Dialectical Argument Game Proof Theories for Classical Logic

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Abstract

Argument game-based proof theories provide procedural structures capable of determining the status of an argument. Given an argumentation framework, argument games identify the membership of an argument in a specific extension simulating a dispute between two opposing contenders. The semantics intended to be captured dictate the rules of the played game, which serve to describe how the players can achieve victory. Dialectical Classical logic Argumentation (Dialectical Cl-Arg) is a recent approach that provides real-world dialectical characterisations of Cl-Arg arguments by resource-bounded agents while preserving the rational criteria established by the rationality postulates and practical desiderata. This paper combines both subjects and introduces argument games for Dialectical Cl-Arg, highlighting the properties and benefits enjoyed by these games in comparison with the standard ones. The result will be a proof theory better equipped to approximate real-world non-monotonic single-agent reasoning processes.

1 Introduction

Since Aristotle's Organon [1, 33] and its considerable influence on the history of Western thought, rich scholarly literature has been investigating the intertwined notions of arguments, reasoning, and logic. For example, Walton claimed that *"logic is the evaluation of reasoning in arguments"* [35], whereas Mercier and Sperber emphasised the argumentative characterisation of reasoning:

> *"Reasoning is generally seen as a means to improve knowledge and make better decisions. However, much evidence shows that reasoning often*

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> *leads to epistemic distortions and poor decisions. This suggests that the function of reasoning should be rethought. Our hypothesis is that the function of reasoning is argumentative. It is to devise and evaluate arguments intended to persuade."* [23]

Trying to consolidate possessed information by formulating reasons (via arguments) that challenge or defend them is an ordinary procedure in which humans engage. This process is not only common but even necessary: how could it be possible, otherwise, to decide what to believe or trust without being misled by a non-reliable source of information? This 'scaffolding' (as defined in [24]) role of dialogues and arguments can be seen in social and lone thinking practices where the reasoner(s) evaluates the possessed information by constructing counter-arguments that assess their acceptability. Thanks to its important role, argumentation has been developed as a rich, interdisciplinary area of research spanning Philosophy, Linguistics, Psychology and Artificial Intelligence. Able to characterize a promising paradigm for modelling reasoning in the presence of conflict and uncertainty, formal-logical accounts of the argumentation theory have come to be increasingly central as a core study within Artificial Intelligence. According to such a theory, in order to determine if a piece of information is acceptable, it will suffice to prove that the argument (in which the considered information is embedded) is justified under specific semantics. A way of doing this is to show the membership of the argument in a winning strategy of an argument game (as described, for example, in [25, 34] and [9]). Indeed, argument game-based proof theories provide procedural structures capable of determining the status of an argument according to the semantics intended to be captured.

Dung's abstract argumentation framework (AF) [17] has been considered the formalism from which stemmed most of the subsequent studies in this fruitful research field. Nevertheless, although a plethora of works has successfully shown various additions and instantiations of Dung's abstract AF and achieved different goals, none of these approaches managed to provide a full rational account for real-world resource-bounded agents. Undoubtedly, the introduction of the rationality postulates [6, 7], as well as desiderata for practical applications [20], have allowed eschewing the arising of counter-intuitive results in AFs instantiations. However, such requirements demand a consumption of resources that typically far exceed the availability of real-world agents.

1.1 Contribution

The main contribution of this research paper is the development of argument games for Dialectical Classical Logic Argumentation (Dialectical Cl-Arg [15]), a recent approach that provides real-world dialectical characterisations of AFs by resource-bounded agents. This approach satisfies the practical desiderata and the rationality postulates (under minimal requirements) and revolves around the core notion of *dialectical defeats*. Such defeats enable argumentative interactions more aligned with the dialectical reasoning of real-world resource-bounded agents. Thus, their presence requires the implementation of *dialectical argument game* proof theories capable of conveying the same idea as single-agent reasoning processes.

1.2 Paper Overview

The paper is organized as follows. Section 2 outlines an overview of the main definitions of Dung's argumentation framework, the standard argument games and Dialectical Cl-Arg. Section 3 provides the first contributions by establishing the general formal background that characterises the dialectical argument games. The other contributions occur in Sections 4 and 5, where (a) the protocol of the dialectical admissible game (which also yields the credulous preferred game) and (b) the protocol of the dialectical grounded game are given along with (c) their respective soundness and completeness results. The specific properties enjoyed by dialectical games in comparison with the standard ones are illustrated in Section 6, whereas Section 7 introduces potential efficiency improvements that may be embedded in the developed protocols. Section 8 presents the related works and some promising research paths that might be investigated in the future. Finally, Section 9 draws the conclusions and summarizes the paper findings.

2 Background

Argumentation has been developed as a theory able to characterize the essence of non-monotonic reasoning via the dialectical interplay of arguments. According to Dung's seminal paper [17], an Argumentation Framework (AF) is composed of a set of arguments 'AR' and a binary relation called *'attacks'*, which denotes conflicts existing between arguments in AR, i.e., AF = $\langle$AR, *attacks*$\rangle$. Various semantics have also been presented and each of them specifies the status of (*sceptically* or *credulously*) justified (i.e., acceptable) arguments. Several works stemmed from [17], some of which introduced different ways of structuring arguments and instantiating Dung's abstract AF [18, 31, 27]. For example, Classical Logic Argumentation (Cl-arg) [21, 2] is one such instantiation that builds AFs using classical logic as its underlying language.

2.1 Dialectical Classical Logic Argumentation

Unlike the standard formalisation of Cl-Arg, real-world agents behave pragmatically and do not need to: (i) always construct every argument defined by a base, (ii) enforce consistency and subset minimality checks on their arguments (nor do they have enough computational power to do these checks, given their limited resources). Dialectical Cl-Arg provides a formalisation of real-world modes of dialectical reasoning from resource-bounded agents whilst satisfying both the rationality postulates [6, 7] and practical desiderata [20].

Definition 1. [Dialectical Arguments] [15] $X = (\Delta, \Gamma, \alpha)$ *is a* dialectical argument *defined by a base* $\mathcal{B}$ *of classical wff, if* $(\Delta \cup \Gamma) \subseteq \mathcal{B}$*,* $\Delta \cap \Gamma = \emptyset$*, and* $\Delta \cup \Gamma \vdash_c \alpha$*. If* $\alpha = \curlywedge$ *then* X *is said to be a* falsum *argument. If* $\Gamma = \emptyset$ *then* X *is said to be* unconditional*; else* X *is* conditional*. Finally, if* $\Delta = \emptyset$ *then* X *is said to be* unassailable*.*

Δ*,* Γ *and* α *are respectively referred to as the* premises $(\mathit{Prem}(X))$*,* suppositions $(\mathit{Supp}(X))$ *and* conclusion $(\mathit{Con}(X))$ *of* $X = (\Delta, \Gamma, \alpha)$*. Also, the union of premises and suppositions of* X *can be referred to as the* assumptions $(\mathit{Assumptions}(X))$ *of the argument.*

Attacks and defeats for Dialectical Cl-Arg work differently than their respective counterparts for Classical Logic Argumentation (Cl-Arg). The reason is the presence of suppositions embedded in the internal structure of the arguments. Intuitively, it is common practice for interlocutors in dialogues to differentiate between their own arguments' premises, regarded as true, and their opponents' premises that they want to challenge: "by considering what I deem to be valid and supposing what you have committed to, I can show your premises inconsistency". This motivates such an epistemic distinction between information considered true (i.e., $\mathit{Prem}(X)$, the *premises* of an argument X) and opponents' information supposed true (i.e., $\mathit{Supp}(X)$, the *supposition* of an argument X) which proves useful also in solving the so-called 'foreign commitment problem'[1].

Definition 2. [Attacks and Defeats][15] *Let* AR *be a set of dialectical arguments defined by a base* $\mathcal{B}$*. The attack relation 'attacks'* $\subseteq AR \times AR$ *is defined as follows. For any* $X = (\Delta, \Gamma, \alpha)$*,* $Y = (\Pi, \Sigma, \beta) \in AR$*: attacks*$(X, Y)$ *iff:*

- *if* $\alpha \neq \curlywedge$ *then* $\overline{\alpha} \in \Pi$ *(*X *attacks* Y *on* $\overline{\alpha}$*, equivalently on* $Y' = (\{\overline{\alpha}\}, \emptyset, \overline{\alpha})$*);*

[1] As extensively explained in [8], the foreign commitment problem is the issue that arises in dialogical applications when agents are forced to commit to the premises of their interlocutors in order to challenge their arguments.

- *if* $\alpha = \curlywedge$ *(X attacks Y on any* $\phi \in \Gamma \cap \Pi$*, equivalently on any* $Y' = (\{\phi\}, \emptyset, \phi)$*).*

Let $\prec$ *be a strict partial ordering over AR. Then, for every* X*,* Y *such that attacks*(X, Y)*, defeats*(X, Y) *iff exactly one of the following holds:*

- *either* X *is an argument of the form* $(\emptyset, \Gamma, \curlywedge)$*;*

- *else,* $\exists \psi \in$ *Prem*(Y) *such that attacks*(X, Y) *on* ψ*, and* $X \not\prec (\{\psi\}, \emptyset, \psi)$*.*

$X \Rightarrow Y$ *will stand for "defeats*(X, Y)*", and* $X \not\Rightarrow Y$ *will stand for "*$\neg$*defeats*(X, Y)*".*

The description of Dialectical Cl-Arg formalism provided in [15] accounts only for *undermine* attacks and the ensuing defeats based upon this type of conflict. Undermines are those kinds of attacks that occur when the conclusion of the attacking argument targets the premises of the challenged argument. Nevertheless, the literature (e.g., [29, 32]) presents *undercuts* and *rebuttals* as additional categories of conflicts. The first denotes arguments arguing against the defeasible inference rule used to derive the attackee's conclusion, whereas the second depicts a disagreement towards the attackee's defeasible conclusion. However, none of these conflicts can be transposed in Dialectical Cl-Arg since no defeasible rules (but only the classical entailment $\vdash_c$) are employed in the construction of the arguments.

The strict partial ordering of Definition 2 refers to the Elitist Preference Ordering. In addition, the authors of [15] show that such preference is also *'redundance-coherent'* in the sense that arguments are not strengthened when redundantly weakening with syntactically disjoint assumptions[2]. This is an important property that ensures the satisfaction of the non-contamination (i.e., Non-Interference and Crash Resistance) rationality postulates for Dialectical Cl-Arg.

Definition 3. [Elitist Preference Ordering]

Let X, Y *be dialectical classical logic arguments defined by a base* $\mathcal{B}$*, and* $\leq$ *a partial preordering over* $\mathcal{B}$*. Then:*

(i) $X \prec Y$ *iff* $\exists \alpha \in$ *Assumptions*(X) *such that* $\forall \beta \in$ *Assumptions*(Y)*,* $\alpha < \beta$*.*

(ii) $\prec$ *is redundance-coherent iff:* $\forall X, X', Y$ *such that* $X = (\Gamma, \emptyset, \alpha)$*,* $X' = (\Delta \cup \Gamma, \emptyset, \alpha)$*, and* $\Delta \parallel \Gamma \cup \{\alpha\}$*: if* $X \prec Y$ *then* $X' \prec Y$*.*

[2]Here 'weakening' denotes that a logical entailment from, say, Δ continues to be valid when adding some Γ to Δ. Also, we consider 'syntactically disjoint' (denoted by using '$\parallel$') two sets of formulae that do not have symbols in common.

Cl-Arg assumes instantiation of an AF by all arguments defined by a base $\mathcal{B}$ of classical wff, a task that proves to be unfeasible for agents with limited resources. As such, dialectical arguments (Definition 1) along with the described defeat relation (Definition 2) allow us to introduce a *dialectical AF* as an argumentation framework $\langle$AR, *defeats*$\rangle$ where AR is any subset of the dialectical arguments defined by a base $\mathcal{B}$.

$A_1 = (\{a\}, \emptyset, a)$	$B_1 = (\{b\}, \emptyset, b)$
$F_1 = (\{b, \neg a \vee \neg b\}, \emptyset, \neg a)$	$G_1 = (\{a, \neg a \vee \neg b\}, \emptyset, \neg b)$
$F_2 = (\{b\}, \{\neg a \vee \neg b\}, \neg a)$	$G_2 = (\{a\}, \{\neg a \vee \neg b\}, \neg b)$
$F_3 = (\{\neg a \vee \neg b\}, \{b\}, \neg a)$	$G_3 = (\{\neg a \vee \neg b\}, \{a\}, \neg b)$
$N_1 = (\{a \supset b\}, \{\neg b\}, \neg a)$	$N_2 = (\{a \supset b, \neg b\}, \emptyset, \neg a)$
$N_3 = (\{a \supset b, a\}, \emptyset, b)$	$O_1 = (\{\neg(a \supset b)\}, \emptyset, \neg(a \supset b))$
$L_1 = (\{\neg b\}, \emptyset, \neg b)$	$X_3 = (\{b\}, \{\neg b\}, \curlywedge)$
$C_1 = (\{\neg a \vee \neg b\}, \emptyset, \neg a \vee \neg b)$	$H_1 = (\{a, b\}, \emptyset, \neg(\neg a \vee \neg b))$
$X_1 = (\emptyset, \{a, b, \neg a \vee \neg b\}, \curlywedge)$	$X_2 = (\{a, b, \neg a \vee \neg b\}, \emptyset, \curlywedge)$

Table 1: Example of dialectical arguments defined by a base $\mathcal{B} = \{a,\ b,\ \neg a \vee \neg b,\ \neg b,\ a \supset b,\ \neg(a \supset b)\}$.

