots, L_j^{l,q_j})$ to denote a decomposition factor of E_j^l, i.e., $E_j^l = (L_j^l)^\top L_j^l$. Note further that the sets $\mathcal{V}_j, j = 1, \ldots, m$, in (18) are assumed to be nonempty and bounded.

Corollary 2.1 (Stable exact SDP relaxations with intersection of ellipsoids uncertainty) *Let* $F := \{x \in \mathbb{R}^n \mid g_j(x, v_j) \leq 0,\ \forall v_j \in \mathcal{V}_j, j = 1, \ldots, m\} \neq \emptyset$, *where* $g_j, j = 1, \ldots, m$, *are given as in* (14), *and* $\mathcal{V}_j, j = 1, \ldots, m$, *are given as in* (18). *Then, the conclusion of Theorem 2.2 holds with* $w_j^0 A_j^0 + \sum_{i=1}^{q_j} w_j^i A_j^i \succeq 0, j = 1, \ldots, m$ *being replaced by* $w_j^0 \begin{pmatrix} I_{q_j} & 0 \\ 0 & -\beta_j^l \end{pmatrix} + \sum_{i=1}^{q_j} w_j^i \begin{pmatrix} 0 & L_j^{l,i} \\ (L_j^{l,i})^\top & -b_j^{l,i} \end{pmatrix} \succeq 0, l = 1, \ldots, s_j, j = 1, \ldots, m$, *where* $w_j^0 \in \mathbb{R}_+, w_j^i \in \mathbb{R}, i = 1, \ldots, q_j, j = 1, \ldots, m$.

Proof Let $j \in \{1, \ldots, m\}$ and $w_j^0 \in \mathbb{R}_+, w_j^i \in \mathbb{R}, i = 1, \ldots, q_j$. In this setting, we can verify that $w_j^0 A_j^0 + \sum_{i=1}^{q_j} w_j^i A_j^i \succeq 0$ is nothing else but $w_j^0 \begin{pmatrix} I_{q_j} & 0 \\ 0 & -\beta_j^l \end{pmatrix} + \sum_{i=1}^{q_j} w_j^i \begin{pmatrix} 0 & L_j^{l,i} \\ (L_j^{l,i})^\top & -b_j^{l,i} \end{pmatrix} \succeq 0, l = 1, \ldots, s_j$. So, the proof is complete. $\square$

Remark 2.3 It is worth mentioning here that the exact SDP relaxation result in Corollary 2.1 is verified by the closedness of the characteristic cone, and so it is more favourable than that of [15, Theorem 3.2], which was obtained under the so-called *KKT Qualification Condition*.

Parametric robust convex quadratic problems. Let us now consider a parametric robust *convex quadratic* problem that is defined as follows: For a convex quadratic function $Q : \mathbb{R}^n \to \mathbb{R}$ (i.e., $Q(x) := x^\top Cx + c^\top x + \varsigma$, where $C \succeq 0$, $c \in \mathbb{R}^n$ and $\varsigma \in \mathbb{R}$ for $x \in \mathbb{R}^n$), one has a robust convex quadratic problem of the form:

$$\inf_{x\in\mathbb{R}^n} \{Q(x) \mid Q_j^0(x) + \sum_{i=1}^{q_j} v_j^i Q_j^i(x) \le 0, \ \forall v_j := (v_j^1, \ldots, v_j^{q_j}) \in \mathcal{V}_j, j = 1, \ldots, m\}. \tag{P_Q}$$

Here, $Q_j^i : \mathbb{R}^n \to \mathbb{R}$, $i = 0, 1, \ldots, q_j, j = 1, \ldots, m$ are quadratic functions that are defined by $Q_j^i(x) := x^\top B_j^i x + (b_j^i)^\top x + \beta_j^i$, where $b_j^i \in \mathbb{R}^n, \beta_j^i \in \mathbb{R}$ and $B_j^i, i = 0, 1, \ldots, q_j, j = 1, \ldots, m$ are symmetric matrices satisfying $B_j^0 + \sum_{i=1}^{q_j} v_j^i B_j^i \succeq 0$ for all $v_j := (v_j^1, \ldots, v_j^{q_j}) \in \mathcal{V}_j$, and $\mathcal{V}_j, j = 1, \ldots, m$ are spectrahedra given as in (15).

The following result provides a characterization of exact stable SDP relaxations for the parametric robust convex quadratic problem defined by (P_Q). As above, we denote $g_j(x, v_j) := Q_j^0(x) + \sum_{i=1}^{q_j} v_j^i Q_j^i(x)$, $x \in \mathbb{R}^n$, for $j = 1, \ldots, m$.

Corollary 2.2 (Characterization of stable exact SDP relaxations of (P_Q)**)** *Assume that* $F := \{x \in \mathbb{R}^n \mid Q_j^0(x) + \sum_{i=1}^{q_j} v_j^i Q_j^i(x) \le 0, \ \forall v_j \in \mathcal{V}_j, j = 1, \ldots, m\} \neq \emptyset$, *where* $B_j^0 + \sum_{i=1}^{q_j} v_j^i B_j^i \succeq 0$ *for all* $v_j \in \mathcal{V}_j$ *and* $\mathcal{V}_j, j = 1, \ldots, m$, *are given as in* (15). *Then, the conclusion of Theorem 2.2 holds with* $f := Q$, $g_j^i := Q_j^i$, $i = 0, \ldots, q_j$, $j = 1, \ldots, m$, $d := 2$ *and* $f + \sum_{j=1}^{m} \sum_{i=0}^{q_j} w_j^i g_j^i - t \in \Sigma_d^2$ *being replaced by* $\begin{pmatrix} 2(C + \sum_{j=1}^{m} \sum_{i=0}^{q_j} w_j^i B_j^i) & c + \sum_{j=1}^{m} \sum_{i=0}^{q_j} w_j^i b_j^i \\ (c + \sum_{j=1}^{m} \sum_{i=0}^{q_j} w_j^i b_j^i)^\top & 2(\varsigma + \sum_{j=1}^{m} \sum_{i=0}^{q_j} w_j^i \beta_j^i - t) \end{pmatrix} \succeq 0$, *where* $t \in \mathbb{R}, w_j^0 \in \mathbb{R}_+, w_j^i \in \mathbb{R}, i = 1, \ldots, q_j, j = 1, \ldots, m$.

