Zonotopic Distributed Fusion for Nonlinear Networked Systems with Bit Rate Constraint

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Abstract

In this paper, the distributed fusion estimation problem is studied for a class of nonlinear networked systems subject to unknown-but-bounded (UBB) noises. A bit rate constraint is introduced to quantify the limited bandwidth of the communication channel, under which a bit rate allocation protocol is further designed by solving a certain off-line optimization problem. Based on the received data from the network, several local extended-Kalman-type estimators are constructed and zonotopic sets confining local estimation errors are then obtained. By designing the local estimator parameters, the F-radii of the obtained zonotopic sets are minimized. Subsequently, with the calculated local estimates and zonotopic sets, a zonotopes-based distributed fusion estimator is put forward by means of the matrix-weighted fusion method, and the global zonotope (i.e., the zonotope encompassing the error between the system state and the fused estimate) is derived. Moreover, under the proposed zonotopes-based fusion framework, the distributed fusion estimators are designed based on, respectively, the scalar-weighted fusion method and the diagonal-matrix-weighted fusion method. Finally, the effectiveness of the proposed distributed fusion method is illustrated through a numerical example.

Keywords: Nonlinear networked systems, distributed fusion, bit rate constraint, set-membership state estimation, zonotopes.

Abbreviations and Notations

MSIF	Multi-sensor information fusion
UBB	Unknown-but-bounded
SMSE	Set-membership state estimation

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$\mathbb{N}$	The set of nonnegative integers
$\mathbb{N}^*$	The set of positive integers
$\mathbb{R}^n$	The n-dimensional Euclidean space
$\mathbb{R}^{n\times m}$	The set of all $n \times m$ real matrices
0	Zero matrix of compatible dimension
I	Identity matrix of compatible dimension
$\text{diag}\{\cdot\}$	The block-diagonal matrix
$\text{diag}_v\{\chi\}$	The diagonal matrix $\text{diag}\{\chi_1, \cdots, \chi_n\}$ with $\chi_\imath$ being the $\imath$-th component of the vector $\chi = \begin{bmatrix}\chi_1 & \cdots & \chi_n\end{bmatrix}^T$
$X_1 > X_2$	$X_1 - X_2$ is positive definite
X^T	The transpose of the matrix X
$\|\cdot\|_F$	The Frobenius norm of the matrix "·"
$\|\cdot\|_\infty$	The infinity norm of the matrix "·"
X^{-1}	The inverse of the matrix X
$\text{tr}\{\cdot\}$	The trace of the matrix "·"
$\lvert\cdot\rvert$	The element-to-element absolute value operator for the matrix "·"
$\mathbf{1}$	The column vector $\begin{bmatrix}1 & 1 & \cdots & 1\end{bmatrix}^T$ of compatible dimension
$\|\cdot\|_\infty$	The infinity norm of the vector "·"
$\lfloor\cdot\rfloor$	The maximum integer less than or equal to the real number "·"
$D_1 \oplus D_2$	The Minkowski sum of sets D_1 and D_2, i.e., $\{d_1 + d_2 : \ d_1 \in D_2, \ d_2 \in D_2\}$
$M \odot D$	The set $\{Md : \ d \in D\}$ for $M \in \mathbb{R}^{m\times n}$ and $D \subset \mathbb{R}^n$

1. INTRODUCTION

Multi-sensor information fusion (MSIF), whose aim is to combine local measurements (or estimates) to generate an estimate with higher accuracy than that of using a single sensor, has drawn tremendous attention owing to its wide applications in a variety of areas such as target tracking, integrated navigation, environment observation, communications and robotics [18, 35, 44, 39, 15]. As a result, a rich body of MSIF-related literature has been published, see e.g., [30, 54, 3, 7, 14, 31, 49, 2, 13] and the references therein.

Generally speaking, MSIF can be categorized into the centralized fusion and distributed fusion according to whether the raw measurement information is utilized directly for fusion or not [39]. It is well known that the centralized fusion can achieve the optimal estimation in certain cases by directly using all original measurements [52, 31]. The distributed fusion, on the other hand, is capable of separately processing the sensor measurements, thereby reducing the computational burden and improving the reliability [51, 38].

For decades, the distributed fusion has attracted much research enthusiasm leading to the development of a great number of efficient algorithms, see [38, 37, 9] and the references therein. For example, in [38], a distributed optimal linear fusion estimator with feedback has been proposed, which has the same accuracy as the centralized fusion estimator. In [9], by using the dimensionality reduction strategy, a distributed fusion estimator has been designed for stochastic uncertain systems with fading measurements under Denial-of-Service attacks and deception attacks. In [37], the distributed fusion is investigated for linear time-varying systems over sensor networks with probabilistic constraints and stochastic perturbations.

Recently, the distributed fusion under unknown-but-bounded (UBB) noises has received some initial research attention and, based on the ellipsoidal set-membership state estimation (SMSE) technique, some remarkable works have been reported in [46, 56]. Notice that the ellipsoidal SMSE method, though effective in handling UBB noises, suffers inevitably from certain loss of accuracy in calculating the Minkowski sum and the linear image (two widely-used operations in SMSE). Hence, there remains much room for further improving the distributed fusion schemes in the presence of UBB noises.