Defeats and dialectical defeats for dialectical AFs present an important difference: the reference to a set $\mathcal{S}$ of arguments. The general idea is that, when challenging the acceptability of an argument with respect to a set $\mathcal{S}$, the defeating argument can also suppose premises from all the arguments in $\mathcal{S}$. Whereas, the argument that defends $\mathcal{S}$ can only suppose the premises of the defeating argument. This new kind of defeat compelled the authors of [15] to adjust the standard semantics accordingly.

Definition 4. [Dialectical defeats and semantics for dialectical AFs][15] *Let $\langle AR$, defeats$\rangle$ be a dialectical AF, $\mathcal{S} \subseteq AR$ and X, $Y \in AR$. Then:*

1) X dialectically defeats Y with respect to $\mathcal{S}$, *denoted* $X \Rightarrow_{\mathcal{S}} Y$, *if defeats*$(X, Y)$ *and Supp*$(X) \subseteq$ *Prem*$(\mathcal{S} \cup \{Y\})$.
2) $\mathcal{S}$ *is* conflict-free *if* $\forall Z$, $Y \in \mathcal{S}$, $Z \not\Rightarrow_{\mathcal{S}} Y$.
3) Y *is* acceptable *with respect to* $\mathcal{S}$ *if* $\forall X$ *such that* $X \Rightarrow_{\mathcal{S}} Y$, $\exists Z \in \mathcal{S}$ *such that* $Z \Rightarrow_{\{X\}} X$.
4) *Let $\mathcal{S}$ be conflict-free. Then $\mathcal{S}$ is: an* admissible *extension iff $X \in \mathcal{S}$ implies X is acceptable with respect to $\mathcal{S}$; a* complete *extension iff $\mathcal{S}$ is admissible and*

if X is acceptable with respect to $\mathcal{S}$ then $X \in \mathcal{S}$; a preferred *extension iff it is a set inclusion maximal complete extension; the* grounded *extension iff it is the set inclusion minimal complete extension.*

The following example depicts a scenario that clarifies the role of dialectical defeats while also providing a comparison between Dialectical Cl-Arg and Cl-Arg arguments. Since rigorous Cl-Arg formal definitions can be found in [2, 21, 15], for simplicity, Example 1 will consider such arguments as being identical to Dialectical Cl-Arg arguments devoided of suppositions.

Example 1. *Consider Figure 1. Let A_1, $B_1 \in \mathcal{S}$ be the dialectical arguments introduced in Table 1, and let $Z_1 = (\{a \supset \neg b\}, \{a\}, \neg b)$ be a dialectical argument that defeats B_1 with respect to $\mathcal{S}$, i.e., $Z_1 \Rightarrow_{\mathcal{S}} B_1$. Notice that such defeat occurs only due to the presence of the formula $a \in Prem(A_1)$. The supposition of the formula a by the dialectical argument Z_1 (i.e., $Supp(Z_1) \subseteq Prem(\mathcal{S} \cup \{B_1\})$ allows concluding $\neg b$, hence defeating argument B_1. However, $Z_0 = (\{a \supset \neg b\}, a \supset \neg b)$, the Cl-Arg argument that has the same premises as Z_1, is not capable of moving the same defeat to B_1. Indeed, the absence of the formula a among the premises prevents Z_0 from classically entailing the conclusion $\neg b$, hence precluding the defeat of argument B_1. This example shows how, by supposing formulae (from single arguments or sets), additional attacks and defeats may arise for Dialectical Cl-Arg arguments in comparison with Cl-Arg arguments.*

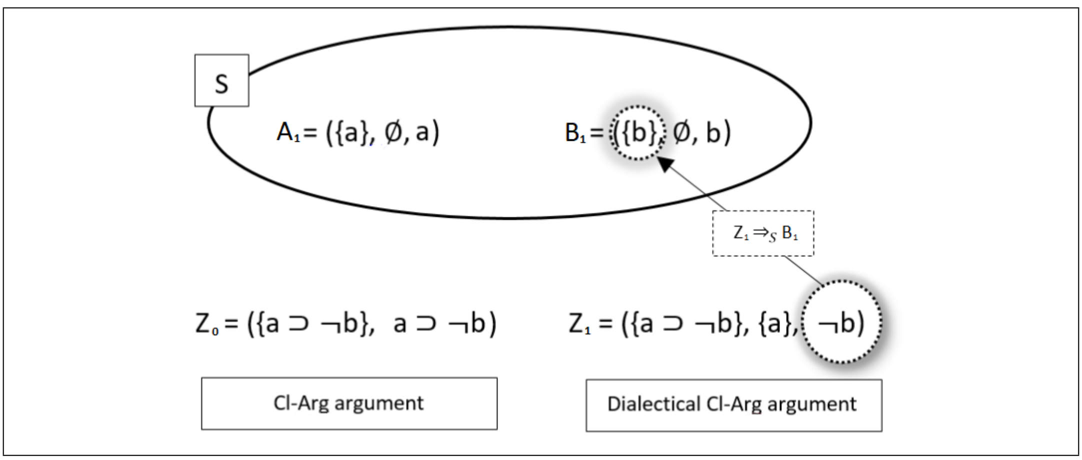

Figure 1: An example of differences between Cl-Arg and Dialectical Cl-Arg.

The conclusions of an extension in Dialectical Cl-Arg may derive from conditional arguments that only suppose the truth of the premises without any commitment. As such, we should account for a more restrictive definition of conclusions. That is to say, once the extensions are defined, we detach only the conclusions of *unconditional* arguments all of whose assumptions are premises presumed true.

Definition 5. [Conclusions of an Extension in Dialectical Cl-Arg] *Let E be an extension of a dialectical AF. Then $C(E) = \{\phi \mid (\Delta, \emptyset, \phi) \in E\}$.*

Dialectical AFs enjoy some specific properties, as explained in [15]. Here we are going to outline five of them (P1, P2, P3, P4, P4′), which will be used later in the next sections.

Proposition 1. *Given a dialectical $AF = \langle AR,\ defeats \rangle$:*

($P1$) $\forall X \in AR$: $\alpha \in Prem(X)$ *implies that* $(\{\alpha\}, \emptyset, \alpha) \in AR$ (*where* $(\{\alpha\}, \emptyset, \alpha)$ *is denoted as the* 'elementary argument' *of X defined by α);*

($P2$) $\forall X \in AR$: *if* $X' \in [X]$, *that is to say, if X' is the* logically equivalent argument *of X (i.e., the only difference between X and X' is the different distribution of premises and supposition), then $X' \in AR$;*

($P3$) *If* $(\Delta, \emptyset, \alpha) \in AR$ *and* $(\Gamma, \emptyset, \overline{\alpha}) \in AR$, *then either* $(\Delta, \emptyset, \curlywedge) \in AR$ *or* $(\Gamma, \emptyset, \curlywedge) \in AR$ *or* $(\Delta \cup \Gamma, \emptyset, \curlywedge) \in AR$;

($P4$) *If* $(\Gamma, \emptyset, \alpha) \in AR$, $\Delta \subseteq \Gamma$, $\Delta \neq \emptyset$ *and* $\Delta \parallel \Gamma \setminus \Delta \cup \{\alpha\}$, *then either* $(\Delta, \emptyset, \curlywedge) \in AR$ *or* $(\Gamma \setminus \Delta, \emptyset, \alpha) \in AR$;

($P4'$) *If* $(\Gamma, \emptyset, \alpha) \in AR$, $\Delta \subseteq \Gamma$, $\Delta \neq \emptyset$ *and* $\Delta \parallel \Gamma \setminus \Delta \cup \{\alpha\}$, *then* $(\Delta, \emptyset, \curlywedge) \in AR$.

We can now refer to ⟨AR, *defeats*⟩ as a *partially instantiated dialectical AF* (*pdAF*) if AR corresponds to any subset of the dialectical arguments defined by a base $\mathcal{B}$ such that AR satisfies $P1$, $P2$, $P3$ and $P4$.

A *non-redundant pdAF* is, instead, a pdAF such that AR satisfies $P1$, $P2$, $P3$, $P4'$ and there are no redundantly contaminated arguments[3].

[3] A redundantly contaminated argument is an argument that employs redundant assumptions, that is to say, a subset of the assumptions is unnecessary for drawing the argument conclusion. This may occur due to the fact that Dialectical Cl-Arg drops subset minimality checks. To avoid violation of the non-contamination postulates, the adopted preference relation has to be 'redundance-coherent'. Indeed, this is the case of the Elitist preference of Definition 3.

2.1.1 Rationality Postulates for Dialectical Cl-Arg

The rationality postulates are specific properties whose satisfaction ensures that any concrete instantiations of an argumentation framework fulfil some rational criteria. Dialectical Cl-arg satisfies the rationality postulates and does so by requiring that the AF enjoys $P1$-$P4$. This would impose minimally restrictive assumptions[4] as to the arguments that agents should be able to construct, thus providing a rational account of arguments more suited for the limited availability of resources that characterises real-world agents. A detailed report of the postulates, along with lemmas, theorems and respective proofs of their validity, is given in [15].

Theorem 1. **[Sub-argument Closure]** *Let E be a complete extension of a dialectical $AF = \langle AR, \mathit{defeats}\rangle$ such that AR satisfies P1. Let $X \in E$. Then if $\alpha \in \mathit{Prem}(X)$ then $(\{\alpha\}, \emptyset, \alpha) \in E$. That is to say, all the elementary arguments associated with $\mathit{Prem}(X)$ are in E.*

Theorem 2. **[Direct Consistency]** *Let E be an admissible extension of a dialectical $AF = \langle AR, \mathit{defeats}\rangle$. If AR satisfies P1, P2 and P3, then $\forall \alpha, \beta \in C(E)$, $\alpha \neq \curlywedge$ and $\beta \neq \overline{\alpha}$. That is to say, no conflicting or unconditional falsum arguments are in E.*

Theorem 3. **[Premise Consistency]** *Let $\langle AR, \mathit{defeats}\rangle$ be a dialectical AF such that AR satisfies P2. If for some $\Delta \subseteq \mathit{Prem}(E)$: $(\Delta, \emptyset, \curlywedge) \in AR$, then E cannot be an admissible extension of $\langle AR, \mathit{defeats}\rangle$.*

Closure under Strict Rules for Dialectical Cl-Arg slightly differs from its standard version. That is caused by the limited availability of resources that characterises real-world agents. Indeed, although it may be the case that $C(E) \vdash_c \alpha$, it may not be that there exists an $X \in E$ such that X concludes α, given that agents are not logically omniscient and do not construct all arguments from a base. Hence, the following version of the postulate:

Theorem 4. **[Closure under Strict Rules]** *Let E be a complete extension of a dialectical $AF = \langle AR, \mathit{defeats}\rangle$, where AR satisfies P1. Let $E' \subseteq E$ and $C(E') \vdash_c \alpha$. If there exists an $X = (\Delta, \emptyset, \alpha) \in AR$ such that $\Delta = \mathit{Prem}(E')$, then $X \in E$.*

Non-contamination postulates provide means for eschewing different kinds of contaminations that may negatively affect the dialectical AFs. In particular, the satisfaction of *Non-Interference* ensures that no syntactically disjoint bases $\mathcal{B}$ (i.e., bases that do not share predicate or function symbols) influence each other's argumentation defined inferences. On the other hand, *Crash Resistance* guarantees

[4] Especially the satisfaction of $P1$-$P3$.

that no set of formulae yields the same outcome when merged with a syntactically disjoint set of formulae.

Theorem 5. [**Non-Interference**] *Non-interference is satisfied by (non-redundant) pdAFs.*

Theorem 6. [**Crash Resistance**] *Crash Resistance is satisfied if there does not exist a contaminating base $\mathcal{B}$ for pdAFs and non-redundant pdAFs.*

2.1.2 Dung's Fundamental Lemma and Monotonicity of the Characteristic Function for Dialectical Cl-Arg

Among the most important key results of Dung's seminal paper [17] are the *fundamental lemma* and the *monotonicity of the AF's characteristic function* $\mathcal{F}_{AF}$ (that yields the constructive definition of the grounded extension via its iterations). However, unlike Dung's standard AFs, these properties cannot be straightforwardly shown, since when determining the acceptability of X w.r.t. E, the defeats on X are not independent of the set E under consideration. For dialectical AFs, the defeats on X w.r.t. E may be a subset of the defeats on X w.r.t. $E' \supset E$ (due to the additional premises committed to in E'). To avoid this issue, the authors of [15] have devised specific 'epistemically maximal' sets of arguments by means of whose it is possible to show the desired properties.