Proof Let $t \in \mathbb{R}, w_j^0 \ge 0, w_j^i \in \mathbb{R}, j = 1, \ldots, m, i = 1, \ldots, q_j$. In this setting, we can verify that the relation $Q + \sum_{j=1}^{m} \sum_{i=0}^{q_j} w_j^i Q_j^i - t \in \Sigma_2^2$ is equivalent to the matrix inequality

$$\begin{pmatrix} \varsigma + \sum_{j=1}^{m} \sum_{i=0}^{q_j} w_j^i \beta_j^i - t & \frac{1}{2}(c^\top + \sum_{j=1}^{m} \sum_{i=0}^{q_j} w_j^i b_j^{i\top}) \\ \frac{1}{2}(c + \sum_{j=1}^{m} \sum_{i=0}^{q_j} w_j^i b_j^i) & C + \sum_{j=1}^{m} \sum_{i=0}^{q_j} w_j^i B_j^i \end{pmatrix} \succeq 0,$$

and so the proof is complete. □

We now give an example, which shows how one can find the optimal value of a robust SOS-convex polynomial problem by solving its corresponding SDP relaxation problem using the characterization of stable exact SDP relaxations established in Theorem 2.2.

Example 2.1 Consider a parametric robust SOS-convex polynomial problem of the form:

$$\inf_{x\in\mathbb{R}} \big\{ f(x) \mid x^4 + 2v_1 x + v_2 - 4 \leq 0,\ \forall (v_1, v_2) \in \mathcal{V},$$
$$x^4 + 2x^2 + v_2 x + 2v_1 - 8 \leq 0,\ \forall (v_1, v_2) \in \mathcal{V} \big\}, \tag{EP1}$$

where f is an SOS-convex polynomial and $\mathcal{V}$ is an uncertainty set given by

$$\mathcal{V} := \{ v := (v_1, v_2) \in \mathbb{R}^2 \mid \frac{v_1^2}{4} + \frac{v_2^2}{8} \leq 1, v_1 \leq 0, v_2 \leq 0 \}.$$

The problem (EP1) can be expressed in terms of problem (P_f), where the spectrahedra are given as $\mathcal{V}_1 := \mathcal{V}_2 := \mathcal{V}$ with

$$A_j^0 := \begin{pmatrix} 4 & 0 & 0 & 0 & 0 \\ 0 & 8 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},\ A_j^1 := \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},\ A_j^2 := \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix},\ j = 1, 2,$$

and the polynomials are given by $g_1^0(x) := x^4 - 4, g_1^1(x) := 2x, g_1^2(x) := 1, g_2^0(x) := x^4 + 2x^2 - 8, g_2^1(x) := 2, g_2^2(x) := x, x \in \mathbb{R}$.

In this setting, the Slater constraint qualification holds, and thus, the characteristic cone $\widetilde{C} := \operatorname{cone}\{(0,1) \cup \operatorname{epi} g_j^*(\cdot, v) \mid v \in \mathcal{V}, j = 1, 2\}$ is closed, where $g_j(x, v) := g_j^0(x) + \sum_{i=1}^{2} v_i g_j^i(x), j = 1, 2$ for $x \in \mathbb{R}, v := (v_1, v_2) \in \mathcal{V}$. So, we assert by Theorem 2.2 that the exact SDP relaxation holds for the robust SOS-convex polynomial program (EP1) for any SOS-convex polynomial f on $\mathbb{R}$ whenever $\inf$ (EP1) $> -\infty$.

Let us now consider the problem (EP1) with the objective function $f(x) := x^4 + 2x^2 + 2$, i.e,

$$\inf_{x\in\mathbb{R}} \big\{ x^4 + 2x^2 + 2 \mid x^4 + 2v_1 x + v_2 - 4 \leq 0,\ \forall (v_1, v_2) \in \mathcal{V},$$
$$x^4 + 2x^2 + v_2 x + 2v_1 - 8 \leq 0,\ \forall (v_1, v_2) \in \mathcal{V} \big\}, \tag{EP1-1}$$

where $\mathcal{V}$ is defined as above. In this case, it can be checked that $\inf$ (EP1-1) $\in \mathbb{R}$, and so, the exact SDP relaxation holds for the problem (EP1-1), i.e.,

$$\inf\,(\text{EP1-1}) = \max\,(\text{SDP1}), \tag{19}$$

where (SDP1) is the relaxation problem for (EP1-1) given by

$$\max_{(t,w_j^0,w_j^i)} \{t \mid f+\sum_{j=1}^{2}\big(w_j^0 g_j^0+\sum_{i=1}^{2} w_j^i g_j^i\big)-t\in \Sigma_4^2, \tag{20}$$

$$w_j^0 A_j^0+\sum_{i=1}^{2} w_j^i A_j^i \succeq 0, t\in\mathbb{R}, w_j^0\in\mathbb{R}_+, w_j^i\in\mathbb{R}, i=1,2, j=1,2\}. \tag{SDP1}$$

From (20), there exists $\sigma_0\in\Sigma_4^2$ such that

$$f+\sum_{j=1}^{2}\big(w_j^0 g_j^0+\sum_{i=1}^{2} w_j^i g_j^i\big)-t=\sigma_0. \tag{21}$$

