# Counting Cocircuits and Convex Two-Colourings is #P-complete


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**Abstract**

We prove that the problem of counting the number of colourings of the vertices of a graph with at most two colours, such that the colour classes induce connected subgraphs is #P-complete. We also show that the closely related problem of counting the number of cocircuits of a graph is #P-complete.


## 1 Introduction

A convex colouring of a graph $G$ is an assignment of colours to its vertices so that for each colour $c$ the subgraph of $G$ induced by the vertices receiving colour $c$ is connected. We consider a graph with no vertices to be connected. The purpose of this paper is to resolve a question of Makowsky [3] by showing that counting the number of convex colourings using at most two colours is #P-complete. More precisely we show that the following problem is #P-complete.

#CONVEX TWO-COLOURINGS
**Input:** Graph $G$.
**Output:** The number of $f : V(G) \to \{0,1\}$ such that both $G : f^{-1}(0)$ and $G : f^{-1}(1)$ are connected.


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For the definition of the complexity class #P, see [2] or [4].

Note that the number of convex colourings using at most two colours is equal to zero if $G$ has three or more connected components and equal to two if $G$ has exactly two connected components. So we may restrict our attention to connected graphs.

## 2 Reductions

All our graphs will be simple. We begin with a few definitions. Let $X$ and $Y$ be disjoint sets of vertices of a graph $G$. The set of edges of $G$ that have one endpoint in $X$ and the other in $Y$ is denoted by $\delta(X, Y)$. Given a connected graph $G$, a cut is a partition of $V(G)$ into two (non-empty) sets called its *shores*. The *crossing set* of a cut with shores $X$ and $Y$ is $\delta(X, Y)$. A cut is a *cocircuit* if no proper subset of its crossing set is the crossing set of a cut. Let $G$ be a connected graph. Then a cut of $G$ with crossing set $A$ is a cocircuit if and only if the graph $G \setminus A$ obtained by removing the edges in $A$ from $G$ has exactly two connected components.

Note that our terminology is slightly at odds with standard usage in the sense that the terms cut and cocircuit usually refer to what we call the crossing set of respectively a cut and a cocircuit. Our usage prevents some cumbersome descriptions in the proofs. We will however abuse our notation by saying that a cut or cocircuit has size $k$ if its crossing set has size $k$.

We consider the complexity of the following problems.

#COCIRCUITS
**Input:** Simple connected graph $G$.
**Output:** The number of cocircuits of $G$.

#REQUIRED SIZE COCIRCUITS
**Input:** Simple connected graph $G$, strictly positive integer $k$.
**Output:** The number of cocircuits of $G$ of size $k$.

#MAX CUT
**Input:** Simple connected graph $G$, strictly positive integer $k$.
**Output:** The number of cuts of $G$ of size $k$.

#MONOTONE 2-SAT
**Input:** A Boolean formula in conjuctive normal form in which each clause contains two variables and there are no negated literals.
**Output:** The number of satisfying assignments.

It is easy to see that each of these problems is a member of #P.

The following result is from Valiant's seminal paper on #P [6].

**Theorem 1.** #MONOTONE 2-SAT *is #P-complete.*

We will establish the following reductions.

#MONOTONE 2-SAT $\propto$ #MAX CUT $\propto$ #REQUIRED SIZE COCIRCUITS
$\propto$ #COCIRCUITS $\propto$ #CONVEX TWO-COLOURINGS.

Combining Theorem 1 with these reductions shows that each of the five problems that we have discussed is #P-complete. As far as we are aware, each of these reductions is new. We have not been able to find a reference showing that #Max Cut is $\#P$ complete. Perhaps it is correct to describe this result as 'folklore'. In any case our first reduction will establish this result. Some similar problems, but not exactly what we consider here, are shown to be $\#P$ complete in [5].

**Lemma 1.** #Monotone 2-SAT $\propto$ #Max Cut.

*Proof.* Suppose we have an instance $I$ of #Monotone 2-SAT with variables $x_1, \ldots, x_n$ and clauses $\mathcal{C} = \{C_1, \ldots, C_m\}$. We construct a corresponding instance $M(I) = (G, k)$ of #Max Cut by first defining a graph $G$ with vertex set

$$\{x\} \cup \{x_1, \ldots, x_n\} \cup \bigcup \{\{c_{i,1}, \ldots, c_{i,6}\} : 1 \leq i \leq m\}$$

For each clause we add nine edges to $G$. Suppose $C_j$ is $x_u \vee x_v$. Then we add the edges

$$xc_{j,1}, c_{j,1}c_{j,2}, c_{j,2}x_u, x_uc_{j,3}, c_{j,3}c_{j,4}, c_{j,4}x_v, x_vc_{j,5}, c_{j,5}c_{j,6}, c_{j,6}x.$$

Distinct clauses correspond to pairwise edge-disjoint circuits, each of size 9. Now let $k = 8|\mathcal{C}|$. Clearly $M(I)$ may be constructed in polynomial time. We claim that the number of solutions of instance $M(I)$ of #Max Cut is equal to $2^{|\mathcal{C}|}$ times the number of satisfying assignments of $I$.

Given a solution of $I$, let $L_1$ be the set of variables assigned the value true and $L_0$ the set of variables assigned false together with $x$. Observe that for each clause $C_j = x_u \vee x_v$ there are two choices of how to add the vertices $c_{j,1}, \ldots, c_{j,6}$ to either $L_0$ or $L_1$ so that exactly eight edges of the circuit corresponding to $C_j$ have one endpoint in $L_0$ and the other in $L_1$. Clearly the choices for each clause are independent and distinct satisfying assignments result in distinct choices of $L_0$ and $L_1$. Any of the choices of $L_0$ and $L_1$ constructed in this way may be taken as the shores of a cut of size $8|\mathcal{C}|$. Hence we have constructed $2^{|\mathcal{C}|}$ solutions of $M(I)$ corresponding to each satisfying assignment of $I$.