In the past few years, zonotopic SMSE problems have received considerable research interest with applications to various systems, e.g., [28, 27, 26] for linear systems, [1, 53] for nonlinear systems, [43, 42, 41] for singular systems, [6] for uncertain systems and [10] for switched systems. Compared with its ellipsoidal counterpart, the zonotopic SMSE algorithm enjoys the merits of maintaining the estimation accuracy (when performing the Minkowski sum and the linear image operations [6, 43]) and mitigating the computational burden (by using the order reduction technique of zonotopes [22]). Hence, a seemingly natural idea is to apply the zonotopic SMSE technique to the distributed fusion problem under UBB noises.

Networked control systems (NCSs) have become a research forefront for two decades with a great many representative results reported in the literature, see, e.g., [55, 5, 34, 21, 16, 4, 33, 20]. In particular, the nonlinear NCSs have attracted some recent research interest due mainly to the ubiquitous existence of nonlinear dynamics in almost all practical systems [8, 12, 11]. For instance, in [52], the sequential fusion estimation problem has been investigated for nonlinear networked systems by using the unscented Kalman filtering method. The distributed non-fragile SMSE problem has been studied in [33] for nonlinear networked systems under fading channels and bias injection attacks by means of the recursive linear matrix inequality technique.

In a typical NCS, the measurement output needs to be quantized and coded before being transmitted through the communication network [47, 32, 12]. The quality of digital transmissions is inevitably affected by the limited bandwidth of the communication network as characterized by *the bit rate*. In this case, each sensor node in an NCS is allocated a fraction of the total (but limited) bit rate, and this is referred to as the bit rate constraint [25, 36, 24, 40]. Note that the bit rate constraint in digital transmissions, to the best of the authors' knowledge, has never been investigated in the context of NCS-based distributed fusion, not to mention the simultaneous consideration of the nonlinear dynamics, and such a gap motivates the current research.

Based upon the above discussions, the main purpose of this paper is to deal with the distributed fusion estimation problem for nonlinear networked systems subject to the effects of UBB noises and bit rate constraints. Such a research problem appears to be substantially difficult owing mainly to the following three aspects:

1. how to allocate the limited bit rate to each sensor node in a reasonable way?
2. how to calculate zonotopes enclosing each local estimation error for nonlinear networked systems subject to UBB noises and bit rate constraints?
3. how to fuse the local estimates under the zonotopes-based fusion framework?

It is, therefore, our main objective in this paper to provide satisfactory responses to the above three questions. Correspondingly, the contributions of this paper are summarized as follows.

1. The distributed fusion estimation problem is investigated, for the first time, for nonlinear networked systems with UBB noises and bit rate constraints.
2. An allocation protocol is designed to address the bit rate constraints such that the F-radius of the zonotope encompassing the decoding error is minimized, where the minimum problem can be solved off-line in advance.
3. By using the mean value theorem and the first-order Taylor-series expansion, respectively, two zonotopes are derived that restrain the estimation error of the nonlinear function. By taking the intersection of these two zonotopes, a tight zonotope is obtained that helps enhance the estimation accuracy.
4. The gain matrix of each local estimator is designed such that the F-radius of the zonotopic set restraining the local estimation error is minimized.
5. Under a novel zonotopes-based fusion framework, three distributed fusion estimators are designed that are based, respectively, on the matrix-weighted method, the scalar-weighted fusion method and the diagonal-matrix-weighted fusion method.

The rest of this paper is organized as follows. In Section 2, the distributed fusion estimation problem is formulated for nonlinear systems subject to the UBB noises and bit rate constraints. In Section 3, an allocation protocol subject to the bit rate constraint is designed. Then, the zonotopes containing the local estimation errors are derived and the local estimator parameters are designed by minimizing the F-radii of the derived zonotopes. Moreover, three distributed fusion estimators weighted by matrices, scalars and diagonal matrices are designed. An illustrative example is given in Section 4, and this paper is concluded in Section 5.

2. PRELIMINARIES AND PROBLEM FORMULATION

2.1. Zonotopes

In this paper, we use zonotopes to characterize the bounds on system states and UBB noises. The definition of zonotopes is given as follows.

Definition 1. *[23] Let a center vector $d \in \mathbb{R}^{\ell}$ and a generator matrix $D \in \mathbb{R}^{\ell \times \jmath}$ be given. A $\jmath$-order zonotope $\langle d, D \rangle \subset \mathbb{R}^{\ell}$ is defined by*

$$\langle d, D \rangle \triangleq \{d + Ds : \|s\|_{\infty} \leq 1\}.$$

2.2. System Model

Consider a nonlinear system described by the following model:

$$x(s+1) = g(x(s)) + B(s)w(s) \tag{1}$$

where $x(s) \in \mathbb{R}^p$ is the system state; the nonlinear function $g(\cdot) : \mathbb{R}^p \mapsto \mathbb{R}^p$ is known and twice continuously differentiable; $w(s) \in \mathbb{R}^r$ is the UBB process noise; and $B(s)$ is a known matrix of appropriate dimension.