We conclude by $\sigma_0\in\Sigma_4^2$ that there exists a symmetric (3×3) matrix B such that $\sigma_0=X^\top BX$ and $B\succeq 0$, where $X:=(1,x,x^2)$ (see e.g., [35, Lemma 3.33]). Letting $B:=\begin{pmatrix} B_1 & B_2 & B_3\\ B_2 & B_4 & B_5\\ B_3 & B_5 & B_6\end{pmatrix}$, we derive from (21) that $B_1=2-4w_1^0+w_1^2-8w_2^0+2w_2^1-t, B_2=w_1^1+\frac{1}{2}w_2^2, 2B_3+B_4=2+2w_2^0, B_5=0, B_6=1+w_1^0+w_2^0$. Putting $B_3:=w\in\mathbb{R}$, we see that $B_4=2+2w_2^0-2w$. Then, the problem (SDP1) becomes the following semidefinite programming problem

$$\max\{t \mid \begin{pmatrix} 2-4w_1^0+w_1^2-8w_2^0+2w_2^1-t & w_1^1+\frac{1}{2}w_2^2 & w\\ w_1^1+\frac{1}{2}w_2^2 & 2+2w_2^0-2w & 0\\ w & 0 & 1+w_1^0+w_2^0\end{pmatrix}\succeq 0,$$

$$\begin{pmatrix} 4w_1^0 & 0 & w_1^1 & 0 & 0\\ 0 & 8w_1^0 & w_1^2 & 0 & 0\\ w_1^1 & w_1^2 & w_1^0 & 0 & 0\\ 0 & 0 & 0 & -w_1^1 & 0\\ 0 & 0 & 0 & 0 & -w_1^2\end{pmatrix}\succeq 0, \begin{pmatrix} 4w_2^0 & 0 & w_2^1 & 0 & 0\\ 0 & 8w_2^0 & w_2^2 & 0 & 0\\ w_2^1 & w_2^2 & w_2^0 & 0 & 0\\ 0 & 0 & 0 & -w_2^1 & 0\\ 0 & 0 & 0 & 0 & -w_2^2\end{pmatrix}\succeq 0, \tag{SDP2}$$

$$\begin{pmatrix} w_1^0 & 0\\ 0 & w_2^0\end{pmatrix}\succeq 0, t\in\mathbb{R}, w\in\mathbb{R}, w_j^0\in\mathbb{R}, w_j^i\in\mathbb{R}, i=1,2, j=1,2\Big\}.$$

Using the Matlab toolbox CVX [19], we solve the SDP problem (SDP2), and the solver returns its optimal value as 2.000.

Now, taking into account the validation of the exact SDP relaxation for the robust SOS-convex polynomial program (EP1-1) given by (19), we conclude that the optimal value of problem (EP1-1) is $\inf(\text{EP1-1})=\max(\text{SDP1})=2$.

3 Stable Exact SOCP Relaxations for Classes of Parametric Robust Convex Quadratic Problems

In this section, we employ stable exact SDP relaxations obtained earlier to derive characterizations of stable exact second-order cone programming (SOCP) relaxations for some classes of parametric robust convex quadratic problems under ellipsoidal uncertainty.

Parametric robust convex separable quadratic problems. Let us first focus on a parametric robust convex *separable* quadratic problem that is defined as follows: For $D := \mathrm{diag}(\omega_1, \ldots, \omega_n)$, $\omega_k \geq 0$, $c := (c_1, \ldots, c_n) \in \mathbb{R}^n$ and $d \in \mathbb{R}$, one has a robust convex *separable* quadratic problem of the form

$$\inf_{x \in \mathbb{R}^n} \{x^\top Dx + c^\top x + d \mid x^\top M_j(v_j)x + a_j(v_j)^\top x + b_j(v_j) \leq 0, \forall v_j \in \mathcal{V}_j, j = 1, \ldots, m\}, \tag{SQP}$$

where $\mathcal{V}_j \subset \mathbb{R}^{q_j}$, $j = 1, \ldots, m$, are ellipsoidal uncertainty sets given by

$$\mathcal{V}_j := \left\{v_j \in \mathbb{R}^{q_j} \mid v_j^\top E_j v_j \leq 1\right\}, j = 1, \ldots, m, \tag{22}$$

with symmetric $(q_j \times q_j)$ matrices $E_j \succ 0, j = 1, \ldots, m$, and $M_j : \mathbb{R}^{q_j} \to \mathbb{R}^{n \times n}, a_j : \mathbb{R}^{q_j} \to \mathbb{R}^n$ and $b_j : \mathbb{R}^{q_j} \to \mathbb{R}, j = 1, \ldots, m$ are affine functions defined respectively by

$$M_j(v_j) := M_j^0 + \sum_{i=1}^{q_j} v_j^i M_j^i,\ a_j(v_j) := a_j^0 + \sum_{i=1}^{q_j} v_j^i a_j^i,\ b_j(v_j) := b_j^0 + \sum_{i=1}^{q_j} v_j^i b_j^i \tag{23}$$

for $v_j := (v_j^1, \ldots, v_j^{q_j}) \in \mathbb{R}^{q_j}$ with $M_j^i := \mathrm{diag}(u_{j1}^i, \ldots, u_{jn}^i), u_{jk}^i \in \mathbb{R}, k = 1, \ldots, n,$ $a_j^i := (a_{j1}^i, \ldots, a_{jn}^i) \in \mathbb{R}^n, b_j^i \in \mathbb{R}, i = 0, 1, \ldots, s, j = 1, \ldots, m$ fixed.