In any graph the intersection of a set of edges forming a circuit and a crossing set of a cut must always have even size. So in a solution of $M(I)$ each of the edge-disjoint circuits making up $G$ and corresponding to clauses of $I$ must contribute exactly eight edges to the cut. Suppose $U$ and $V \setminus U$ are the shores of a cut of $G$ of size $8|\mathcal{C}|$. Then it can easily be verified that for any clause $C = x_u \vee x_v$ both $U$ and $V \setminus U$ must contain at least one element from $\{x_u, x_v, x\}$. So it is straightforward to see that this solution of $M(I)$ is one of those constructed above corresponding to the satisfying assignment where a variable is false if and only if the corresponding vertex is in the same set as $x$. □

**Lemma 2.** #Max Cut $\propto$ #Required Size Cocircuits.

*Proof.* Suppose $(G, k)$ is an instance of #Max Cut. We construct an instance $(G', k')$ of #Required Size Cocircuits as follows. Suppose $G$ has $n$ vertices. To form $G'$ add new vertices $x, x', x_1, \ldots, x_{n^2}$ to $G$. Now add an edge from $x$ to every other vertex of $G'$ except $x'$ and similarly add an edge from $x'$ to every other vertex of $G'$ except $x$. Let $k' = n^2 + n + k$. Clearly $G'$ may be constructed in polynomial time. From each solution of the #Max Cut instance $(G, k)$

we construct $2^{n^2+1}$ solutions of the #Required Size Cocircuits instance $(G', k')$. Suppose $C = (U, V(G) \setminus U)$ is a solution of $(G, k)$ then we may freely choose to add $x, x', x_1, \ldots, x_{n^2}$ to either $U$ or $V(G) \setminus U$, with the sole proviso that $x$ and $x'$ are not both added to the same set, to obtain a cut in $G'$ of size $k' = n^2 + n + k$. Furthermore this cut is a cocircuit because both shores contain exactly one of $x$ and $x'$ and so they induce connected subgraphs.

Conversely suppose $C = (U, V(G') \setminus U)$ is a cocircuit in $G'$ of size $k'$. Consider the pair of edges incident with $x_j$. Note that the partition $(x_j, V(G') \setminus x_j)$ is a cocircuit. So if both of the edges incident with $x_j$ are in the crossing set of $C$ then because of its minimality we must have $C = (x_j, V(G') \setminus x_j)$ which is not possible because $C$ would then have size $2 < k'$. Now suppose that neither edge incident with $x_j$ is in the crossing set of $C$. Then both $x$ and $x'$ lie in the same block of the partition constituting $C$. But since $G$ is a simple graph, the maximum possible size of such a cocircuit is at most $2n + \binom{n}{2} < n^2 + n + k$. Hence precisely one of the edges adjacent to $x_j$ is in the crossing set. So $x$ and $x'$ are in different shores of $C$. Hence the crossing set of $C$ contains: for each $j$ precisely one edge incident to $x_j$ ($n^2$ edges in total), for each $v \in V(G)$ precisely one of edges $vx$ and $vx'$ ($n$ edges in total) and $k$ other edges with both endpoints in $V(G)$. So the partition $C' = (U \cap V(G), V(G) \setminus U)$ is a cut of $G$ of size $k$ and hence $C$ is one of the cocircuits constructed in the first part of the proof. Consequently the number of solutions of the instance $(G', k')$ of #Required Size Cocircuits is $2^{n^2+1}$ multiplied by the number of solutions of the instance $(G, k)$ of #Max Cut. □

**Lemma 3.** #Required Size Cocircuits $\propto$ #Cocircuits

*Proof.* Given a graph $G$ let $N_k(G)$ denote the number of cocircuits of size $k$ and $N(G)$ denote the total number of cocircuits. Let $G_l$ denote the $l$-stretch of $G$, that is, the graph formed from $G$ by replacing each edge of $G$ by a path with $l$ edges. Let $m = |E(G)|$. Then we claim that

$$N(G_l) = \sum_{k=1}^{m} l^k N_k(G) + \binom{l}{2} m.$$

To see this suppose that $C$ is a cocircuit of $G_k$. If the crossing set of $C$ contains two edges from one of the paths corresponding to an edge of $G$ then by the minimality of the crossing set of $C$ we see that it contains precisely these two edges. The number of such cocircuits is $\binom{l}{2}m$.

Otherwise the crossing set $C$ contains at most one edge from each path in $G_l$ corresponding to an edge of $G$. Suppose the crossing set of $C$ contains $k$ such edges. Let $A$ denote the corresponding edges in $G$. Then $A$ is the crossing set of a cocircuit in $G$ of size $k$. From each such cocircuit we can constuct $l^k$ cocircuits of $G_l$ by choosing one edge from each path corresponding to an edge in $A$. The claim then follows.

If we compute $N(G_1), \ldots, N(G_m)$ then we may retrieve $N_1(G), \ldots, N_m(G)$ by using Gaussian elimination because the matrix of coefficients of the linear equations is an invertible Vandermonde matrix. The fact that the Gaussian elimination may be carried out in polynomial time follows from [1]. □

**Lemma 4.** #Cocircuits $\propto$ #Convex Colourings.

*Proof.* The lemma is easily proved using the following observation. When two colours are available, there are two convex colourings of a connected graph using just one colour and the number of convex colourings using both colours is equal to twice the number of cocircuits. □

The preceding lemmas imply our main result.

**Theorem 2.** #CONVEX COLOURINGS *is #P-hard.*

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