The system (1) is measured by N sensor nodes, and the measurement output of the j-th sensor node is described as:

$$y_j(s) = C_j(s)x(s) + v_j(s) \tag{2}$$

where, for $j \in \{1, 2, \cdots, N\}$, $y_j(s) \in \mathbb{R}^{q_j}$ represents the measurement output; $C_j(s)$ is the measurement matrix of compatible dimension; and $v_j(s)$ stands for the UBB measurement noise.

Assumption 1. *The initial state $x(0)$ belongs to the zonotope $\langle \bar{x}(0), \bar{X}(0) \rangle$ where $\bar{x}(0) \in \mathbb{R}^p$ is a known vector and $\bar{X}(0) \in \mathbb{R}^{p \times p}$ is a known diagonal positive-definite matrix.*

Assumption 2. *The process noise $w(s)$ and the measurement noise $v_j(s)$ are, respectively, confined to zonotopes $\langle 0, W(s) \rangle$ and $\langle 0, V_j(s) \rangle$ where $W(s) \in \mathbb{R}^{r \times r}$ and $V_j(s) \in \mathbb{R}^{q_j \times q_j}$ are known matrices.*

2.3. Coding-Decoding under Bit Rate Constraint

In this paper, due to the needs of digital transmission, an encoding-decoding approach is adopted. As is shown in Fig. 1, the measurements $y_j(s)$ $(j = 1, 2, \cdots, N)$ are first coded and then sent to the corresponding decoders through a communication network whose bit rate is limited as a result of the resource constraints on practical signal transmissions. In the following, let us introduce the coding-decoding procedure in detail.

a) Coding Procedure

For the j-th sensor node, let the codeword generated by coding the measurement output $y_j(s)$ be

$$\tilde{y}_j(s) = \begin{bmatrix} \tilde{y}_j^{(1)}(s) & \tilde{y}_j^{(2)}(s) & \cdots & \tilde{y}_j^{(q_j)}(s) \end{bmatrix}^T.$$

For $l = 1, 2, \cdots, q_j$, the component $\tilde{y}_j^{(l)}(s)$ of the codeword $\tilde{y}_j(s)$ is given by

$$\tilde{y}_j^{(l)}(s) = \mathscr{C}_j\left(y_j^{(l)}(s)\right).$$

Here, $y_j^{(l)}(s)$ is the l-th component of $y_j(s)$, and $\mathscr{C}_j(\cdot)$ is the coding mechanism defined as follows:

$$\begin{aligned} & \mathscr{C}_j\left(y_j^{(l)}(s)\right) \\ \triangleq & \begin{cases} 1, & -b_j \leq y_j^{(l)}(s) < -b_j + \frac{2b_j}{\lfloor 2^{\vec{R}_j/q_j} \rfloor} \\ 2, & -b_j + \frac{2b_j}{\lfloor 2^{\vec{R}_j/q_j} \rfloor} \leq y_j^{(l)}(s) < -b_j + \frac{4b_j}{\lfloor 2^{\vec{R}_j/q_j} \rfloor} \\ & \vdots \\ \lfloor 2^{\vec{R}_j/q_j} \rfloor, & b_j - \frac{2b_j}{\lfloor 2^{\vec{R}_j/q_j} \rfloor} \leq y_j^{(l)}(s) \leq b_j \end{cases} \end{aligned} \tag{3}$$

where b_j is a given positive scalar and $\vec{R}_j$ is the bit rate allocated to the j-th sensor node which is to be determined later.

The codeword $\tilde{y}_j(s)$ is sent to the side of the decoder through a communication network to generate decoded signal used to estimate the system state. To reflect the practical situation that the network bandwidth is limited, the total bit rate of the network is assumed to be $\vec{R} \in \mathbb{N}^*$. As such, one has the following bit rate constraint