We assume that $M_j(v_j) \succeq 0$ for all $v_j \in \mathcal{V}_j, j = 1, \ldots, m$. In what follows, we use the notation L_j to denote a decomposition factor of E_j (i.e., $E_j = L_j^\top L_j$) for each $j \in \{1, \ldots, m\}$, and put $g_j(x, v_j) := x^\top M_j(v_j)x + a_j(v_j)^\top x + b_j(v_j)$ for $x \in \mathbb{R}^n$ and $v_j \in \mathcal{V}_j, j = 1, \ldots, m$.

We are now ready to derive a characterization of stable exact second-order cone programming (SOCP) relaxations for the parametric robust convex separable quadratic problem defined by (SQP).

Theorem 3.1 (Characterization of stable exact SOCP relaxations of (SQP)**)** *Assume that* $F := \{x \in \mathbb{R}^n \mid x^\top M_j(v_j)x + a_j(v_j)^\top x + b_j(v_j) \leq 0,\ \forall v_j \in \mathcal{V}_j, j = 1, \ldots, m\} \neq \emptyset$*, where* $\mathcal{V}_j, j = 1, \ldots, m$*, are given as in* (22)*. Then, the closedness of the characteristic cone* $\widetilde{C}$ *is equivalent to the following statement: For any* $D := \mathrm{diag}(\omega_1, \ldots, \omega_n)$ *with* $\omega_k \geq 0,\ k = 1, \ldots, n,\ c := (c_1, \ldots, c_n) \in \mathbb{R}^n$*, and* $d \in \mathbb{R}$ *with* $\inf \text{(SQP)} > -\infty$*,* *one has*

$$\begin{aligned}\inf(\mathrm{SQP}) = \max_{(t,\lambda_j^0,\lambda_j^i,\mu_k)} \Big\{ t \mid\ & \sum_{k=1}^n \mu_k - \sum_{j=1}^m (\lambda_j^0 b_j^0 + \sum_{i=1}^{q_j} \lambda_j^i b_j^i) + t - d \le 0, t \in \mathbb{R}, \\ & \|\big(\mu_k - \omega_k - \sum_{j=1}^m (\lambda_j^0 u_{jk}^0 + \sum_{i=1}^{q_j} \lambda_j^i u_{jk}^i), c_k + \sum_{j=1}^m (\lambda_j^0 a_{jk}^0 + \sum_{i=1}^{q_j} \lambda_j^i a_{jk}^i)\big)\|_2 \\ & \le \mu_k + \omega_k + \sum_{j=1}^m (\lambda_j^0 u_{jk}^0 + \sum_{i=1}^{q_j} \lambda_j^i u_{jk}^i), \\ & \omega_k + \sum_{j=1}^m (\lambda_j^0 u_{jk}^0 + \sum_{i=1}^{q_j} \lambda_j^i u_{jk}^i) \ge 0, \mu_k \ge 0, k = 1, \ldots, n, \\ & \|L_j(\lambda_j^1, \ldots, \lambda_j^{q_j})\|_2 \le \lambda_j^0, \lambda_j^0 \in \mathbb{R}_+, \lambda_j^i \in \mathbb{R}, i = 1, \ldots, q_j, j = 1, \ldots, m \Big\},\end{aligned}$$

where L_j is the decomposition factor of E_j for each $j \in \{1, \ldots, m\}$.

Proof Let $L_j := (L_j^1, \ldots, L_j^{q_j})$ and

$$A_j^0 := I_{q_j+1}, \quad A_j^i := \begin{pmatrix} 0 & L_j^i \\ (L_j^i)^\top & 0 \end{pmatrix}, \ i = 1, \ldots, q_j, \ j = 1, \ldots, m. \tag{24}$$

As we have seen in the proof of Corollary 2.1, one has $v_j := (v_j^1, \ldots, v_j^{q_j}) \in \mathcal{V}_j$ if and only if $A_j^0 + \sum_{i=1}^{q_j} v_j^i A_j^i \succeq 0$. It means that the ellipsoids in (22) are expressed as spectrahedra in (15). Moreover, by letting $Q(x) := x^\top D x + c^\top x + d$ and $Q_j^i(x) := x^\top M_j^i x + (a_j^i)^\top x + b_j^i, x \in \mathbb{R}^n, i = 0, 1, \ldots, q_j, j = 1, \ldots, m$, we see that the problem (SQP) is a particular case of problem (P_Q).

Invoking Corollary 2.2 and keeping in mind Remark 2.1, we conclude that the closedness of the cone $\widetilde{C}$ is equivalent to the assertion that for any $D := \mathrm{diag}(\omega_1, \ldots, \omega_n)$ with $\omega_k \ge 0$, $k = 1, \ldots, n$, $c := (c_1, \ldots, c_n) \in \mathbb{R}^n$, and $d \in \mathbb{R}$ with $\inf(\mathrm{SQP}) > -\infty$, one has

$$\begin{aligned}\inf(\mathrm{SQP}) = \max_{(t,\lambda_j^0,\lambda_j^i)} \{ t \mid & \begin{pmatrix} 2(D + \sum_{j=1}^m \sum_{i=0}^{q_j} \lambda_j^i M_j^i) & c + \sum_{j=1}^m \sum_{i=0}^{q_j} \lambda_j^i a_j^i \\ (c + \sum_{j=1}^m \sum_{i=0}^{q_j} \lambda_j^i a_j^i)^\top & 2(d + \sum_{j=1}^m \sum_{i=0}^{q_j} \lambda_j^i b_j^i - t) \end{pmatrix} \succeq 0, \\ & \lambda_j^0 A_j^0 + \sum_{i=1}^{q_j} \lambda_j^i A_j^i \succeq 0, \\ & t \in \mathbb{R}, \lambda_j^0 \in \mathbb{R}_+, \lambda_j^i \in \mathbb{R}, i = 1, \ldots, q_j, j = 1, \ldots, m\}.\end{aligned} \tag{25}$$