Definition 6. [**Epistemically maximal sets**] *Let $\langle AR, \mathit{defeats} \rangle$ be a dialectical AF. Then $E \subseteq AR$ is* epistemically maximal (em) *iff:*

$$\textit{If } X = (\Delta, \Gamma, \alpha) \in E,\ \Gamma' \subseteq (\Gamma \cap \mathit{Prem}(E)),\ \textit{then } X' = (\Delta \cup \Gamma', \Gamma \setminus \Gamma', \alpha) \in E \qquad (\bullet)$$

The function $Cl_{em} : 2^{AR} \to 2^{AR}$ maps any E to its epistemically maximal set. As such, $Cl_{em}(E)$ denotes the smallest superset of E that is closed under condition $(\bullet)$.

Notice that adding all arguments up to some i to a set E, and then closing, yields the same result as adding each argument one by one and closing prior to each subsequent addition [15]. It is now possible to prove a variant of the fundamental lemma that involves *em* sets:

Lemma 1. [**Fundamental Lemma for Dialectical Cl-Arg**][15] *Let X, X' be acceptable w.r.t. an admissible extension E of a dialectical $AF = \langle AR,\ \mathit{defeats} \rangle$. Then:*

(1) *$Cl_{em}(E \cup \{X\})$ is admissible, and*

(2) *X' is acceptable w.r.t. $Cl_{em}(E \cup \{X\})$*

Lemma 1 entails:

Proposition 2. *Every admissible extension of a dialectical AF is a subset of a preferred extension.*

Proposition 2 guarantees that it suffices to show that an argument X is in an admissible extension, in order to prove that X is credulously justified under the preferred semantics (exactly as Dung's standard AFs).

Finally, by employing a variant of the framework characteristic function, i.e., $\mathcal{F}_p$, whose domain is composed of sets E that are *em* admissible and that returns $Cl_{em}(\mathcal{F}(E))$, we can also show the constructive definition of the grounded extension. Indeed, starting with the empty set and iteratively applying $\mathcal{F}_p$, the monotonically increasing sequence approximates, and in the case of a *finitary* dialectical AF, it constructs, the least fixed point of $\mathcal{F}_p$, i.e., the grounded extension:

Proposition 3. [15] *Let $\langle AR, defeats\rangle$ be a dialectical AF, and $F^0 = \emptyset$, $F^{i+1} = \mathcal{F}_p(F^i)$. Let E be the grounded extension of $\langle AR, defeats\rangle$. Then:*

1. $E \subseteq \bigcup_{i=0}^{\infty}(F^i)$.
2. *If $\langle AR, defeats\rangle$ is* finitary*, i.e., $\forall X \in AR$, the set $\{Y \mid defeats(Y, X)\}$ is finite, then $E = \bigcup_{i=0}^{\infty}(F^i)$.*

In the remainder of the paper, we are going to see how harnessing the properties and formalism thus far introduced will shape the dialectical characterisation of standard argument games.

2.2 Standard Argument Games

Before moving forward, let us now review the fundamental notions of the standard argument games as described in [25]. Notice that these definitions have been modified to accommodate dialectical AFs (which is a fair straightforward adaptation). However, recall that the main contributions of this paper concern the development of argument games for Dialectical Cl-Arg that involves *dialectical defeats* (Definition 4): this entails a non-trivial modification of the standard games.

In a nutshell, an argument game is played by two players, PRO (for *proponent*) and OPP (for *opponent*), each of which is referred to as the other's 'counterpart'. PRO starts the game by moving an initial argument X that it wants to test. After that, both players take turns in moving arguments against their counterpart's moves. This generates disputes:

Definition 7. [Dispute] *A sequence of moves in which each player moves against its counterpart's argument is referred to as a* dispute*. Formally, $d = X$—Y—Z— $\cdots$ is a dispute, and X—Y denotes a player moving argument Y against an argument X played by its counterpart (similarly, Y—Z). A* sub-dispute *d' of a dispute d is any sub-sequence of d that starts with the same initial argument. For example, if $d = X$—Y—Z, then $d' = X$—Y would be a sub-dispute of d.*

Notice that, to avoid ambiguity, each argument of a dispute will be labelled with either P or O (that stands for either one of the two players, PRO or OPP). Hence, $d =$ (P)X—(O)Y—(P)Z is a dispute where PRO moves the argument X, followed by Y played by OPP and countered by another move from PRO, Z.

We can now introduce the notion of the (unique) dispute tree, which represents the 'playing field' of the standard argument games. In other words, the dispute tree is the data structure that contains all the potential moves (and sequences of moves) available to the players.

Definition 8. [Dispute Tree] *Let $AF = \langle AR, defeats\rangle$ be a finite dialectical argumentation framework, and let* A $\in AR$*. The* dispute tree *induced by* A *in the AF is the (upside-down) tree $\mathcal{T}$ of arguments, such that $\mathcal{T}$'s root node is* A*, every branch of the tree (from root to leaf) is a different dispute, and $\forall X, Y \in AR$: X is a child of Y in $\mathcal{T}$ iff $defeats(X, Y)$.*

From here on, we are going to write PRO($*$) and OPP($*$) to denote the sets of all PRO and OPP arguments in $*$, where $*$ can be replaced with d, $\mathcal{T}$ or any other tree that will be introduced in the remainder of the paper. Also, LAST(d) will identify the last argument played in a dispute d.

An argument game is said to be won by the proponent only if it has a winning strategy. That is to say, only if it can successfully defend the argument it wants to test (i.e., the root of $\mathcal{T}$) against any possible arguments moved by the opponent. PRO loses otherwise. In other words, this may be interpreted as a formalisation of the simple principle already emphasised by Dung: *"The one who has the last word laughs best"* [17].

Definition 9. [Winning Strategy] *Let $\mathcal{T}$ be the dispute tree induced by* A *in a finite dialectical $AF = \langle AR, defeats\rangle$. Let also d be a dispute in $\mathcal{T}$. Then, a* winning strategy *$\mathcal{T}'$ for* A *is the dispute tree $\mathcal{T}$ pruned in a way such that:*

(9.1) *The set $\mathcal{T}'_D$ of disputes in $\mathcal{T}'$ is a non-empty finite set such that each dispute $d \in \mathcal{T}'_D$ is finite and is won by PRO (i.e., $LAST(d) \in PRO(\mathcal{T})$);*

(9.2) $\forall d \in \mathcal{T}'_D$, $\forall d'$ *such that* d' *is some sub-dispute of* d, $LAST(d') = X$ *and* $X \in PRO(\mathcal{T})$, *then* $\forall Y \in OPP(\mathcal{T})$ *such that* $Y \Rightarrow X$, *there is a* $d'' \in \mathcal{T}'_D$ *such that* $d' - Y$ *is a sub-dispute of* d''.

Informally, the previous definition states that a winning strategy is the dispute tree $\mathcal{T}$ pruned in a way such that (9.1) $\mathcal{T}'_D$ is a non-empty finite set, its disputes are finite, end with a PRO argument and (9.2) are such that OPP has moved exhaustively (i.e., all the moves that OPP could have played, had been played) and also PRO has countered every defeating argument moved by OPP.

3 Developing Dialectical Argument Games

In the following sections, we are going to develop argument games for Dialectical Cl-Arg that accommodate the dialectical defeats and semantics introduced in Definition 4. The resulting proof theory will present some specific features that will distinguish it from the standard argument games, although the general structure remains similar. Intuitively, winning a dialectical game for an argument A means having a 'dialectical procedure' (depending on the semantics that the proof theory is meant to capture) for defending the information contained in A, hence showing the admissibility of the encoded data.

The main difference between a dispute tree $\mathcal{T}$ and a *dialectical dispute tree* $\mathcal{D}$ can be identified with the additional reference to a subset $\mathcal{S} \subseteq \text{PRO}(\mathcal{T})$. That is to say, $\mathcal{S}$ represents a candidate admissible set of PRO arguments such that PRO commits to their premises. Recall once again that, when challenging the acceptability of an argument with respect to a set $\mathcal{S}$, the defeating argument can suppose premises from all the arguments in $\mathcal{S}$. Whereas, the argument that defends $\mathcal{S}$ can only suppose the premises of the defeating argument. Another important difference between standard and dialectical games is that the latter handles *partially instantiated dialectical AFs* (*pdAFs*)[5]. As a consequence, each dialectical game enjoys specific properties that encapsulate the dialectical uses of arguments by real-world resource-bounded agents, thus succeeding in better approximating a process capable of bridging formal (proof-theoretical) and informal (real-world exchange of arguments) single-agent reasoning.

We can now formally introduce the (unique) dialectical dispute tree induced by A wrt a set $\mathcal{S}$:

Definition 10. **[Dialectical Dispute Tree]** *Let* $\mathcal{T}$ *be the dispute tree induced by* A *in a finite* $pdAF = \langle AR, defeats\rangle$. *Let also* $\mathcal{S} \subseteq PRO(\mathcal{T})$. *Then, the* dialectical

[5] Refer to Proposition 1.

dispute tree $\mathcal{D}$ *induced by* A *with respect to* $\mathcal{S}$ *is the dispute tree* $\mathcal{T}$ *pruned in a way such that* $\forall X, Y \in AR$*: X is a child of Y in $\mathcal{D}$ iff defeats*(X, Y) *and:*

1. *If* $X \in PRO(\mathcal{D})$ *and* $Y \in OPP(\mathcal{D})$*, then* $X \Rightarrow_{\{Y\}} Y$*, i.e.* X *defeats* Y *and* $Supp(X) \subseteq Prem(Y)$*;*
2. *If* $X \in OPP(\mathcal{D})$ *and* $Y \in PRO(\mathcal{D})$*, then* $X \Rightarrow_{\mathcal{S}} Y$*, i.e.* X *defeats* Y *with respect to* $\mathcal{S}$ *and* $Supp(X) \subseteq Prem(\mathcal{S} \cup \{Y\})$*.*

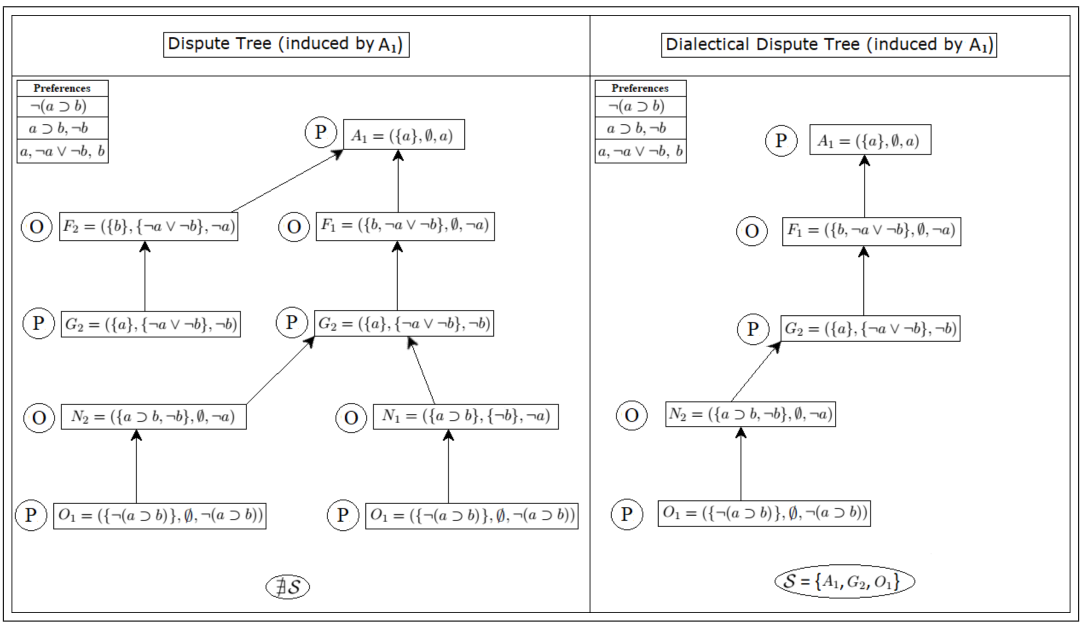

Figure 2: The (incomplete) dispute tree $\mathcal{T}$ (on the left) induced by A_1 in a finite pdAF = $\langle$AR, *defeats*$\rangle$ and the corresponding (incomplete) dialectical dispute tree $\mathcal{D}$ (on the right) induced by A_1 wrt $\mathcal{S} = \{A_1, G_2, O_1\}$ in the same pdAF = $\langle$AR, *defeats*$\rangle$.

The 'playing field' of the dialectical argument games (i.e., the data structure on the basis of which the games are played) is still depicted by the dispute tree $\mathcal{T}$. Indeed, the relationship existing between the dispute tree $\mathcal{T}$ induced by A in a finite pdAF and the dialectical dispute tree $\mathcal{D}$ induced by A wrt $\mathcal{S}$ is such that $\mathcal{D}$ is 'contained' in $\mathcal{T}$ (since $\mathcal{D}$ is a pruned version of $\mathcal{T}$), as shown in the following example.