$$\sum_{j=1}^{N} \vec{R}_j \leq \vec{R}. \tag{4}$$

b) Decoding Procedure

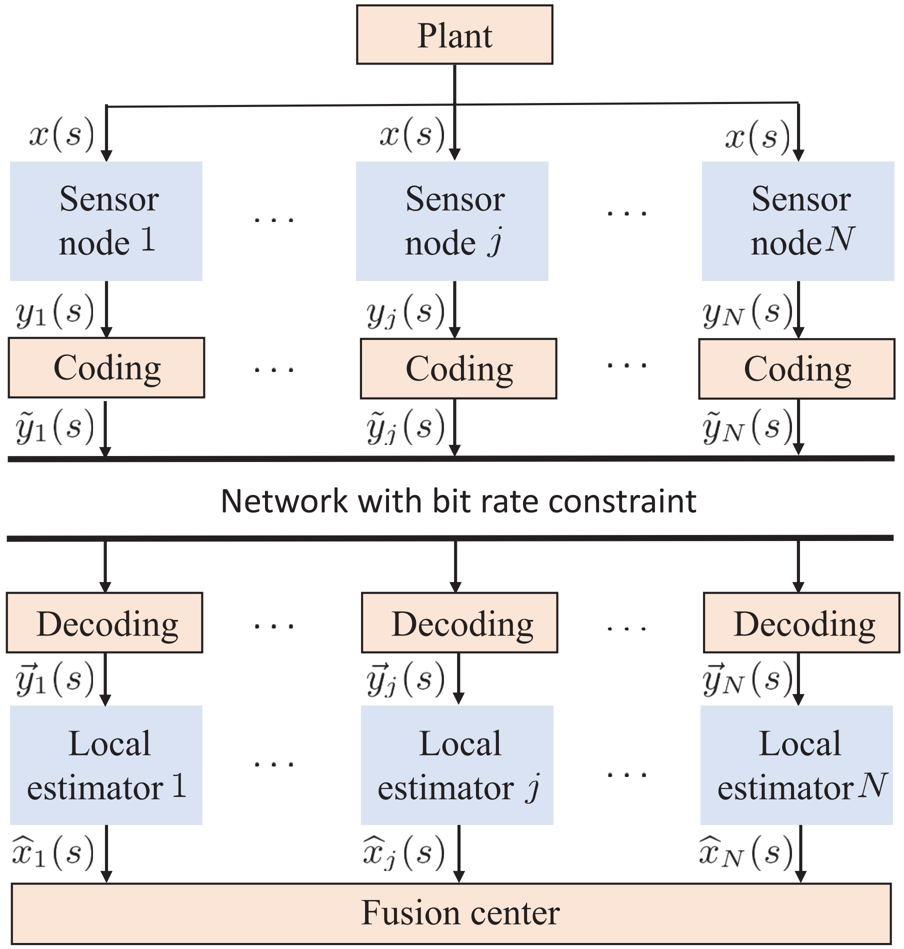

Figure 1: Block diagram of the distributed fusion with bit rate constraint.

At the side of the decoder, the codeword $\tilde{y}_j(s)$ is utilized to produce the following decoded signal

$$\vec{y}_j(s) = \left[\vec{y}_j^{(1)}(s) \quad \vec{y}_j^{(2)}(s) \quad \cdots \quad \vec{y}_j^{(q_j)}(s)\right]^T.$$

Here, the l-th component of $\vec{y}_j(s)$ is obtained by

$$\vec{y}_j^{(l)}(s) = \mathscr{D}_j\left(\tilde{y}_j^{(l)}(s)\right)$$

with $\mathscr{D}_j(\cdot)$ being the decoding mechanism given as follows:

$$\mathscr{D}_j\left(\tilde{y}_j^{(l)}(s)\right) \triangleq -b_j + \frac{2b_j\tilde{y}_j^{(l)}(s) - b_j}{\lfloor 2^{\vec{R}_j/q_j} \rfloor}. \tag{5}$$

Let $\varsigma_j(s) \triangleq \vec{y}_j(s) - y_j(s)$ be the decoding error of the j-th decoder. It is easy to see from (3) and (5) that $\varsigma_j(k)$ satisfies

$$\left|\varsigma_j^{(l)}(s)\right| \leq \frac{b_j}{\lfloor 2^{\vec{R}_j/q_j} \rfloor} \tag{6}$$

where $\varsigma_j^{(l)}(s)$ is the l-th component of $\varsigma_j(s)$.

Remark 1. *The coding-decoding procedures introduced in (3)-(6) are based on the uniform quantization scheme, where the saturation value of the quantization is affected by the bit rate allocated to the j-th sensor node (i.e., $\vec{R}_j$). For the reason that the uniform quantization mechanism is often much easier to be operated than other quantization schemes, the uniform-quantization-based coding-decoding method has found some applications in the networked systems subject to the constrained bit rates such as [24, 25]. In this paper, the coding-decoding procedures given in (3)-(6) are applied to the distributed fusion estimation of nonlinear systems.*

2.4. Local Estimators

Based on the output of the decoder $\vec{y}_j(s)$, the following extended-Kalman-type local estimator is constructed:

$$\check{x}_j(s+1) = g\left(\widehat{x}_j(s)\right), \tag{7}$$

$$\begin{aligned} \widehat{x}_j(s+1) = \check{x}_j(s+1) + F_j(s+1)\Big(&\vec{y}_j(s+1) \\ &- C_j(s+1)\check{x}_j(s+1)\Big) \end{aligned} \tag{8}$$

where $\widehat{x}_j(s+1)$ denotes the estimate of $x(s+1)$ with $\widehat{x}_j(0) = \bar{x}(0)$, $\check{x}_j(s+1)$ represents the one-step prediction, and $F_j(s+1)$ is the estimator gain to be designed.

For the j-th local estimator, let the one-step prediction error and the estimation error be $\check{e}_j(s) \triangleq x(s) - \check{x}_j(s)$ and $\widehat{e}_j(s) \triangleq x(s) - \widehat{x}_j(s)$, respectively. Then, it follows from (1), (2), (7) and (8) that

$$\check{e}_j(s+1) = \tilde{g}(\widehat{e}_j(s)) + B(s)w(s), \tag{9}$$

$$\begin{aligned} \widehat{e}_j(s+1) = &\ \Xi_j(s+1)\check{e}_j(s+1) - F_j(s+1)\varsigma_j(s+1) \\ &- F_j(s+1)v_j(s+1) \end{aligned} \tag{10}$$

where

$$\tilde{g}(\widehat{e}_j(s)) \triangleq g(x(s)) - g(\widehat{x}_j(s)),$$
$$\Xi_j(s+1) \triangleq I - F_j(s+1)C_j(s+1).$$

2.5. Problem Statement

Definition 2. *[41] The F-radius of a given zonotope $\langle d, D\rangle$ is defined as*

$$\|D\|_F \triangleq \sqrt{\mathrm{tr}\{DD^T\}}. \tag{11}$$

The aim of this paper is fourfold as detailed below.