Granting this, we can justify that

$$\inf(\mathrm{SQP}) = \max(\mathrm{SCP}), \tag{26}$$

where (SCP) is the following problem

$$\sup_{(t,\lambda_j^0,\lambda_j^i,\mu_k)} \Big\{ t \mid \sum_{k=1}^{n} \mu_k - \sum_{j=1}^{m} (\lambda_j^0 b_j^0 + \sum_{i=1}^{q_j} \lambda_j^i b_j^i) + t - d \leq 0, t \in \mathbb{R},$$

$$\|\big(\mu_k - \omega_k - \sum_{j=1}^{m} (\lambda_j^0 u_{jk}^0 + \sum_{i=1}^{q_j} \lambda_j^i u_{jk}^i), c_k + \sum_{j=1}^{m} (\lambda_j^0 a_{jk}^0 + \sum_{i=1}^{q_j} \lambda_j^i a_{jk}^i)\big)\|_2$$

$$\leq \mu_k + \omega_k + \sum_{j=1}^{m} (\lambda_j^0 u_{jk}^0 + \sum_{i=1}^{q_j} \lambda_j^i u_{jk}^i),$$

$$\omega_k + \sum_{j=1}^{m} (\lambda_j^0 u_{jk}^0 + \sum_{i=1}^{q_j} \lambda_j^i u_{jk}^i) \geq 0, \mu_k \geq 0, k = 1, \ldots, n,$$

$$\|L_j(\lambda_j^1, \ldots, \lambda_j^{q_j})\|_2 \leq \lambda_j^0, \lambda_j^0 \in \mathbb{R}_+, \lambda_j^i \in \mathbb{R}, i = 1, \ldots, q_j, j = 1, \ldots, m \Big\}. \tag{SCP}$$

The proof is completed by combining (25) and (26). □

Parametric robust convex quadratic problems with affine constraints. We now examine a parametric robust *convex* quadratic program with *affine constraints* that is defined as follows: For $C \succeq 0, c \in \mathbb{R}^n$ and $d \in \mathbb{R}$, one has a robust convex quadratic program with *affine constraints* as

$$\inf_{x \in \mathbb{R}^n} \{x^\top C x + c^\top x + d \mid a_j(v_j)^\top x \leq b_j(v_j), \forall v_j \in \mathcal{V}_j, j = 1, \ldots, m\}, \tag{QLP}$$

where $\mathcal{V}_j \subset \mathbb{R}^{q_j}, j = 1, \ldots, m$, are ellipsoidal uncertainty sets given as in (22) and $a_j : \mathbb{R}^{q_j} \to \mathbb{R}^n, b_j : \mathbb{R}^{q_j} \to \mathbb{R}, j = 1, \ldots, m$ are affine functions given as in (23).

It is well-known that any symmetric matrix is diagonalizable and thus, we can assume that the matrix C given in the problem (QLP) can be decomposed as

$$C = U^\top D U, \tag{27}$$

where $U := (U_1, \ldots, U_n)$ is an orthogonal $(n \times n)$ matrix and $D := \operatorname{diag}(\omega_1, \ldots, \omega_n)$ with $\omega_k \geq 0, k = 1, \ldots, n$.

The following corollary provides a characterization of stable exact second-order cone programming (SOCP) relaxations for the parametric robust convex quadratic program with affine constraints defined by (QLP).

Corollary 3.1 (Characterization of stable exact SOCP relaxations of (QLP)**)** *Assume that* $F := \{x \in \mathbb{R}^n \mid a_j(v_j)^\top x \leq b_j(v_j),\ \forall v_j \in \mathcal{V}_j, j = 1, \ldots, m\} \neq \emptyset$, *where* $\mathcal{V}_j, j = 1, \ldots, m$, *are given as in* (22). *Then, the closedness of the characteristic cone* $\widetilde{C} := \operatorname{cone}\big\{(0_n, 1) \cup \big(a_j(v_j), b_j(v_j)\big) \mid v_j \in \mathcal{V}_j, j = 1, \ldots, m\big\}$ *is equivalent to the following statement: For any* $C \succeq 0, c \in \mathbb{R}^n$ *and any* $d \in \mathbb{R}$ *with* $\inf$ (QLP) $> -\infty$, *one* *has*

$$\inf(\text{QLP}) = \max_{(t,\lambda_j^0,\lambda_j^i,\mu_k)} \Big\{ t \mid \sum_{k=1}^{n} \mu_k + \sum_{j=1}^{m} (\lambda_j^0 b_j^0 + \sum_{i=1}^{q_j} \lambda_j^i b_j^i) + t - d \le 0, t \in \mathbb{R},$$

$$\|\Big(\mu_k - \omega_k, U_k^\top \big(c + \sum_{j=1}^{m} (\lambda_j^0 a_j^0 + \sum_{i=1}^{q_j} \lambda_j^i a_j^i)\big)\Big)\|_2 \le \mu_k + \omega_k,$$

$$\mu_k \in \mathbb{R}_+, k = 1, \ldots, n,$$

$$\|L_j(\lambda_j^1, \ldots, \lambda_j^{q_j})\|_2 \le \lambda_j^0, \lambda_j^0 \in \mathbb{R}_+, \lambda_j^i \in \mathbb{R}, i = 1, \ldots, q_j, j = 1, \ldots, m\Big\}, \tag{28}$$

where $U := (U_1, \ldots, U_n)$ is the orthogonal matrix given as in (27) *and L_j is the decomposition factor of E_j for each $j \in \{1, \ldots, m\}$.*