Example 2. *Figure 2 presents the (incomplete) dispute tree* $\mathcal{T}$ *induced by* A_1 *in a finite pdAF* $= \langle AR, defeats\rangle$ *and the corresponding (incomplete) dialectical dispute* *tree $\mathcal{D}$ induced by A_1 wrt $\mathcal{S} = \{A_1, G_2, O_1\}$ in the same pdAF. Both trees are incomplete since the purpose of the example is just to show the relationship existing between them. For the same reason, we also avoid listing all the arguments of the pdAF.*

Observe that, unlike $\mathcal{T}$, where no set is taken into consideration, the defeats in $\mathcal{D}$ are parametrized to the set $\mathcal{S}$. This implies that, when defeating PRO's arguments, OPP can only suppose the premises of the arguments in the set $\mathcal{S}$ (besides the premises of the targeted argument). No such restrictions exist for $\mathcal{T}$. Notice that, even if we keep extending both trees, dispute $d = (P)A_1 — (O)F_2 — (P)G_2$ will never be part of $\mathcal{D}$. This is because, according to Definition 10 (which also emphasizes how dialectical defeats work), PRO can move G_2 only if $Supp(G_2) \subseteq Prem(F_2)$. However, this is never going to be the case since the formula $\neg a \vee \neg b \notin Prem(F_2)$. Therefore, even if the two trees were identical in every other branch, the absence of dispute d will still make $\mathcal{D}$ 'contained' in $\mathcal{T}$.

Dialectical argument games share with the standard argument games the notion of a winning strategy: in order to win the game for an argument A, PRO must have a winning strategy for it. It will lose otherwise. However, the two definitions slightly differ since a dialectical winning strategy has to take into account the set $\mathcal{S}$ targeted by the dialectical defeats:

Definition 11. [Dialectical Winning strategy] *Let $\mathcal{D}$ be the dialectical dispute tree induced by* A *wrt $\mathcal{S}$ in a finite pdAF $= \langle AR, defeats \rangle$ and let d be a dispute in $\mathcal{D}$. Then, a* dialectical winning strategy *$\mathcal{W}$ for* A *corresponds to the dialectical dispute tree $\mathcal{D}$ pruned in a way such that:*

(11.1) *The set $\mathcal{W}_D$ of disputes in $\mathcal{D}$ is a non-empty finite set such that each dispute $d \in \mathcal{W}_D$ is finite and is won by PRO (i.e., $LAST(d) \in PRO(\mathcal{D})$);*

(11.2) *$\forall d \in \mathcal{W}_D$, $\forall d'$ such that d' is some sub-dispute of d, $LAST(d') = X$ and $X \in PRO(\mathcal{D})$, then $\forall Y \in OPP(\mathcal{D})$ such that $Y \Rightarrow_{\mathcal{S}} X$, there is a $d'' \in \mathcal{W}_D$ such that $d' — Y$ is a sub-dispute of d''.*

Similarly to Definition 9, the previous definition states that a dialectical winning strategy corresponds to the dialectical dispute tree $\mathcal{D}$ pruned in a way such that (11.1) $\mathcal{W}_D$ is a non-empty finite set, its disputes are finite, end with a PRO argument and are such that (11.2) OPP has moved exhaustively and also PRO has countered each defeating argument moved by OPP. The difference is in the dialectical defeats: the nodes are no more connected by means of the defeats relations among arguments, but through dialectical defeats among arguments that target the set $\mathcal{S}$.

We now have all the elements needed to formally introduce the protocol of the dialectical admissible/preferred game. Similar to a list of instructions, this protocol determines the legal moves that can be performed by the players. The game unfolds as a result of the legal arguments played and terminates when there are no more valid moves available. When this happens, the status of the root of the tree is evaluated. The presence of a winning strategy for such an argument assigns the victory to PRO. Strictly speaking, OPP never wins: its purpose is to counter each argument moved by the proponent in order to assist it in testing the admissibility of the root argument (indeed, argument games are formalisations of single-agent reasoning processes). Nevertheless, OPP can still prevent PRO's victory by invalidating its winning strategy.

3.1 Progressively Constructing Dialectical Dispute Trees

When we play a Φ-dialectical game we are increasingly building, starting from the root A and following the legal moves licensed by the protocol Φ, a dialectical dispute tree denoted as Φ-$\mathcal{D}^n$. Each node of such a tree corresponds to an argument progressively played by either PRO or OPP that is labelled with a positive integer i (with $1 \leq i \leq n$). These additional labels allow identifying the order in which the arguments have been played, hence, also determining the current stage (i.e., the nth-stage) of the Φ-dialectical game. Recall that the dispute tree $\mathcal{T}$ induced by A represents the playing field of the games, and every Φ-dialectical game for A is contained within its data structure (i.e., Φ-$\mathcal{D}^n$ is a 'pruned-version' of $\mathcal{T}$). Moreover, being a dialectical dispute tree, even Φ-$\mathcal{D}^n$ is constructed wrt a set $\mathcal{S} \subseteq$ PRO($\mathcal{T}$), however, such $\mathcal{S}$ can gradually increase with each new move made by PRO during the game. Indeed, $\mathcal{S}$ is composed of the same arguments moved by PRO in Φ-$\mathcal{W}^n$ (i.e., a dialectical winning strategy for A of Φ-$\mathcal{D}^n$), which can be extended while the game proceeds[6]. As it will be shown, observe also that $\mathcal{S}$ is still a different set than PRO(Φ-$\mathcal{W}^n$), meaning that it will modify its members according to the changes in PRO(Φ-$\mathcal{W}^n$), but it will never be empty even if there is no winning strategy Φ-$\mathcal{W}^n$.

In order to formally describe a Φ-dialectical game, we first need to define a *partial dialectical dispute tree* $\mathcal{D}^n$ which will stand as a potential 'game template' deprived of a protocol:

Definition 12. [Partial dialectical dispute tree] *A partial dialectical dispute tree*

[6]Although the set $\mathcal{S}$ can increase the number of its members while the game goes on, it can never exceed the size of PRO($\mathcal{T}$). Indeed, keep in mind that every Φ-dialectical game for A is contained in the dispute tree $\mathcal{T}$ induced by A (since $\mathcal{T}$ corresponds to the playing field of the game).

$\mathcal{D}^n$ *induced by* A *wrt* $\mathcal{S} \subseteq PRO(\mathcal{T})$ *(with* $\mathcal{S} \neq \emptyset$*) in a finite pdAF* $= \langle AR, defeats \rangle$ *is the (upside-down) tree that starts from the argument* A*, and it is progressively built up to the nth-move by one of the two players, such that each node of the tree is labelled with a positive integer* i *(for* $1 \leq i \leq n$*). Moreover, every branch of the tree (from root to leaf) constitutes a different dispute. Also* $\forall X, Y \in AR$*:* X *is a child of* Y *in* $\mathcal{D}^n$ *iff defeats*(X, Y) *and:*

1. *If* $X \in PRO(\mathcal{D}^n)$ *and* $Y \in OPP(\mathcal{D}^n)$*, then* $X \Rightarrow_{\{Y\}} Y$*, i.e.* X *defeats* Y *and Supp*$(X) \subseteq$ *Prem*(Y)*;*
2. *If* $X \in OPP(\mathcal{D}^n)$ *and* $Y \in PRO(\mathcal{D}^n)$*, then* $X \Rightarrow_{\mathcal{S}} Y$*, i.e.* X *defeats* Y *with respect to a set* $\mathcal{S}$ *and Supp*$(X) \subseteq$ *Prem*$(\mathcal{S} \cup \{Y\})$*.*

Finally, $\mathcal{W}^n$ *will denote a dialectical winning strategy for* A *of* $\mathcal{D}^n$ *as per Definition 11 (substituting* $\mathcal{D}$ *with* $\mathcal{D}^n$*).*

Every stage of a Φ-dialectical game can then be identified with a specific dialectical dispute tree Φ-$\mathcal{D}^n$, i.e., a *partial dialectical dispute tree* of Definition 12 where each of its nodes also fulfils the legal move requirements according to the protocol Φ. Consider that every such stage of the game is not unique: playing the same game multiple times does not necessarily hold the same Φ-$\mathcal{D}^n$ at identical stages n. They can indeed differ depending on the way in which the legal arguments have been deployed by the players. As we are going to see, this notion is essential for a proper account of the dialectical defeats in the game protocol[7].

3.2 Disqualified Defeats

It is interesting to notice that, during a Φ-dialectical game, a dialectical defeat that occurred in an early stage of the game might not take place in a more advanced phase of the same game. This can be caused by an update of the current $\mathcal{S}$, the set parametrized by OPP for performing dialectical defeats. We denote this anomaly as ‘disqualified defeats’.

Definition 13. [Disqualified dialectical defeats] *Let* Φ-$\mathcal{D}^n$ *be the dialectical dispute tree of a* Φ*-dialectical game built up to the nth-move where* X *and* Y *denote*

[7]Observe that it is possible for one (or more, depending on the protocol) dialectical winning strategy Φ-$\mathcal{W}^n$ for A of Φ-$\mathcal{D}^n$ to exist, although there is no dialectical winning strategy $\mathcal{W}$ for A of $\mathcal{D}$. This can happen, for example, when $\mathcal{D}$ is composed only by infinite disputes (recall that we need finite disputes to have winning strategies, as stated by Definition 11.1), whilst Φ-$\mathcal{D}^n$ is composed by finite disputes, due to the restrictions imposed by the protocol Φ. In this situation, it is possible to identify in Φ-$\mathcal{D}^n$ a winning strategy Φ-$\mathcal{W}^n$. Such an example is illustrated in Figure 3(b).

arguments played respectively by OPP and PRO in Φ-$\mathcal{D}^n$*. Let also* $X \Rightarrow_{\mathcal{S}} Y$ *by supposing* $\alpha \in Prem(\mathcal{S})$*. If, after the game goes on, we will reach a stage* Φ-$\mathcal{D}^{n+k}$ *(for* $k > 0$*) where* $\alpha \notin Prem(\mathcal{S})$*, then the defeat moved by* X *against* Y *will be invalidated and will be denoted as 'disqualified'. As such,* X *and all the arguments following it in the same dispute will be (temporarily) pruned from the tree.*

Consider indeed that the status of disqualified defeats might be temporary and be updated again in a further stage of the game (when these defeats will become valid once more). Definition 13 entails the following proposition:

Proposition 4. *Let* Φ-$\mathcal{D}^n$ *be the dialectical dispute tree of a* Φ*-dialectical game built up to the nth-move:*

(I) *If the nth-move is an argument* X *played by OPP, then moving* X *cannot disqualify the dialectical defeat that* X *performs against a PRO argument.*

(II) *The presence of OPP arguments whose defeats have been disqualified will not affect the dialectical winning strategy.*

Proof.

(I) Since X is the last argument (legally) played in Φ-$\mathcal{D}^n$, it trivially does not comply with Definition 13.

(II) Even if the dialectical defeats moved by OPP arguments have been disqualified (hence are no more a threat for PRO), the requirements of the dialectical winning strategy have not changed. That is to say, every dispute of Φ-$\mathcal{W}^n$ must terminate with a PRO argument (Definition (11.1)).

□

Notice that every dialectical game protocol Φ takes into account disqualified defeats, which are then also contemplated by the dialectical dispute tree Φ-$\mathcal{D}^n$ (and dialectical winning strategy Φ-$\mathcal{W}^n$).

4 Dialectical Admissible/Preferred Games

We can now formally introduce the protocol for the dialectical admissible/preferred game. As already stated, during each dialectical argument game, the players have to comply with a protocol Φ that identifies the legal moves allowed.

Definition 14. [Dialectical Admissible Game legal moves] *Let $\mathcal{D}^n$ and $\mathcal{W}^n$ be defined as in Definition 12, let d be a dispute of $\mathcal{D}^n$ and d' be a sub-dispute of d. Let also $(PL_n)X$ (for $n > 0$) denote the argument X played by either one of the two players (P or O) as the (last) nth-move. Then Φ_P identifies legal moves in the following way:*

(14.0) *PRO moves the first argument.*

(14.1) *If $(PL_n)X$ and $n = 2k$ (for $k > 0$), then the next move $n+1$, say Y, is by PRO and it is such that:*

(*a*) *$Y \Rightarrow_{\{Z\}} Z$, where $Z \in OPP(\mathcal{D}^n)$;*

(*b*) *There exists a $\mathcal{W}^{n+1}$ for A of $\mathcal{D}^{n+1}$.*

(14.2) *If $(PL_n)X$ and $n = 2k + 1$ (for $k \geq 0$), then the next move $n+1$, say Y, is by OPP and it is such that:*

(*a*) *$Y \Rightarrow_{\mathcal{S}} Z$, where $Z \in \mathcal{S}$ and $\mathcal{S} := PRO(\mathcal{W}^n)$*[8]*;*

(*b*) *If $d = d'—Z$, then $Y \notin OPP(d')$;*

(*c*) *For each $d = d'—J—\cdots$, where $J \in OPP(\mathcal{D}^n)$ and its defeat has been disqualified, then $LAST(d) = LAST(d')$ until next OPP's turn.*

A Φ_P-dialectical game is said to be terminated when, during its turn, the corresponding player runs out of the legal moves identified by (14.1(*a*-*b*)) or (14.2(*a*-*b*)) of the protocol Φ_P. PRO wins only if it has a winning strategy once the game terminates. It loses otherwise.