1) Design a bit rate allocation protocol subject to the constraint (4) such that a zonotope containing the decoding error $\varsigma(s) \triangleq \begin{bmatrix} \varsigma_1^T(s) & \varsigma_2^T(s) & \cdots & \varsigma_N^T(s) \end{bmatrix}^T$ is guaranteed and the F-radius of such a zonotope is minimized.
2) Look for a sequence of matrices $\widehat{Z}_j(s)$ ($s \in \mathbb{N}$) such that the local estimation error $\widehat{e}_j(s)$ satisfies $\widehat{e}_j(s) \in \left\langle 0, \widehat{Z}_j(s) \right\rangle$ for all $s \in \mathbb{N}$.
3) Design the estimator parameter $F_j(s+1)$ such that the F-radius of $\left\langle 0, \widehat{Z}_j(s+1) \right\rangle$ is minimized.
4) Fuse the local estimates $\widehat{x}_j(s)$ ($j = 1, 2, \cdots, N$) under a zonotopes-based fusion criterion.

Remark 2. *In the zonotopic SMSE, the F-radii of zonotopes are widely used to measure their sizes. This is because the F-radius of a zonotope can reflect the lengths of the segments generating the zonotope. For more details, we refer the readers to [22]. Note that the F-radius of a zonotope is much easier to calculate in practice than other measures of the zonotope such as the volume and the radius. Hence, in this paper, F-radii of zonotopes are employed to evaluate the estimation performance.*

3. MAIN RESULTS

In this section, a bit rate allocation protocol is designed first under the constraint (4). Then, a zonotope containing the local estimation error $\widehat{e}_j(s+1)$ is derived at each time instant $s \in \mathbb{N}$ for each $j \in \{1, 2, \cdots, N\}$, and the F-radius of such a zonotope is subsequently minimized by properly designing the estimator parameter $F_j(s+1)$. Finally, based on the obtained results on local estimators, three distributed fusion estimators are designed under a zonotopes-based fusion criterion by using the matrix-weighted method, the scalar-weighted method and the diagonal-matrix-weighted method, respectively.

3.1. Preliminary Lemmas

To begin with, some useful lemmas are given as follows.

Lemma 1. *[22] Let zonotopes $\langle d_1, D_1\rangle, \langle d_2, D_2\rangle \subset \mathbb{R}^n$ and a matrix $M \in \mathbb{R}^{m\times n}$ be given. Then, the following relationships hold:*

$$\begin{aligned} \langle d_1, D_1\rangle \oplus \langle d_2, D_2\rangle &= \left\langle d_1 + d_2, \begin{bmatrix} D_1 & D_2 \end{bmatrix} \right\rangle, & (12)\\ M \odot \langle d_1, D_1\rangle &= \langle Md_1, MD_1\rangle, & (13)\\ \langle d_1, D_1\rangle &\subset \langle d_1, \mathrm{diag}_v\{|D_1|\mathbf{1}\}\rangle . & (14) \end{aligned}$$

Lemma 2. *Consider the term $\tilde{g}(\widehat{e}_j(s))$ given in (9). Suppose that the system state $x(s)$ satisfies*

$$x(s) \in \left\langle \widehat{x}_j(s), \widehat{Z}_j(s) \right\rangle \tag{15}$$

with $\widehat{x}_j(s)$ being a given local estimate and $\widehat{Z}_j(s)$ being a known matrix. Then, $\tilde{g}(\widehat{e}_j(s))$ satisfies

$$\tilde{g}(\widehat{e}_j(s)) \in \left\langle 0, \widehat{\mathcal{G}}_j(s) \right\rangle \tag{16}$$

where

$$\widehat{\mathcal{G}}_j(s) \triangleq \text{diag}\left\{\widehat{\mathcal{G}}_j^{(1)}(s), \widehat{\mathcal{G}}_j^{(2)}(s), \cdots, \widehat{\mathcal{G}}_j^{(p)}(s)\right\},$$

$$\widehat{\mathcal{G}}_j^{(i)}(s) \triangleq \min\Bigg\{ \left\|\widehat{Z}_j(s)\right\|_\infty \underline{m}_j^{(i)}(s), \left|\frac{\partial g^{(i)}(x(s))}{\partial x(s)}\Big|_{x(s)=\widehat{x}_j(s)} \widehat{Z}_j(s)\right| \mathbf{1} + \frac{p}{2}\left\|\widehat{Z}_j(s)\right\|_\infty^2 \bar{m}_j^{(i)}(s)\Bigg\},\ i = 1, 2, \cdots, p,$$