Proof Let $y := Ux, \widehat{c} := Uc, \widehat{a}_j(v_j) := Ua_j(v_j)$ for $x \in \mathbb{R}^n, v_j \in \mathcal{V}_j, j = 1, \ldots, m$. We see that the problem (QLP) becomes

$$\inf_{y \in \mathbb{R}^n} \{y^\top D y + \widehat{c}^\top y + d \mid \widehat{a}_j(v_j)^\top y - b_j(v_j) \le 0, \forall v_j \in \mathcal{V}_j,\ j = 1, \ldots, m\}, \tag{AQL}$$

which is a particular case of problem (SQP) due to $D := \text{diag}(\omega_1, \ldots, \omega_n)$ with $\omega_k \ge 0, k = 1, \ldots, n$. Let $\widehat{g}_j(x, v_j) := \widehat{a}_j(v_j)^\top x - b_j(v_j)$ for $x \in \mathbb{R}^n$ and $v_j \in \mathcal{V}_j, j = 1, \ldots, m$. In view of Theorem 3.1, we assert that the closedness of the cone $\widehat{C} := \text{cone}\{(0_n, 1) \cup \text{epi}\, \widehat{g}_j^*(\cdot, v_j) \mid v_j \in \mathcal{V}_j, j = 1, \ldots, m\}$ is equivalent to the assertion that for any $D := \text{diag}(\omega_1, \ldots, \omega_n)$, where $\omega_k \ge 0, k = 1, \ldots, n$, and any $\widehat{c} \in \mathbb{R}^n$ with $\inf(\text{AQL}) > -\infty$, one has

$$\inf(\text{AQL}) = \max_{(t,\lambda_j^0,\lambda_j^i,\mu_k)} \Big\{ t \mid \sum_{k=1}^{n} \mu_k + \sum_{j=1}^{m} (\lambda_j^0 b_j^0 + \sum_{i=1}^{q_j} \lambda_j^i b_j^i) + t - d \le 0, t \in \mathbb{R},$$

$$\|\big(\mu_k - \omega_k, \widehat{c}_k + \sum_{j=1}^{m} (\lambda_j^0 \widehat{a}_{jk}^0 + \sum_{i=1}^{q_j} \lambda_j^i \widehat{a}_{jk}^i)\big)\|_2 \le \mu_k + \omega_k,$$

$$\mu_k \in \mathbb{R}_+, k = 1, \ldots, n,$$

$$\|L_j(\lambda_j^1, \ldots, \lambda_j^{q_j})\|_2 \le \lambda_j^0, \lambda_j^0 \in \mathbb{R}_+, \lambda_j^i \in \mathbb{R}, i = 1, \ldots, q_j, j = 1, \ldots, m\Big\},$$

where $\widehat{c}_k := U_k^\top c, \widehat{a}_{jk}^0 := U_k^\top a_j^0, \widehat{a}_{jk}^i := U_k^\top a_j^i, i = 1, \ldots, q_j, k = 1, \ldots, n, j = 1, \ldots, m.$

In this setting, it holds that $\widehat{C} = \text{cone}\big\{(0_n, 1) \cup \big(\widehat{a}_j(v_j), b_j(v_j)\big) \mid v_j \in \mathcal{V}_j, j = 1, \ldots, m\big\}$. Moreover, we can show that the closedness of the cone $\widehat{C}$ is equivalent to the closedness of the cone $\widetilde{C}$. So, the proof is complete due to $\inf(\text{AQL}) = \inf(\text{QLP})$. □

It is worth mentioning here that the closedness of the characteristic cone $\widetilde{C}$ implies the exact SOCP relaxation in (28), which was obtained in [12, Theorem 3.1] by using a dual approach of quadratic semi-infinite programming problems.

Parametric robust linear programs with convex quadratic constraints. Let us now focus on a parametric robust *linear* program with *convex quadratic* constraints that is defined as follows: For $c \in \mathbb{R}^n$ and $d \in \mathbb{R}$, one has a robust *linear* program with *convex quadratic* constraints as

$$\inf_{x\in\mathbb{R}^n} \{c^\top x + d \mid x^\top Bx + a_j(v_j)^\top x + b_j(v_j) \leq 0, \forall v_j \in \mathcal{V}_j, j = 1,\ldots,m\}, \tag{LQP}$$

where $\mathcal{V}_j \subset \mathbb{R}^{q_j}, j = 1,\ldots,m$, are ellipsoidal uncertainty sets given as in (22), $B \succeq 0$ and $a_j : \mathbb{R}^{q_j} \to \mathbb{R}^n, b_j : \mathbb{R}^{q_j} \to \mathbb{R}, j = 1,\ldots,m$ are affine functions given as in (23).

As said earlier, we may assume without loss of generality that the matrix B given in the problem (LQP) can be decomposed as

$$B = W^\top M^0 W, \tag{29}$$

where $W := (W_1,\ldots,W_n)$ is an orthogonal $(n \times n)$ matrix and $M^0 := \mathrm{diag}(u_1^0,\ldots,u_n^0)$ with $u_k^0 \geq 0, k = 1,\ldots,n$. Let $g_j(x,v_j) := x^\top Bx + a_j(v_j)^\top x + b_j(v_j)$ for $x \in \mathbb{R}^n$ and $v_j \in \mathcal{V}_j, j = 1,\ldots,m$.

In this case, we obtain a characterization of stable exact second-order cone programming (SOCP) relaxations for the parametric robust linear program with convex quadratic constraints defined by (LQP).