The previous protocol can be informally summarised as follows. PRO starts the game by playing the first argument [(14.0)] and, after that, OPP will make its move. Then, the two players alternate in playing only one argument at a time to reply to one of their counterpart's arguments. Observe that when $\mathcal{S}$ is initialized in the game and, subsequently, every time its arguments are updated by the changes in PRO($\mathcal{W}^n$) [(14.2(a))], it is always the beginning of OPP's turn. This means that the condition for which $\mathcal{S} \neq \emptyset$ is continuously respected[9].

[8]The symbol ':=' denotes a variable initialization rather than an equivalence relation. That is to say, at the beginning of each OPP's turn, the content of $\mathcal{S}$ is initialized to the current PRO($\mathcal{W}^n$), i.e, the arguments member of $\mathcal{S}$ are the same as PRO($\mathcal{W}^n$). This operation overwrites the previous contents of $\mathcal{S}$.

[9]That is because a situation in which $\mathcal{S}$ = PRO($\mathcal{W}^n$) = $\emptyset$ never occurs at the beginning of OPP's turn.

Notice that the established protocol allows *backtracking* to other arguments. That is to say, when PRO moves it can either target the last argument played by OPP or another argument moved by OPP in the dialectical dispute tree generated thus far (i.e., an argument member of the set OPP($\mathcal{D}^n$)) [(14.1(a))]. Similarly, when OPP moves it can either target the last argument played by PRO or another argument moved by PRO in the current dialectical winning strategy (i.e., an argument member of the set PRO($\mathcal{W}^n$)) [(14.2(a))]. The *relevance conditions* [(14.1(b)) for PRO; (14.2(a)) for OPP] ensure that: after PRO has made its move, there will be a winning strategy $\mathcal{W}^{n+1}$, hence providing the victory to PRO; after OPP has moved, instead, the previous winning strategy will cease to exist, thus preventing PRO from winning. That is to say, PRO will be forced to generate a dialectical winning strategy during each of its turns, while OPP will have to invalidate such a winning strategy during every one of its turns. Backtracking and relevance conditions are strictly connected. Although it is possible for a player to defeat an argument other than the one previously posited by its counterpart, such a move needs to comply with the protocol relevance conditions. This combination ensures that both participants exhaustively account for every option available, otherwise restricted around the last argument played (which may be unassailable, hence preventing further move against it). Indeed, given the goal of changing the winning status at the end of their respective turns, PRO and OPP may choose which argument to defeat, thus leaving for a later moment the other (if still available) alternatives.

The restriction (14.2(b)) on the moves played by OPP is necessary (as also shown in the standard games of [25, 34] and [9]). Indeed, allowing OPP to repeat its arguments, since OPP is required to move exhaustively, could imply the generation of infinite disputes. To see why let us suppose that $(\mathrm{PL}_n)X$ (for $n > 1$) identifies an argument X played by either one of the two players (denoted as P or O) as its nth move in a Φ-dialectical game. Then, there could be an infinite dispute d like the following:

$$d = (P_1)A\text{—}\cdots\text{—}(O_n)Y\text{—}(P_{n+1})Z\text{—}(O_{n+2})Y\text{—}(P_{n+3})Z\text{—}(O_{n+4})Y\text{—}\cdots$$

Intuitively, since Z is capable of defending itself by defeating Y, there is no need to further extend the dispute by repeating the same arguments: this is because Z has already shown its acceptability wrt PRO($\mathcal{W}^{n+1}$). Therefore, the only way for avoiding infinite disputes (and infinite dialectical admissible/preferred games) is to prevent OPP from repeating its arguments in the same disputes.

Finally, (14.2(c)) ensures that the disqualified defeats (Definition 13) are taken into account throughout the game. That is to say, whenever a dialectical defeat moved by an argument J is disqualified, the protocol guarantees the pruning of J and all the arguments that follow in the same dispute, until the next turn of OPP, when a new check for disqualified defeats will occur.

Remark 1. *Similarly to the standard argument games presented in* [*25*]*, the protocol of the dialectical admissible games is identical to the protocol of the dialectical credulous preferred games. Indeed, it suffices to show the membership of an argument* A *in an admissible extension to show also that* A *is credulously justified under the preferred semantics as well. That is because every admissible extension of a dialectical AF is a subset of a preferred extension. This is a consequence of the Fundamental Lemma* (*Lemma 1*) *and its entailed property* (*Proposition 2*)*.*

4.1 Soundness and Completeness

As it has been defined, the admissible/preferred game satisfies the properties of soundness and completeness. This proves the equivalence existing between the victory of the Φ_P-dialectical game for an argument A and the membership of the same A to an admissible/preferred extension of the corresponding finite pdAF.

Theorem 7. *Let* Φ_P*-*$\mathcal{D}^n$ *identifies a terminated* Φ_P*-dialectical game for* A*.*
Then, there exists a dialectical winning strategy Φ_P*-*$\mathcal{W}^n$ *for* A*, such that the set PRO(*Φ_P*-*$\mathcal{W}^n$*) of arguments moved by PRO in* Φ_P*-*$\mathcal{W}^n$ *is conflict-free, iff* A *is included in an admissible extension* Adm *of the pdAF.*

Proof.

Soundness. We have to prove that if A is a member of the conflict-free set PRO(Φ_P-$\mathcal{W}^n$), then $A \in \mathsf{Adm}$. To simplify the notation, let E = PRO(Φ_P-$\mathcal{W}^n$). Assume that A is a member of the conflict-free set E, then:

- By Definition 11.2, the existence of the winning strategy implies that: each argument played by OPP against arguments moved by PRO in the winning strategy has been successfully countered by PRO. That is to say, $\forall X \in$ E, if $\exists Y \in$ AR such that $Y \Rightarrow_E X$, then $\exists Z \in$ E, such that $Z \Rightarrow_{\{Y\}} Y$, ensuring in this way that X is acceptable wrt E.
- Recall that the set of disputes of Φ_P-$\mathcal{W}^n$ is finite and composed of finite disputes (by Definition 11.1). As such, E is composed of a finite number of arguments.

We have thus shown that E is a finite, conflict-free set and every argument in E is acceptable wrt it. Therefore, E corresponds to an admissible extension, hence, if A is a member of the conflict-free set PRO(Φ_P-$\mathcal{W}^n$), then $A \in \mathsf{Adm}$.

Completeness. We show that if $A \in \mathsf{Adm}$, then A is a member of the conflict-free set PRO(Φ_P-$\mathcal{W}^n$). We are going to do this by constructing a dialectical winning strategy Φ_P-$\mathcal{W}^n$ for A.

– Assume that $A \in \mathsf{Adm}$. Since the pdAF is finite, then it is also finitary, meaning that every argument in Adm has a finite number of defeaters. Then we can build a dialectical winning strategy Φ_P-$\mathcal{W}^n$ for A if PRO starts the game with A and, for each argument Y dialectically defeating A and moved by OPP, PRO chooses one argument X from Adm (even A itself) such that X$\Rightarrow_{\{Y\}}$Y. Notice that the generation of infinite disputes is prevented by the admissible/preferred protocol (Definition 14.2(b)). This procedure can be repeated for every argument Z dialectically defeating X, and so on, until OPP runs out of legal moves according to the protocol Φ_P (which will happen for sure since A is a member of an admissible set).

The result will be a dialectical winning strategy Φ_P-$\mathcal{W}^n$ for A, hence, A is a member of the conflict-free set PRO(Φ_P-$\mathcal{W}^n$). We have thus shown that, if $A \in \mathsf{Adm}$, then A is a member of the conflict-free set PRO(Φ_P-$\mathcal{W}^n$).

□

5 Dialectical Grounded Games

The dialectical grounded game protocol Φ_G enjoys the same notations and definitions introduced thus far, but presents also important differences compared to the dialectical admissible/preferred game. Indeed, the protocol should be designed such that, when the game terminates and PRO is the winner, the set PRO(Φ_G-$\mathcal{W}^n$) of arguments moved by PRO in a dialectical winning strategy Φ_G-$\mathcal{W}^n$ is a subset of the grounded extension Grd of the pdAF. In this way, by iterating the framework characteristic function $\mathcal{F}$ from PRO(Φ_G-$\mathcal{W}^n$), we are able to obtain the grounded extension Grd. However, recall that it is the monotonicity of the function, in the case of a finitary pdAF[10], that ensures the construction of the least fixed point of $\mathcal{F}$ which corresponds to the grounded extension.

In Dialectical Cl-Arg [15] the monotonicity of $\mathcal{F}$ holds only under the domain of *epistemically maximal* (*em*) admissible sets of arguments (described in Definition 6). Then, to get the grounded extension via the iteration of $\mathcal{F}$ from the set PRO(Φ_G-$\mathcal{W}^n$), we will need PRO(Φ_G-$\mathcal{W}^n$) to be *em*. Otherwise, we might have to face a situation in which argument A, whose membership in Grd we wanted to

[10]Being finitary, it can be shown that $\mathcal{F}$ is also $\omega-$continuous (as explained in [17] for standard AFs and in [15] for pdAFs).

test via the dialectical grounded game, is not acceptable wrt Grd, although $A \in$ PRO(Φ_G-$\mathcal{W}^n$). To address this issue, we are going to adapt the protocol Φ_G accordingly.

Definition 15. [Dialectical Grounded Game legal moves] *Let $\mathcal{D}^n$ and $\mathcal{W}^n$ be characterized as in Definition 12, let d be a dispute of $\mathcal{D}^n$ and d' be a sub-dispute of d. Let also $(PL_n)X$ (for $n > 0$) denote the argument X played by either one of the two players (P or O) as the (last) nth-move. Then Φ_G identifies legal moves in the following way:*

(15.0) *PRO moves the first argument.*

(15.1) *If $(PL_n)X$ and $n = 2k$ (for $k > 0$), then the next move n+1, say Y, is by PRO and it is such that:*

- (a) *$Y \Rightarrow_{\{Z\}} Z$, where $Z \in$ OPP($\mathcal{D}^n$);*
- (b) *There exists a $\mathcal{W}^{n+1}$ for A of $\mathcal{D}^{n+1}$;*
- (c) *If $d = d'$—Z, then $Y \notin$ PRO(d').*

(15.2) *If $(PL_n)X$ and $n = 2k + 1$ (for $k \geq 0$), then the next move n+1, say Y, is by OPP and it is such that:*

- (a) *$Y \Rightarrow_{\mathcal{S}} Z$, where $Z \in \mathcal{S}$ and $\mathcal{S} :=$ PRO($\mathcal{W}^n$).*
- (b) *For each $d = d'$—J—$\cdots$, where $J \in$ OPP($\mathcal{D}^n$) and its defeat has been disqualified, then LAST(d) = LAST(d') until next OPP's turn.*

(15.3) *If, at the beginning of its turn, OPP cannot perform the move described by (15.2(a)), then apply function Cl_{em} (Definition 6) on PRO($\mathcal{W}^n$).*

Notice that a Φ_G-dialectical game is said to be terminated when, during its turn, at least one player runs out of the legal moves identified by (15.1(a-c)) or (15.2(a)) of the protocol Φ_G. PRO wins only if it has a winning strategy once the game terminates. It loses otherwise.

As per Definition 14, the previous protocol can be informally summarised as follows. PRO starts the game by playing the first argument [(15.0)] and after that OPP will make its move. Then, the two players alternate in playing only one argument at a time to reply to one of their counterpart's arguments. Observe that when $\mathcal{S}$ is initialized in the game and, subsequently, every time its arguments are updated by the changes in PRO($\mathcal{W}^n$) [(15.2(a))], it is always the beginning of OPP's turn. This means that the condition for which $\mathcal{S} \neq \emptyset$ is continuously respected.

Notice also that the established protocol allows *backtracking* to other arguments. That is to say, when PRO moves it can either target the last argument played by OPP or another argument moved by OPP in the dialectical dispute tree generated thus far (i.e., an argument member of the set OPP($\mathcal{D}^n$)) [(15.1(a))]. Similarly, when OPP moves it can either target the last argument played by PRO or another argument moved by PRO in the current dialectical winning strategy (i.e., an argument member of the set PRO($\mathcal{W}^n$)) [(15.2(a))]. The *relevance conditions* [(15.1(b)) for PRO; (15.2(a)) for OPP] ensure that: after PRO has made its move, there will be a winning strategy $\mathcal{W}^{n+1}$, hence providing the victory to PRO; after OPP has moved, instead, the previous winning strategy will cease to exist, thus preventing PRO from winning. That is to say, PRO will be forced to generate a dialectical winning strategy during each of its turns, while OPP will have to invalidate such a winning strategy during every one of its turns. Observe also that backtracking and relevance conditions are strictly correlated (similarly to Definition 14).

The restriction (15.1(c)) emphasises the additional burden of proof entailed by the membership to the grounded extension. This is intuitively captured by the idea that in defending an argument X's membership to the grounded extension Grd, PRO must 'appeal to' some argument other than X itself. This is reflected in the game by the fact that PRO cannot repeat the arguments it has already moved in the same disputes.