$$g^{(i)}(x(s)) \triangleq \Big[\underbrace{0 \ \cdots \ 0}_{i-1} \ \ 1 \ \ \underbrace{0 \ \cdots \ 0}_{p-i}\Big] g(x(s)),$$

$$\underline{m}_j^{(i)}(s) \triangleq \max_{\chi \in \left\langle \widehat{x}_j(s), \widehat{Z}_j(s)\right\rangle} \left\| \frac{\partial g^{(i)}(x(s))}{\partial x(s)}\Big|_{x(s)=\chi} \right\|_\infty,$$

$$\bar{m}_j^{(i)}(s) \triangleq \max_{\chi \in \left\langle \widehat{x}_j(s), \widehat{Z}_j(s)\right\rangle} \left\| \frac{\partial^2 g^{(i)}(x(s))}{\partial x^2(s)}\Big|_{x(s)=\chi} \right\|_\infty.$$

Proof. See Appendix Appendix A.

Remark 3. *In Lemma 2, we reveal that, if the system state $x(s)$ or the local estimation error $\widehat{e}_j(s)$ resides in a zonotope, the estimation error $\tilde{g}(\widehat{e}_j(s))$ of $g(x(s))$ also belongs to a zonotope. Furthermore, we give the explicit expression of the generator matrix of the zonotope containing $\tilde{g}(\widehat{e}_j(s))$. In calculating such a zonotope, we need to determine the value of $\underline{m}_j^{(i)}(s)$ and $\bar{m}_j^{(i)}(s)$ by solving two optimization problems. There are many methods that can be used to solve these two optimization problems such as the particle swarm optimization [48] and the genetic algorithm [45]. Note that, in existing references concerning the SMSE, there are mainly two methods dealing with the twice continuously differentiable nonlinear functions, namely, the mean value theorem and the first-order Taylor expansion. In Lemma 2, by using these two methods, respectively, two zonotopes restraining $\tilde{g}(\widehat{e}_j(s))$ are obtained at each time instant. By taking the intersection of the obtained two zonotopes, a tighter zonotope is obtained in (16), which obviously improve the estimation accuracy.*

Lemma 3. *Given the bit rate $\vec{R}_j$, the decoding error $\varsigma_j(s)$ given in (10) satisfies*

$$\varsigma_j(s) \in \left\langle 0, \frac{b_j}{\lfloor 2^{\vec{R}_j/q_j} \rfloor} I \right\rangle. \tag{17}$$

Proof. See Appendix Appendix B.

3.2. Design of Bit Rate Allocation Protocol

In the following, a zonotope enclosing the decoding error $\varsigma(s)$ will be derived and a bit rate allocation protocol will be designed by minimizing the F-radius of such a zonotope.

From Definition 1 and Lemma 3, it is obvious that

$$\varsigma(s) \in \left\langle 0, \text{diag}\left\{ \frac{b_1}{\lfloor 2^{\vec{R}_1/q_1} \rfloor} I_{q_1}, \cdots, \frac{b_N}{\lfloor 2^{\vec{R}_N/q_N} \rfloor} I_{q_N} \right\} \right\rangle. \tag{18}$$

Then, one can easily obtain that the F-radius of the zonotope given in (18) satisfies

$$\begin{aligned}
&\left\|\text{diag}\left\{\frac{b_1}{\lfloor 2^{\vec{R}_1/q_1}\rfloor}I_{q_1},\cdots,\frac{b_N}{\lfloor 2^{\vec{R}_N/q_N}\rfloor}I_{q_N}\right\}\right\|_F^2\\
&=\text{tr}\left\{\text{diag}\left\{\frac{b_1^2}{\left(\lfloor 2^{\vec{R}_1/q_1}\rfloor\right)^2}I_{q_1},\cdots,\frac{b_N^2}{\left(\lfloor 2^{\vec{R}_N/q_N}\rfloor\right)^2}I_{q_N}\right\}\right\}\\
&=\sum_{j=1}^{N}\frac{q_j b_j^2}{\left(\lfloor 2^{\vec{R}_j/q_j}\rfloor\right)^2}. \qquad (19)
\end{aligned}$$

Hence, the bit rate allocation protocol is designed by solving the following optimization problem

$$\begin{aligned}
&\min_{\vec{R}_1,\vec{R}_2,\cdots,\vec{R}_N}\sum_{j=1}^{N}\frac{q_j b_j^2}{\left(\lfloor 2^{\vec{R}_j/q_j}\rfloor\right)^2}\\
&\text{s.t. } (4),\ \vec{R}_1,\vec{R}_2,\cdots,\vec{R}_N\in\mathbb{N}^*. \qquad (20)
\end{aligned}$$

In the rest of this paper, $(\vec{R}_1,\vec{R}_2,\cdots,\vec{R}_N)$ is set to be a solution to the optimization problem (20).

3.3. Zonotopes Restraining Local Estimation Errors and Design of Local Estimators

In this subsection, the sequence of matrices $\widehat{Z}_j(s)$ ensuring that $\widehat{e}_j(s)\in\left\langle 0,\widehat{Z}_j(s)\right\rangle$ $(s\in\mathbb{N})$ will be given. Moreover, the F-radius of $\left\langle 0,\widehat{Z}_j(s+1)\right\rangle$ will be minimized by appropriately designing the estimator parameter $F_j(s+1)$.