Corollary 3.2 (Characterization of stable exact SOCP relaxations of (LQP)**)** *Assume that* $F := \{x \in \mathbb{R}^n \mid x^\top Bx + a_j(v_j)^\top x + b_j(v_j) \leq 0,\ \forall v_j \in \mathcal{V}_j, j = 1,\ldots,m\} \neq \emptyset$, *where* $\mathcal{V}_j, j = 1,\ldots,m$, *are given as in* (22)*. Then, the closedness of the characteristic cone* $\widetilde{C}$ *is equivalent to the following statement: For any* $c \in \mathbb{R}^n$ *and any* $d \in \mathbb{R}$ *with* $\inf(\mathrm{LQP}) > -\infty$, *one has*

$$\begin{aligned}
\inf(\mathrm{LQP}) = \max_{(t,\lambda_j^0,\lambda_j^i,\mu_k)} \Big\{ t \mid\ & \sum_{k=1}^n \mu_k - \sum_{j=1}^m (\lambda_j^0 b_j^0 + \sum_{i=1}^{q_j} \lambda_j^i b_j^i) + t - d \leq 0, t \in \mathbb{R}, \\
& \|\Big(\mu_k - \sum_{j=1}^m \lambda_j^0 u_k^0, W_k^\top \big(c + \sum_{j=1}^m (\lambda_j^0 a_j^0 + \sum_{i=1}^{q_j} \lambda_j^i a_j^i)\Big)\|_2 \leq \mu_k + \sum_{j=1}^m \lambda_j^0 u_k^0, \\
& \mu_k \in \mathbb{R}_+, k = 1,\ldots,n, \\
& \|L_j(\lambda_j^1,\ldots,\lambda_j^{q_j})\|_2 \leq \lambda_j^0, \lambda_j^0 \in \mathbb{R}_+, \lambda_j^i \in \mathbb{R}, i = 1,\ldots,q_j, j = 1,\ldots,m \Big\},
\end{aligned}$$

where $W := (W_1,\ldots,W_n)$ *is the orthogonal matrix given as in* (29) *and* L_j *is the decomposition factor of* E_j *for each* $j \in \{1,\ldots,m\}$.

Proof Let $y := Wx, \widehat{c} := Wc, \widehat{a}_j(v_j) := Wa_j(v_j)$ for $x \in \mathbb{R}^n, v_j \in \mathcal{V}_j, j = 1,\ldots,m$. We see that the problem (LQP) becomes

$$\inf_{y\in\mathbb{R}^n} \{\widehat{c}^\top y + d \mid y^\top M^0 y + \widehat{a}_j(v_j)^\top y + b_j(v_j) \leq 0, \forall v_j \in \mathcal{V}_j,\ j = 1,\ldots,m\}, \tag{ALQ}$$

which is a particular case of problem (SQP) due to $M^0 := \operatorname{diag}(u_1^0, \ldots, u_n^0)$ with $u_k^0 \geq 0, k = 1, \ldots, n$. Invoking Theorem 3.1 and arguing as in the proof of Corollary 3.1, we arrive at the desired conclusion. □

We close this paper with an example, which illustrates that the stable exact SOCP relaxations hold for a parametric robust convex separable quadratic program without the validation of the Slater constraint qualification. This example also shows how one can find the optimal value and an optimal solution of a robust convex separable quadratic program by employing the characterization of stable exact SOCP relaxations.

Example 3.1 ***(Stable exact SOCP relaxations without the Slater condition)*** Consider the following parametric robust convex separable quadratic program

$$\inf_{x\in\mathbb{R}^2} \Big\{ x^\top Dx + c^\top x + d \mid x_1^2 + (6+v_1)x_2^2 + v_1 x_1 + (1+v_2)x_2 \leq 6,\ -v_2 x_2 \leq 0,\ \forall (v_1, v_2) \in \mathcal{V} \Big\}, \quad \text{(EP2)}$$

where $D := \operatorname{diag}(\omega_1, \omega_2)$, $\omega_k \geq 0$, $k = 1, 2$, $c := (c_1, c_2) \in \mathbb{R}^2$, $d \in \mathbb{R}$, and $\mathcal{V}$ is an ellipsoid uncertainty set given by $\mathcal{V} := \{v := (v_1, v_2) \in \mathbb{R}^2 \mid \frac{v_1^2}{25} + v_2^2 \leq 1\}$.

The problem (EP2) can be viewed in the form of (SQP), where $\mathcal{V}_1 := \mathcal{V}_2 := \mathcal{V}$, and $M_j : \mathbb{R}^2 \to \mathbb{R}^4, a_j : \mathbb{R}^2 \to \mathbb{R}^2, b_j : \mathbb{R}^2 \to \mathbb{R}, j = 1, 2$ are defined by $M_1^0 := \operatorname{diag}(1, 6), M_1^1 := \operatorname{diag}(0, 1), M_1^2 := \operatorname{diag}(0, 0), M_2^0 = M_2^1 = M_2^2 := \operatorname{diag}(0, 0), a_1^0 = a_1^2 := (0, 1), a_1^1 := (1, 0), a_2^0 = a_2^1 := (0, 0), a_2^2 := (0, -1), b_1^0 := -6, b_2^0 := 0, b_j^i := 0, i = 1, 2, j = 1, 2$.