Moreover, (15.2(b)) ensures that the disqualified defeats (Definition 13) are taken into account throughout the game. That is to say, whenever a dialectical defeat moved by an argument J is disqualified, the protocol guarantees the pruning of J and all the arguments that follow in the same dispute, until the next turn of OPP, when a new check for disqualified defeats will occur.

Finally, in light of the previously underlined epistemically maximal requirement, an additional one-time move has been included. Recall that adding all arguments up to some i to a set E, and then *em* closing, yields the same result as adding each argument one by one and closing prior to each subsequent addition. As such, once the game is terminated in favour of PRO and immediately before PRO is declared the winner, it suffices to apply function Cl_{em} (Definition 6) over the resulting set PRO($\mathcal{W}^n$) rendering it *em*, therefore, a subset of the grounded extension of the pdAF.

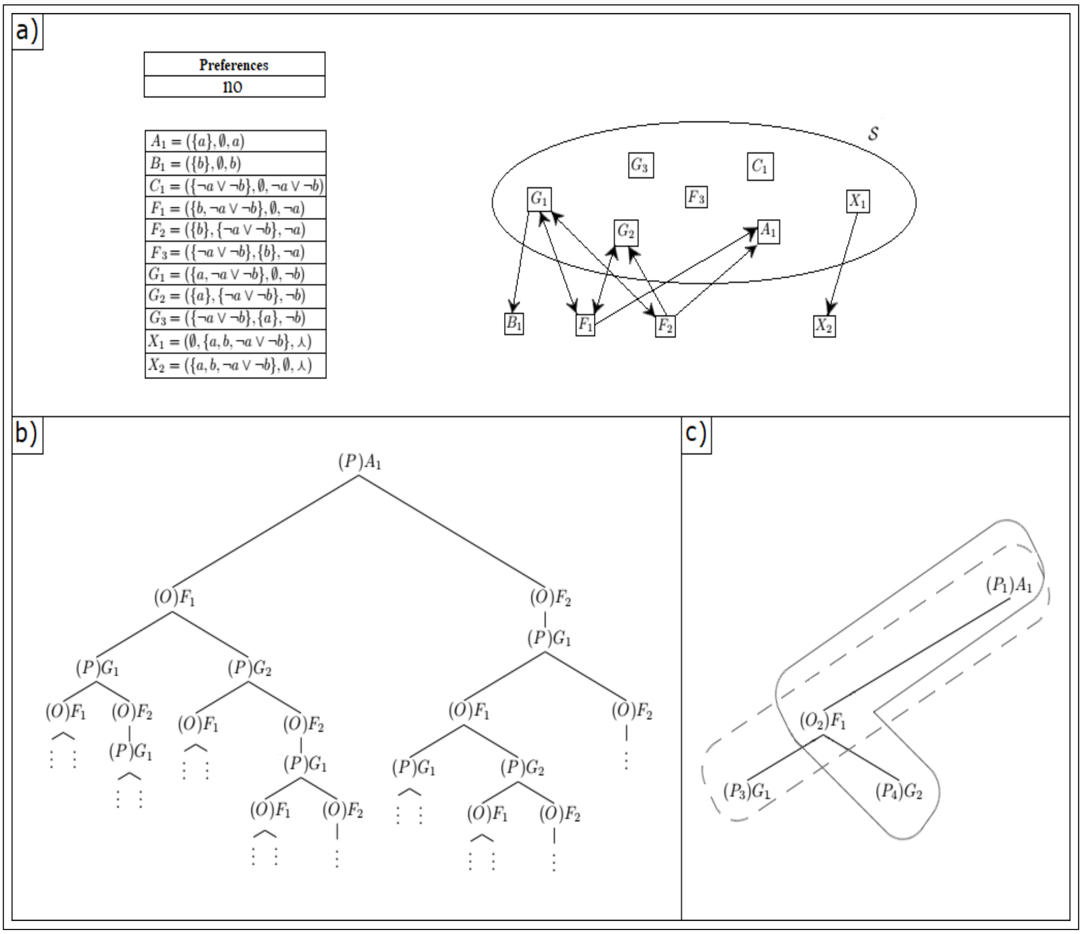

Figure 3: Figure a) illustrates a pdAF with a list of its arguments and the set $\mathcal{S}$ that is parametrized by the dialectical defeats. Consider also that X_2 is defeated by all the arguments of the pdAF, except A_1, B_1, and C_1 (the arrows that should have highlighted such defeats have been omitted to avoid unnecessary graphical confusion). Figure b) displays the dialectical dispute tree $\mathcal{D}$ induced by A_1 wrt $\mathcal{S}$ in the pdAF of Figure a). Notice that $\mathcal{D}$ is composed of infinite disputes (the vertical dots represent the endless continuation of the disputes), as such, it does not have a winning strategy. A dialectical dispute tree Φ-$\mathcal{D}^n$, with $n = 4$, is depicted in Figure c) and corresponds to a Φ-dialectical game played up to the nth-move. Observe that the number of each move (next to the label P or O) represents the order in which the arguments have been played in the game. In this example, we are assuming a protocol Φ that licenses legal moves where PRO can play more than one argument per turn, therefore, Φ-$\mathcal{D}^n$ has two winning strategies (both of which are encircled in the figure).

5.1 Soundness and Completeness

In the following proofs, we are going to employ the framework characteristic function $\mathcal{F}_p$, which iterates over admissible epistemically maximal extensions:

Definition 16. *Let $\langle AR, defeats \rangle$ be a pdAF and AR_p the set of all the em admissible subsets of AR. Then $\mathcal{F}_p : AR_p \mapsto AR_p$, where $\mathcal{F}_p(E) = Cl_{em}(\mathcal{F}(E))$.*

We can now show that the dialectical grounded game satisfies the properties of soundness and completeness.

Theorem 8. *Let Φ_G-$\mathcal{D}^n$ identifies a terminated Φ_G-dialectical game for* A*. Then, there exists a dialectical winning strategy Φ_G-$\mathcal{W}^n$ for* A*, such that the em closure $Cl_{em}(PRO(\Phi_G\text{-}\mathcal{W}^n))$ of the set of arguments moved by PRO in Φ_G-$\mathcal{W}^n$ is conflict-free, iff* A *is included in the grounded extension* Grd *of the pdAF.*

To simplify the notation, let us abbreviate $Cl_{em}(\mathrm{PRO}(\Phi_G\text{-}\mathcal{W}^n))$ in Cl_{em}.

Proof.

Soundness. We have to prove that if A is a member of the conflict-free set Cl_{em}, then $A \in \mathsf{Grd}$. Hence, assuming that A is a member of the conflict-free set Cl_{em}:

– Clearly, all of Φ_G-$\mathcal{W}^n$ leaves, say X_i, are in $\mathcal{F}_p(E_0)$ since they have no defeaters and are then acceptable wrt $\emptyset$. Now, consider that in every branch of Φ_G-$\mathcal{W}^n$, the arguments defended[11] by each X_i are acceptable with respect to $\mathcal{F}_p(E_0)$ and so are in $\mathcal{F}_p(E_1)$. This process can be repeated until, say, $\mathcal{F}_p(E_i)$ when the root A of Φ_G-$\mathcal{W}^n$ is reached. Since $Cl_{em} \subseteq \mathcal{F}_p(E_i)$, and further iterations of $\mathcal{F}_p(E_i)$ will yield the generation of the least fixed point Grd, then A will be a member of Grd.

This suffices to show that if A is a member of the conflict-free set Cl_{em}, then $A \in \mathsf{Grd}$.

Completeness. We have to prove that if $A \in \mathsf{Grd}$, then A is a member of the conflict-free set Cl_{em}. Employing the acceptable arguments in the characteristic function $\mathcal{F}_p$ we are going to show that we can build a Φ_G-winning strategy for A.

[11] Recall that an argument X defends an argument Z iff: when $\exists Y \in AR$ such that Y defeats Z, then X defeats Y.

– Assume that $A \in \mathsf{Grd}$. Since the pdAF is finite, it is also finitary, hence we know that there is a least number i such that $A \in \mathcal{F}_p(E_i)$. Then we will have a dialectical winning strategy Φ_G-$\mathcal{W}^n$ for A if PRO starts the game with A and: for each argument Y dialectically defeating A and moved by OPP, PRO chooses one argument X from $\mathcal{F}_p(E_{i-1})$ such that X$\Rightarrow_{\{Y\}}$Y. This procedure can be iterated for every argument Z dialectically defeating X, and so on, until PRO can choose an argument from $\mathcal{F}_p(E_0)$. $\mathcal{F}_p(E_0)$ has no defeaters and, as such, OPP cannot play any legal move (licensed by the protocol Φ_G) against it. Finally, the grounded game protocol will also ensure the epistemically maximality of the set of arguments moved by PRO in Φ_G-$\mathcal{W}^n$ (15.3).

The result yields a dialectical winning strategy Φ_G-$\mathcal{W}^n$ for A, such that A is a member of the conflict-free set Cl_{em}. We have thus shown that, if $A \in \mathsf{Grd}$, then A is a member of the conflict-free set Cl_{em}.

□

6 Main Features of Dialectical Argument Games

Dialectical argument games hold specific features that differentiate them from the standard argument games of [25, 9, 34] and depend upon their protocols and the properties possessed by each pdAF (especially P1, P2 and P3). Although, for convenience, we are going to outline these features using the dialectical admissible/preferred game (Definition 14), notice that the choice of the protocol is irrelevant.

6.1 Feature 1 (F1)

(F1) The set of all the arguments moved by PRO in a dialectical winning strategy (i.e., PRO(Φ_P-$\mathcal{W}^n$)), is always conflict-free.

Every pdAF = $\langle$AR, *defeats*$\rangle$ prevents any conflicts existing between arguments in a set E $\subseteq$ AR if each argument in E is acceptable with respect to it. Since this has already been formally proven and shown[12], here we will try to explain it through an example. Notice also the rationale underpinning *F1*: due to their limited resources, it would be unrealistic to demand that real-world agents actually perform conflict-free checks on every set E of arguments.

[12]**Lemma 17** of [15] states that: *Let $E \subseteq AR$ such that every argument in E is acceptable w.r.t. E, and AR satisfies $P1, P2$ and $P3$. Then E is conflict-free.*
The proof can be found in the same paper.

Example 3. *Consider a pdAF that includes the arguments listed in Table 1 and such that all the arguments composing the set* $PRO(\Phi_P\text{-}\mathcal{W}^n)$ *are acceptable wrt it. To simplify the notation, let* $E = PRO(\Phi_P\text{-}\mathcal{W}^n)$.
Among the arguments of E*, suppose that there are two conflicting arguments as* $G_2 = (\{a\}, \{\neg a \vee \neg b\}, \neg b)$ *and* $F_1 = (\{b, \neg a \vee \neg b\}, \emptyset, \neg a)$*: we are going to show how this will lead to a contradiction. Due to property P1,* $A_1 = (\{a\}, \emptyset, a) \in AR$*. Hence, by property P3,* $X_2 = (\{a, b, \neg a \vee \neg b\}, \emptyset, \curlywedge) \in AR$ *and by property P2,* $X_1 = (\emptyset, \{a, b, \neg a \vee \neg b\}, \curlywedge) \in AR$*. However, if this is the case,* $X_1 \Rightarrow_E G_2$ *(and, similarly,* $X_1 \Rightarrow_E F_1$*). Since* X_1 *is unassailable,* $\nexists Z \in E$ *such that* $Z \Rightarrow_{\{X_1\}} X_1$ *and this will contradict the assumption that all the arguments members of* E *are acceptable wrt to it. Therefore, since all the arguments that compose the set* $PRO(\Phi_P\text{-}\mathcal{W}^n)$ *are acceptable wrt it,* $PRO(\Phi_P\text{-}\mathcal{W}^n)$ *must be conflict-free.*

6.2 Feature 2 (F2)

(F2) The relevance conditions, i.e., the conditions of the protocol that compel both players to change the outcome of the game at the end of every turn, are essential to the unfolding of the dialectical argument games. This also justifies why the set $\mathcal{S}$ *cannot be initialized with any set other than* $PRO(\Phi_P\text{-}\mathcal{W}^n)$.

The relevance conditions (14.1(b) and 14.2(a) of Definition 14) can be summarised as the conditions that force the two players to change the outcome of the game at the end of every turn[13]. These requirements are fundamental for real-world agents that reason with limited availability of resources. Indeed, it would be illogical to allow such players to move arguments useless for the result of the game: this would simply mean wasting valuable resources[14]. Moreover, the relevance conditions clarify why the set $\mathcal{S}$, referenced in the admissible/preferred protocol, corresponds to the current set of arguments moved by PRO in $\Phi_P\text{-}\mathcal{W}^n$, that is to say, PRO($\Phi_P\text{-}\mathcal{W}^n$). This, in turn, allows avoiding a specific issue that could permanently prevent the victory of PRO, as the following example will show.

Example 4. *The examples of Figures 4, 5 and 6 depict a dialectical admissible game played using the arguments of Table 1, where* $F_1 \not\curlywedge (\{a\}, \emptyset, a)$, $\forall T \in \{G_1, L_1\}$, $T \not\curlywedge (\{b\}, \emptyset, b)$, $\forall V \in \{N_3, X_3\}$, $V \not\curlywedge (\{\neg b\}, \emptyset, \neg b)$*, while* $H_1 \not\curlywedge (\{\neg a \vee \neg b\}, \emptyset, \neg a \vee$

[13]The research presented in [28] introduces a series of relevant properties for dialogue protocols. Property *R1* seems quite similar to our relevance conditions, although our study concerns argument game proof theories rather than dialogues.