Theorem 1. *Let the estimator parameter $F_j(s+1)$ be given. Suppose that the local estimation error $\widehat{e}_j(s)$ satisfies*

$$\widehat{e}_j(s)\in\left\langle 0,\widehat{Z}_j(s)\right\rangle$$

with $\widehat{Z}_j(s)$ being a known matrix. Then, $\widehat{e}_j(s+1)$ satisfies

$$\widehat{e}_j(s+1)\in\left\langle 0,\widehat{Z}_j(s+1)\right\rangle \qquad (21)$$

where

$$\begin{aligned}
\widehat{Z}_j(s+1)\triangleq&\Big[\Xi_j(s+1)\check{Z}_j(s+1)\quad -\tfrac{b_j}{\lfloor 2^{\vec{R}_j/q_j}\rfloor}F_j(s+1)\\
&\quad -F_j(s+1)V_j(s+1)\Big], \qquad (22)\\
\check{Z}_j(s+1)\triangleq&\left[\widehat{\mathcal{G}}_j(s)\quad B(s)W(s)\right]. \qquad (23)
\end{aligned}$$

Proof. See Appendix Appendix C.

Theorem 2. *Assume that the local estimator parameter $F_j(s+1)$ is designed as*

$$F_j(s+1) = \Upsilon_j^T(s+1)\Theta_j^{-1}(s+1) \tag{24}$$

where

$$\begin{aligned}\Upsilon_j(s+1) &\triangleq C_j(s+1)\check{Z}_j(s+1)\check{Z}_j^T(s+1),\\ \Theta_j(s+1) &\triangleq C_j(s+1)\check{Z}_j(s+1)\check{Z}_j^T(s+1)C_j^T(s+1)\\ &\quad + \frac{b_j^2}{\left(\lfloor 2^{\vec{R}_j/q_j}\rfloor\right)^2} I + V_j(s+1)V_j^T(s+1).\end{aligned}$$

Then, the local estimation error $\widehat{e}_j(s)$ satisfies $\widehat{e}_j(s) \in \langle 0, \widehat{Z}_j(s)\rangle$ for $s \in \mathbb{N}$ where $\widehat{Z}_j(s)$ $(s \in \mathbb{N})$ are given by (22)-(23) with

$$\widehat{Z}_j(0) = \bar{X}(0). \tag{25}$$

Moreover, the F-radius of the zonotope $\left\langle 0, \widehat{Z}_j(s+1)\right\rangle$ is minimized with the minimum F-radius satisfying

$$\begin{aligned}&\|\widehat{Z}_j(s+1)\|_F^2\\ &= \mathrm{tr}\Big\{\check{Z}_j(s+1)\check{Z}_j^T(s+1)\\ &\qquad - \Upsilon_j^T(s+1)\Theta_j^{-1}(s+1)\Upsilon_j(s+1)\Big\}.\end{aligned} \tag{26}$$

Proof. See Appendix Appendix D.

Algorithm 1 provides a way to recursively obtain the local estimator parameter and the local zonotopic set containing the local estimation error at each time instant.

3.4. Design of Distributed Fusion Estimator

In this subsection, we focus on the design of the distributed fusion estimator.

Let

$$\widehat{x}(s) \triangleq \sum_{j=1}^{N} \Omega_j(s)\widehat{x}_j(s)$$

be the fused estimate where $\Omega_j(s) \in \mathbb{R}^{p\times p}$ $(j = 1, 2, \cdots, N)$ are the fusion weights to be determined subject to $\sum_{j=1}^{N} \Omega_j(s) = I_p$.

Define $e(s) \triangleq x(s) - \widehat{x}(s)$ as the global estimation error. According to $\widehat{e}_j(s) \in \langle 0, \widehat{Z}_j(s)\rangle$ and Lemma 1, one has

$$\begin{aligned}&e(s)\\ &= \sum_{j=1}^{N} \Omega_j(s)x(s) - \sum_{j=1}^{N} \Omega_j(s)\widehat{x}_j(s) = \sum_{j=1}^{N} \Omega_j(s)\widehat{e}_j(s)\end{aligned}$$

Algorithm 1: Zonotopic SMSE Algorithm

Input: $\langle \bar{x}(0), \bar{X}(0) \rangle$, $\vec{R}_j$.
Output: $\widehat{x}_j(s+1)$, $\langle 0, \widehat{Z}_j(s+1) \rangle$.
Initialization: Given positive integers $s_{\max}$ and M_j, set $s = 0$, $\widehat{x}_j(0) = \bar{x}(0)$, $\widehat{Z}_j(0) = \bar{X}(0)$;
for $s \leq s_{\max}$ **do**
  Calculate $\check{x}_j(s+1)$ by (8);
  Calculate $\widehat{\mathcal{G}}_j(s)$ by (16) ;
  Calculate $\check{Z}_j(s+1)$ by (23);
  Calculate $F_j(s+1)$ by (24) ;
  Update $\widehat{x}_j(s+1)$ by (8) ;
  Update $\widehat{Z}_j(s+1)$ by (22) ;
  if *the number of columns of* $\widehat{Z}_j(s+1)$ *is greater than* M_j **then**
    set $\widehat{Z}_j(s+1) = \text{diag}_v\{|\widehat{Z}_j(s+1)|\mathbf{1}\}$;
  Output $\widehat{x}_j(s+1)$, $\langle 0, \widehat{Z}_j(s+1) \rangle$;