Denote $g_j(x, v) := x^\top M_j(v)x + a_j(v)^\top x + b_j(v), j = 1, 2$ for $x \in \mathbb{R}^2$ and $v \in \mathcal{V}$. By taking $\overline{v} := (-5, 0) \in \mathcal{V}$, we see that $g_2(x, \overline{v}) = 0$ for all $x \in \mathbb{R}^2$, which means that the Slater constraint qualification fails. A direct calculation shows that, for each $v := (v_1, v_2) \in \mathcal{V}$,

$$g_1^*(\cdot, v)(w) = \frac{(w_1 - v_1)^2}{4} + \frac{(w_2 - 1 - v_2)^2}{4(6 + v_1)} + 6, \quad w := (w_1, w_2) \in \mathbb{R}^2,$$

$$g_2^*(\cdot, v)(w) = \begin{cases} 0 \text{ if } w = (0, -v_2) \\ +\infty \text{ otherwise,} \end{cases}$$

and then the characteristic cone $\widetilde{C}$ is computed by

$$\widetilde{C} := \operatorname{cone}\Bigg\{ (0,0,1), (w_1, w_2, \mu), (0, -v_2, \lambda) \mid \mu \geq \frac{(w_1 - v_1)^2}{4} + \frac{(w_2 - 1 - v_2)^2}{4(6 + v_1)} + 6, \lambda \geq 0, \\ (w_1, w_2) \in \mathbb{R}^2, (v_1, v_2) \in \mathbb{R}^2, \frac{v_1^2}{25} + v_2^2 \leq 1 \Bigg\} = \mathbb{R}^2 \times [0, +\infty),$$

which is closed.

So, we assert by Theorem 3.1 that the exact SOCP relaxation holds for the robust convex separable quadratic program (EP2) for any $D := \text{diag}(\omega_1, \omega_2)$, $\omega_k \geq 0$, $k = 1, 2$, $c := (c_1, c_2) \in \mathbb{R}^2$, $d \in \mathbb{R}$, whenever $\inf\,(\text{EP2}) > -\infty$.

Let us now consider the problem (EP2) with the following data for the objective function, $D := \text{diag}(1, 1)$, $c := (-2, 0)$ and $d := -3$, which is given by

$$\inf_{x \in \mathbb{R}^2} \Big\{ x_1^2 + x_2^2 - 2x_1 - 3 \mid x_1^2 + (6 + v_1)x_2^2 + v_1 x_1 + (1 + v_2)x_2 \leq 6,\ -v_2 x_2 \leq 0,\ \forall (v_1, v_2) \in \mathcal{V} \Big\}, \quad \text{(EP2-1)}$$

where $\mathcal{V}$ is defined as above. In this case, it can be checked that $\inf\,(\text{EP2-1}) \in \mathbb{R}$, and so, the exact SOCP relaxation holds for the problem (EP2-1), i.e.,

$$\inf\,(\text{EP2-1}) = \max\,(\text{SCP1}), \tag{30}$$

where (SCP1) is the second-order cone programming relaxation problem for (EP2-1) given as

$$\begin{aligned} \max_{(t, \lambda_j^0, \lambda_j^i, \mu_k)} \Big\{ t \mid & \sum_{k=1}^{2} \mu_k - \sum_{j=1}^{2} (\lambda_j^0 b_j^0 + \sum_{i=1}^{2} \lambda_j^i b_j^i) + t - d \leq 0, t \in \mathbb{R}, \\ & \| \big(\mu_k - \omega_k - \sum_{j=1}^{2} (\lambda_j^0 u_{jk}^0 + \sum_{i=1}^{2} \lambda_j^i u_{jk}^i), c_k + \sum_{j=1}^{2} (\lambda_j^0 a_{jk}^0 + \sum_{i=1}^{2} \lambda_j^i a_{jk}^i) \big) \|_2 \\ & \leq \mu_k + \omega_k + \sum_{j=1}^{2} (\lambda_j^0 u_{jk}^0 + \sum_{i=1}^{2} \lambda_j^i u_{jk}^i), \\ & \omega_k + \sum_{j=1}^{2} (\lambda_j^0 u_{jk}^0 + \sum_{i=1}^{2} \lambda_j^i u_{jk}^i) \geq 0, \mu_k \geq 0, k = 1, 2, \\ & \| L_j(\lambda_j^1, \lambda_j^2) \|_2 \leq \lambda_j^0, \lambda_j^0 \in \mathbb{R}_+, \lambda_j^i \in \mathbb{R}, i = 1, 2, j = 1, 2 \Big\} \end{aligned} \tag{SCP1}$$

with $L_j := \begin{pmatrix} \frac{1}{5} & 0 \\ 0 & 1 \end{pmatrix}$, $j = 1, 2$. Using the Matlab toolbox YALMIP [32], we solve the second-order cone programming problem (SCP1), and the solver returns its optimal value as -4.000. This together with (30) entails that

$$\inf\,(\text{EP2-1}) = -4. \tag{31}$$

Granting this, we can verify independently that $\overline{x} = (1, 0)$ is an optimal solution of problem (EP2-1). Indeed, we can see that $\overline{x}$ is feasible for (EP2-1). Now, let $Q(x) := x_1^2 + x_2^2 - 2x_1 - 3$ for $x = (x_1, x_2) \in \mathbb{R}^2$. It follows from (31) that $-4 = Q(\overline{x}) \geq \inf\,(\text{EP2-1}) = -4$. So, $\overline{x}$ is an optimal solution of problem (EP2-1).

4 Conclusions

In this paper, we have examined classes of parametric robust convex polynomial problems involving uncertainty in the constraint data. A parametric robust convex polynomial problem with convex compact uncertainty sets has been shown to admit stable exact conic relaxations under a characteristic cone constraint qualification. We have proved that such stable exact conic relaxations become stable exact SDP relaxations for the class of parametric robust SOS-convex polynomial problems involving spectrahedral uncertainty sets. Therefore, the characteristic cone constraint qualification can be regarded as the weakest regularity condition that guarantees the validation of exact SDP relaxations for robust SOS-convex polynomial problems. Under the corresponding constraint qualification, we have also derived stable exact SOCP relaxations for some classes of parametric robust convex quadratic programs under ellipsoidal uncertainty data.

It would be of great interest to see how the proposed approach can be developed to provide stable exact SDP relaxations for more general classes of optimization problems such as the class of difference of SOS-convex polynomial programs over SOS-concave matrix polynomial constraints in [30] or the class of SOS-convex semialgebraic programs in [11].

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