[14]Notice that we are dealing with pdAFs, and so, small subsets of the respective overall set of arguments of the considered framework. As such, positing only relevant arguments is not going to be particularly expensive for agents' resources.

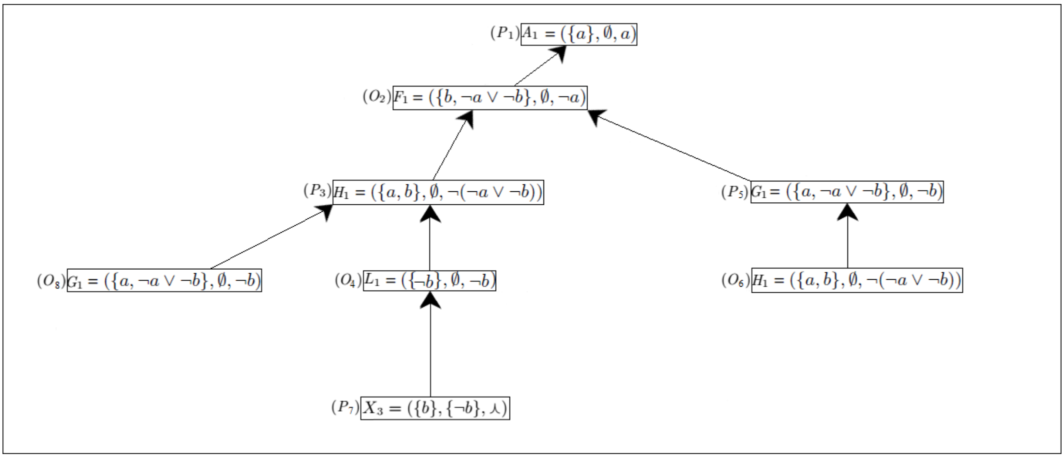

Figure 4: The Figure illustrates a dialectical dispute tree Φ_P-$\mathcal{D}^n$, hence generated following the protocol for the dialectical admissible/preferred games. Notice that the arrows indicate the defeats between the arguments. Starting with the root argument A_1, the other arguments are played according to the order highlighted by the numbers near their labels (P or O). The last player to move is OPP, which moves G_1. Since $G_1 \Rightarrow_{\mathcal{S}} H_1$ (where $\mathcal{S} :=$ PRO(Φ_P-$\mathcal{W}^{n\text{-}1}$), i.e., $\mathcal{S} = \{A_1, H_1, X_3\}$) and $G_1 \not\rightharpoonup (\{b\}, \emptyset, b)$, this ensures OPP invalidates the winning strategy Φ_P-$\mathcal{W}^{n\text{-}1}$. Hence, there is no winning strategy in Φ_P-$\mathcal{D}^n$.

$\neg b$). Starting with the root A_1, the order in which the arguments are played is outlined in the brackets, next to the labels PRO and OPP. The dialectical dispute tree Φ_P-$\mathcal{D}^n$ (Figure 4) has been generated following the protocol for the dialectical admissible/preferred games, however, its extension into Φ-$\mathcal{D}^{n+1}$ (Figure 5) does not take into account PRO's relevance condition (14.1(b) of Definition 14). This immediately raises an issue: without the relevance condition, we could have to face a situation in which PRO is still losing even after its turn has ended (Figure 5). In this circumstance, during the next turn of OPP, there will be no winning strategy, hence no set of arguments moved by PRO in Φ_P-$\mathcal{W}^{n+1}$ (i.e., the set PRO(Φ_P-$\mathcal{W}^{n+1}$)), that can be targeted as $\mathcal{S}$. Suppose, for the sake of the example, that the protocol of the game allows searching for another set $\mathcal{S}$. What could then be the set $\mathcal{S}$ parametrised by the dialectical defeats moved by OPP? Without PRO(Φ_P-$\mathcal{W}^{n+1}$) the only reasonable alternative is to consider a different set $\mathcal{S}$ initialized in a way such that $\mathcal{S} \subseteq$ PRO(Φ_P-$\mathcal{D}^{n+1}$). Nevertheless, notice that if OPP is allowed to suppose the premises of arguments in a non-conflict-free set $\mathcal{S}$, then OPP would have enough resources for playing an unassailable argument (as X_1). As shown in Figure 6, H_1, G_1 $\in \mathcal{S}$, and $B_1 \in AR$ by property P1 of the pdAF. By P3, $X_2 \in AR$, while by property

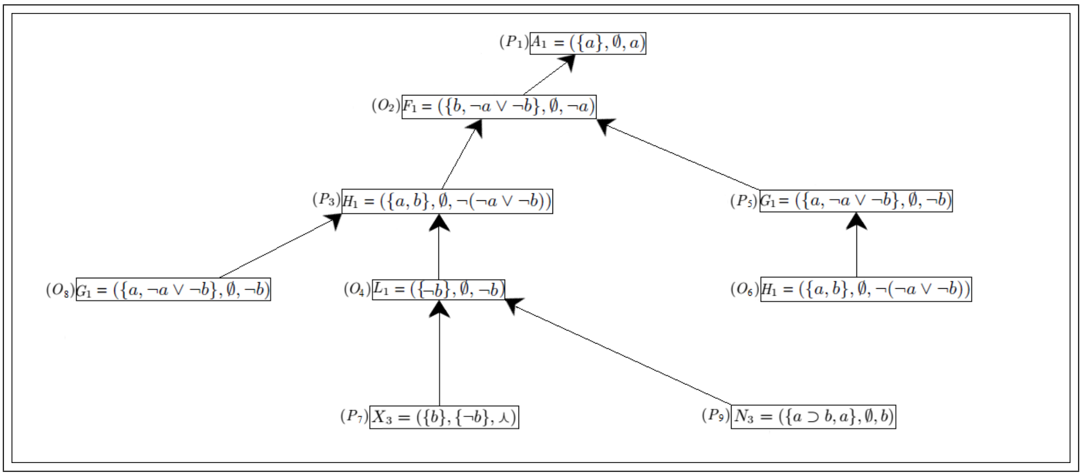

Figure 5: The Figure illustrates the extension of the dialectical dispute tree Φ_P-$\mathcal{D}^n$ into Φ_P-$\mathcal{D}^{n+1}$ due to argument N_3 played by PRO. As we can see, if PRO's relevance condition is dropped, then PRO is free to move any argument and not only the ones that will reinstate the winning strategy. $N_3 \Rightarrow_{\{L_1\}} L_1$ and $N_3 \not\Rightarrow (\{\neg b\}, \emptyset, \neg b)$. However, this implies that, even after PRO moves, there is no winning strategy in Φ_P-$\mathcal{D}^{n+1}$ (because the argument G_1 played by OPP has not yet been defeated).

P2, also $X_1 \in AR$ (since X_1 is the logically equivalent argument of X_2). Argument X_1 constitutes the problem: it defeats A_1 and has empty premises, which implies it cannot be defeated. This means that, by playing X_1, OPP will change the final outcome of the game invalidating any other possible attempt from PRO of reinstating the winning strategy. However, this happened in the example because there was no set PRO(Φ_P-$\mathcal{W}^{n+1}$) and OPP had to suppose the premises of the arguments members of a different set $\mathcal{S} \subseteq$ PRO(Φ_P-$\mathcal{D}^{n+1}$) which was not conflict-free. In other words, unassailable arguments as X_1 can be moved only when (i) arguments that defeat each other or (ii) unconditional arguments with conflicting conclusions are in $\mathcal{S}$. Moving such arguments will immediately trigger property P3 of the pdAF, which will highlight the inconsistency of their premises, while property P2 will ensure the generation of the corresponding unassailable argument.

Nevertheless, without requiring a resource-consuming conflict-free check on every $\mathcal{S} \subseteq$ PRO(Φ_P-$\mathcal{D}^{n+1}$), how would it be possible to ensure the conflict-freeness of the set $\mathcal{S}$? The only set of arguments moved by PRO which satisfies this condition (without requiring a conflict-free check) in a dialectical argument admissible game is the set PRO(Φ_P-$\mathcal{W}^{n+1}$), thanks to property F1. Therefore, $\mathcal{S}$ has to be initialized to PRO(Φ_P-$\mathcal{W}^{n+1}$).

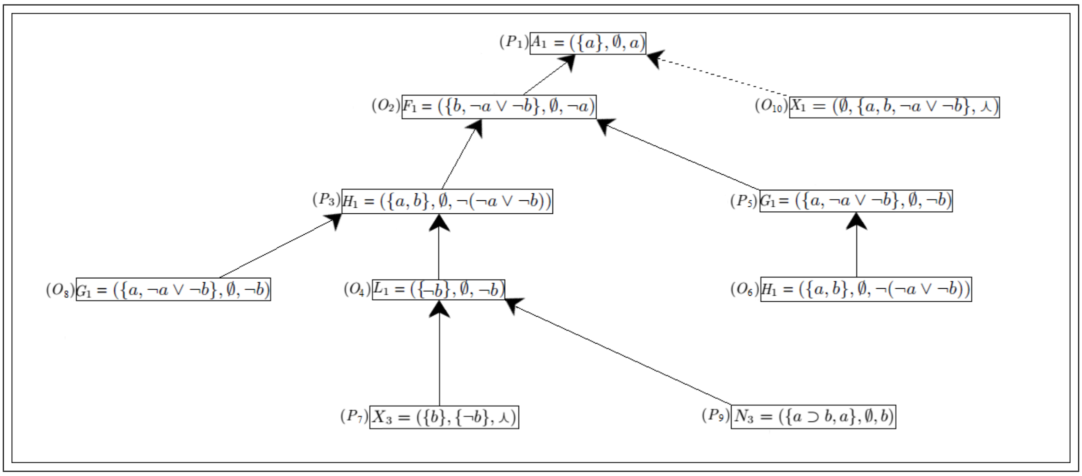

Figure 6: The Figure illustrates the extension of the dialectical dispute tree Φ_P-$\mathcal{D}^{n+1}$ into Φ_P-$\mathcal{D}^{n+2}$ due to argument X_1 played by OPP. It is possible to move X_1 because there is no winning strategy in Φ_P-$\mathcal{W}^{n+1}$, hence there is no set PRO(Φ_P-$\mathcal{W}^{n+1}$): this forces OPP to target the premises of a different set $\mathcal{S}$, initialized in a way such that $\mathcal{S} \subseteq$ PRO(Φ_P-$\mathcal{D}^{n+1}$) (in the case of the example, $\mathcal{S} :=$ PRO(Φ_P-$\mathcal{D}^{n+1}$), i.e., $\mathcal{S} = \{A_1, H_1, G_1, X_3, N_3\}$). The danger of arguments such as X_1 lies in their unassailability and the fact that they always succeed as defeats (underlined by the dashed arrow in the picture and explained in Definition 2). That is to say, the final outcome of the game can then be changed if $\mathcal{S} \neq$ PRO(Φ_P-$\mathcal{W}^{n+1}$) because it can allow OPP to move arguments as X_1 against the root of the tree (preventing PRO from reinstating any other possible winning strategy).

The implication of what has been shown in Example 4 is that the relevance conditions need to be part of the protocols of any dialectical argument game. Indeed, if this is not the case, we could have to face a situation in which PRO is still losing even after its turn has ended. In this circumstance, during the next turn of OPP, there will be no set PRO(Φ_P-$\mathcal{W}^n$) that can be used to initialise $\mathcal{S}$. Hence, once again, the issue outlined in Example 4 could arise and change the final outcome of the game by permanently invalidating PRO's winning strategy. This then means that $\mathcal{S} :=$ PRO(Φ_P-$\mathcal{W}^n$) and cannot be otherwise.

6.3 Feature 3 (F3)

Before the introduction of the third feature (F3) enjoyed by the dialectical admissible/preferred argument games, we need to formally define the *uniqueness* of the dialectical winning strategy regardless of the employed protocol.

Definition 17. [Uniqueness of the dialectical winning strategy] *Let $\mathcal{D}^n$ and* *let $\mathcal{W}^n$ be defined as in Definition 12. Then $\mathcal{W}^n$ is said to enjoy the uniqueness property if there is no other dialectical winning strategy for* A *wrt $\mathcal{S}$ simultaneously present in $\mathcal{D}^n$.*

Let us consider a dialectical dispute tree $\mathcal{D}^n$ identical (although without the implementation of a specific game protocol) to the one in Figure 3(c). This tree has two winning strategies, say $\mathcal{W}_1^n$ and $\mathcal{W}_2^n$, each of which is composed of a single dispute. That is to say: $d_1 = (P_1)A_1$—$(O_2)F_1$—$(P_3)G_1$ and $d_2 = (P_1)A_1$—$(O_2)F_1$—$(P_4)G_2$, such that $\mathcal{W}_1^n$ is composed of d_1, while $\mathcal{W}_2^n$ is composed of d_2. Obviously, $\mathcal{D}^n$ does not enjoy the uniqueness property. Indeed, both G_1 and G_2 defeat the same argument F_1, whereas only one of such defeats is actually needed. This implies that it suffices that either $\mathcal{W}_1^n$ or $\mathcal{W}_2^n$ is present for PRO to win (at least temporarily) the game. For the final outcome of the game, it is pointless to have both winning strategies simultaneously. It is also resource-consuming, meaning that it does not comply well with the Dialectical Cl-Arg purpose of capturing resource-bounded real-world agents' reasoning.

(F3) Any dialectical winning strategy Φ_P-$\mathcal{W}^n$ enjoys the uniqueness property.

Uniqueness is a property enforced on a dialectical winning strategy Φ_P-$\mathcal{W}^n$ by the protocol of the dialectical admissible/preferred argument game. Uniqueness is certainly a desirable property since it allows for shorter and simpler games. This ensures a quicker evaluation of the status of the dialectical dispute tree root.

The following Lemma shows that the protocol of the dialectical admissible/preferred game ensures the uniqueness of Φ_P-$\mathcal{W}^n$.

Lemma 2. *Let Φ_P-$\mathcal{D}^n$ identifies a Φ_P-dialectical game for* A*. Then, there exists only one dialectical winning strategy Φ_P-$\mathcal{W}^n$ for* A *wrt $\mathcal{S}$ that is simultaneously present in Φ_P-$\mathcal{D}^n$.*

Proof. Since the protocol of the admissible/preferred game forces the players to move only one argument per turn, the only other way to have multiple winning strategies simultaneously is by having different arguments moved by PRO (in different turns) that defeat the same argument played by OPP. We are going to show how this case cannot occur under the Φ_P protocol.

Let d_1 be a dispute in Φ_P-$\mathcal{W}^n$ and d' a sub-dispute of d_1. Let also $d_1 = d'$—$(O_{n-i})Y$ —$(P_{n-i+1})X$, for $n - i > 1$. As usual, the index near the player labels denotes the order in which the moves have been played. Suppose now that the last (*nth*) argument moved is an argument $Z \neq X$ from PRO that dialectically defeats Y and generates $d_2 = d'$—$(O_{n-i})Y$—$(P_n)Z$, which is another dispute in Φ_P-$\mathcal{D}^n$ and d' is a sub-dispute of d_2 as well, then it is easy to see that PRO has played against the protocol Φ_P. That is because:

- If PRO defeats an argument without affecting the existing game status it will violate its relevance condition (Definition 14.1(b)).

Playing argument Z will then be prevented by PRO's relevance condition, ensuring in this way the uniqueness of the dialectical winning strategy Φ_P-$\mathcal{W}^n$. □

7 Efficiency Improvements

The protocols thus far developed can benefit from a range of efficiency improvements. They follow from the properties of the dialectical games and Dialectical Cl-Arg in general, which means that they will preserve the already proven soundness and completeness results. In particular, we can obtain shorter games thanks to (I1), which allows us to avoid meaningless repetitions of defeated arguments from OPP. Moreover, (I2) and (I3) show how, due to the features enjoyed by the dialectical games and without additional restrictions on the legal moves available to the players (unlike in [25]), it is possible to obtain other specific efficiency improvements. In the next section, these enhancements will be examined and, when required, also formalised and integrated into the protocols of the dialectical games.

7.1 List of Efficiency Improvements for Dialectical Games

In the admissible/preferred dialectical game, OPP is forbidden to repeat any arguments (and not just in a dispute) which have already been defeated, and not defended or indirectly defended by another argument, in the game.[15]

Let us assume that OPP's argument Y has been defeated, and not defended, by an argument X moved by PRO in a dispute d. If now OPP repeats Y in a different dispute, then PRO can simply repeat X defeating Y once again.

Example 5. *For instance, let Φ_P-$\mathcal{D}^n$ be a dialectical dispute tree and let d be a dispute in Φ_P-$\mathcal{D}^n$. Suppose also that X is an argument moved by PRO in d, while Y is an argument played by OPP in d such that $X \Rightarrow_{\{Y\}} Y$. Then, if the game goes on (up to $n+k$ moves, for $k>1$), whenever Y will 'appear' in a different dispute,*

[15] According to the recursive definition of indirect defence, an argument X *indirectly defends* an argument A if: *i*) X *defends* A; *ii*) X *defends* Z, and Z *indirectly defends* A.

PRO can simply play X again. As such, playing argument Y proves to be just a waste of resources.

We can now formalise this idea by substituting condition ($14.2(b)$) from the protocol Φ_P (Definition 14) with the following constraint ($I1$). The purpose of forbidding such moves is to avoid extending the game by adding useless sequences of arguments to it:

Definition 18 (Improved legal move)**.** *The following additional constraint for OPP* (*where OPP's argument Y is the next move played in the game*) *substitutes* ($14.2(b)$) *from the protocol Φ_P:*

($I1$) *If $\exists J \in OPP(\Phi\text{-}\mathcal{D}^n)$ such that J is defeated and not defended* (*neither directly nor indirectly defended*) *by another argument, then $Y \neq J$.*

The soundness and completeness results of the dialectical games will not be affected by restriction ($I1$), as the following lemma will prove:

Lemma 3. *Let $\Phi_P\text{-}\mathcal{D}^n$ identifies a terminated Φ_P-dialectical game for* A*. Then, there exists a dialectical winning strategy $\Phi_P\text{-}\mathcal{W}_1^n$ for* A*, iff there exists a dialectical winning strategy $\Phi_P\text{-}\mathcal{W}_2^n$ for* A *constructed using a protocol that employs* ($I1$)*.*

Proof.

[$\rightarrow$] If there exists a dialectical winning strategy $\Phi_P\text{-}\mathcal{W}_2^n$, then there also trivially exists a dialectical winning strategy $\Phi_P\text{-}\mathcal{W}_1^n$. Indeed, if OPP cannot repeat its defeated (and not defended) arguments ($I1$), it cannot as well repeat its arguments in the same disputes (($14.2(b)$) of Definition 14). That is to say, $\Phi_P\text{-}\mathcal{W}_2^n$ follows every requirement established by protocol Φ_P.

[$\leftarrow$] We are going to show that every dialectical winning strategy $\Phi_P\text{-}\mathcal{W}_1^n$ can be transformed into a dialectical winning strategy $\Phi_P\text{-}\mathcal{W}_2^n$. Suppose that there is a dispute d in $\Phi_P\text{-}\mathcal{W}_1^n$ in which it appears the sequence J—X of arguments such that J is moved by OPP, X is moved by PRO and $X \Rightarrow_{\{J\}} J$. We also know that J is not defended (or indirectly defended) because, being a dispute in the winning strategy, d terminates with a PRO argument. Notice that, since J is an OPP argument moved in a dispute, it must be preceded by a PRO argument. Hence, if we now remove every other J—X—$\cdots$ sequence (including whatever follows after X) from the dialectical winning strategy, we will not affect PRO's victory and we will generate a new dialectical winning strategy, i.e., $\Phi_P\text{-}\mathcal{W}_2^n$.

□

The following improvements are similar to the ones already introduced in [25], with an important difference. Unlike the standard games, dialectical games do not need to enforce specific restrictions on their protocols in order to benefit from these efficiency enhancements: they are ensured by the properties enjoyed by any dialectical game.

($I2$) *PRO does not move self-defeating arguments (i.e., arguments which defeat themselves).*

Whenever a self-defeating argument, say X, is played by PRO, PRO violates property *F1*. Indeed, even if X reinstates a dialectical winning strategy Φ-$\mathcal{W}^n$, the same X will also conflict with an argument member of PRO(Φ-$\mathcal{W}^n$), i.e., X itself.

($I3$) *PRO does not play an argument that defeats (or is defeated by) an argument in PRO(Φ-$\mathcal{W}^n$).*

That is to say, PRO does not move arguments that conflict with the arguments it has already moved in the winning strategy. Indeed, if PRO plays an argument X defeated by (a member of) PRO(Φ-$\mathcal{W}^n$) or that defeats an argument member of PRO(Φ-$\mathcal{W}^n$), the resulting winning strategy will not be conflict-free. This will then violate property *F1*.

Example 6. *Consider the dialectical dispute tree of Figure 4 and assume that PRO decides to counter its opponent's last move by playing argument* $F_1 = (\{b, \neg a \vee \neg b\}, \emptyset, \neg a)$ *such that* $F_1 \Rightarrow_{\{G_1\}} G_1$ *on* $(\{a\}, \emptyset, a)$*. However, since* F_1 *defeats, hence conflicts, with* $H_1 \in$ *PRO*(Φ-$\mathcal{W}^n$) (F_1 *is also dialectically defeated by* H_1) *this move will violate property* F1 (*the situation will then be similar to the one described in Example 3*).

Remark 2. *Notice that (I3) also subsumes the fact that* PRO does not move an argument X in a dispute d if such an argument has already been played by OPP in d*. Indeed, playing argument X will reinstate the dialectical winning strategy* Φ-$\mathcal{W}^n$*. However, at the same time, X is an argument moved by OPP (hence X complies with OPP's relevance condition). As such, playing X will imply defeating once again an argument in PRO*(Φ-$\mathcal{W}^n$)*, violating property F1*[16].

[16]It is interesting to observe that this is not generally the case if PRO repeats (i) an OPP argument or (ii) an already defeated PRO argument, say X, in a different dispute of the dialectical dispute tree. That is because it might be that the opponent cannot suppose anymore the same premises that (ii) allowed it to defeat X the first time or (i) allowed it to defeat an argument in PRO(Φ-$\mathcal{W}^n$). For example, assume that an argument Y moved by OPP dialectically defeated X

As shown, (I2) and (I3) follow directly from the property *F1*, which is enjoyed by any dialectical game. As such, no modifications to the game protocols are needed, meaning that the soundness and completeness results will be preserved.

8 Related and Future Work

Initially introduced in [14], the dialectical approach of Dialectical Cl-Arg has been subsequently examined from different perspectives. For example, the investigation concerning argumentative characterisations of Brewka's Preferred Subtheories (PS) [3] showed that, compared with the standard approach, the grounded semantics applied to Dialectical Cl-Arg more closely approximates sceptical inference in PS [16]. In addition, the research presented in [26] provides a full rational account of structured (ASPIC$^+$) arguments under resource bounds by adapting the approach of Dialectical Cl-Arg.

Extending further the study commenced in [10] and continued in [11], we plan to increase the range of dialectical argument game protocols investigating the stable [9], semi-stable [4] and ideal semantics [19, 5]. Similarly to the work presented in [25], we could also consider adapting the standard 3-values labelling approach (where each label represents the *IN, OUT*, and *UNDEC* status of an argument with respect to the examined semantics) and devise algorithmic procedures for the enumeration of specific extensions. Starting from the preliminary study proposed in [12], the design of fully-fledged algorithms would also help in additionally assessing the soundness and completeness properties of the dialectical argument games. Finally, another research direction that will be pursued involves generalising the developed dialectical argument games to dialogues, following the guidelines of the already existing literature in the field (mainly [22, 30, 13]). This would have the interesting consequence of allowing to move from non-monotonic single-agent inference to distributed non-monotonic reasoning.

9 Conclusion

The main aspects of the real-world uses of argumentation by resource-bounded agents include: (i) showing the inconsistencies of an opponent's argument by supposing the premises of its arguments; (ii) handling only finite subsets of the arguments of the AFs; (iii) reducing the consumption of resources by employing dialectical

drawing its suppositions α from Prem(PRO(Φ-$\mathcal{W}^n$)). However, after the game goes on, it might be that $\alpha \notin$ Prem(PRO(Φ-$\mathcal{W}^{n+k}$)). Then Y cannot dialectically defeat X anymore (i.e., Y defeat against X is disqualified), therefore X is now a perfectly viable move for PRO.

means (while still satisfying the rationality postulates and practical desiderata) [15]. These features would constitute the hallmarks of an argument game based on Dialectical Cl-Arg, thus capable of better approximating non-monotonic single-agent real-world reasoning processes than the standard argument games. In this paper, we have achieved some important results. We have developed argument game proof theories (denoted as *dialectical argument games*) for the admissible, preferred and grounded semantics of Dialectical Cl-Arg. Incorporating dialectical defeats in the standard structure of the argument games proved to be a non-trivial process which yielded the discovery of interesting properties that differentiate dialectical games from the standard ones. That is to say, dialectical games enjoy (a) specific relevance conditions that characterise their protocols and yield (b) the uniqueness of their winning strategies, whilst property *F1* ensures (c) the conflict-freeness of the set of arguments moved by the proponent in the winning strategy. The last is of particular importance since it provides the games with a various range of efficiency improvements. Without the need to perform any additional checks or to enforce additional restrictions in the protocols (unlike in [25]), *F1* allows each dialectical game to prevent the proponent from: playing self-defeating arguments; playing arguments already moved by the opponent (in the same dispute); and playing arguments that defeat (or are defeated by) other arguments already moved by the proponent. Finally, another efficiency improvement can be obtained if the opponent is forbidden to repeat arguments that have already been defeated in the dialectical admissible/preferred game, such that none of them has also been defended or indirectly defended by other arguments.

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