$$
\begin{aligned}
&\in \left\langle 0, \begin{bmatrix} \Omega_1(s)\widehat{Z}_1(s) & \Omega_2(s)\widehat{Z}_2(s) & \cdots & \Omega_N(s)\widehat{Z}_N(s) \end{bmatrix} \right\rangle \\
&\triangleq \langle 0, \widehat{Z}(s) \rangle.
\end{aligned} \tag{27}
$$

According to (27), the fusion weights are given by solving the following optimization problem

$$
\begin{aligned}
&\min_{\Omega_1(s), \Omega_2(s), \cdots, \Omega_N(s)} \left\| \widehat{Z}(s) \right\|_F^2 \\
&\text{s.t.} \sum_{j=1}^{N} \Omega_j(s) = I_p.
\end{aligned} \tag{28}
$$

To solve the optimization problem (28), the following lemma is useful.

Lemma 4. *Suppose that the matrix* $\frac{\partial g(x(s))}{\partial x(s)}$ *is nonsingular for all* $s \in \mathbb{N}$. *Then, the matrix* $\widehat{Z}_j(s)\widehat{Z}_j^T(s)$ *is nonsingular for all* $j = 1, 2, \cdots, N$ *and* $s \in \mathbb{N}$.

Proof. See Appendix Appendix E.

For the sake of ensuring that the matrix $\widehat{Z}_j(s)\widehat{Z}_j^T(s)$ is nonsingular, we assume that the matrix $\frac{\partial g(x(s))}{\partial x(s)}$ is nonsingular for all $s \in \mathbb{N}$ in the rest of this paper.

Theorem 3. *The solution to the optimization problem (28) is given as follows:*

$$
\Omega_j(s) = \left(\sum_{j=1}^{N} \left(\widehat{Z}_j(s)\widehat{Z}_j^T(s) \right)^{-1} \right)^{-1} \left(\widehat{Z}_j(s)\widehat{Z}_j^T(s) \right)^{-1}. \tag{29}
$$

Proof. See Appendix Appendix F.

In Theorem 3, with the matrix-weighted fusion method, the solution to the optimization problem (28) is provided. Two other widely-used fusion methods in distributed fusion are the scalar-weighted and diagonal-matrix-weighted methods. In the following corollaries, let us give the solution to the optimization problem (28) when adopting these two methods.

Corollary 1. *In the case that the fusion weights are scalars, i.e.,* $\Omega_j(s) = \omega_j(s) I_p$ *with* $\omega_j(s) \in \mathbb{R}$*, the solution to the optimization problem (28) is given by*

$$\begin{aligned} \omega_j(s) = & \left(\sum_{j=1}^{N} \left(\operatorname{tr}\left\{ \widehat{Z}_j(s) \widehat{Z}_j^T(s) \right\} \right)^{-1} \right)^{-1} \\ & \times \left(\operatorname{tr}\left\{ \widehat{Z}_j(s) \widehat{Z}_j^T(s) \right\} \right)^{-1}. \end{aligned} \tag{30}$$

Proof. See Appendix Appendix G.

Corollary 2. *In the case that the fusion weights are diagonal matrices, i.e.,* $\Omega_j(s) = \operatorname{diag}\left\{ \varpi_j^{(1)}(s), \cdots, \varpi_j^{(p)}(s) \right\}$ *with* $\varpi_j^{(i)}(s) \in \mathbb{R}$ $(i = 1, 2, \cdots, p)$*, the solution to the optimization problem (28) is given by*

$$\begin{aligned} \varpi_j^{(i)}(s) = & \left(\sum_{j=1}^{N} \left(\widehat{Z}_j^{(i)}(s) \left(\widehat{Z}_j^{(i)}(s) \right)^T \right)^{-1} \right)^{-1} \\ & \times \left(\widehat{Z}_j^{(i)}(s) \left(\widehat{Z}_j^{(i)}(s) \right)^T \right)^{-1} \end{aligned} \tag{31}$$

with

$$\widehat{Z}_j^{(i)}(s) = \begin{bmatrix} \underbrace{0 \ \cdots \ 0}_{i-1} & 1 & \underbrace{0 \ \cdots \ 0}_{p-i} \end{bmatrix} \widehat{Z}_j(s),$$

Proof. See Appendix Appendix H.

The distributed fusion estimator design procedure is summarized in Algorithm 2.

Algorithm 2: Distributed Fusion Algorithm

Input: $\widehat{x}_j(s)$ and $\langle 0, \widehat{Z}_j(s) \rangle$, $j = 1, 2, \cdots, N$.
Output: $\sum_{j=1}^{N} \Omega_j(s) \widehat{x}_j(s)$, $\langle 0, \widehat{Z}(s) \rangle$.

```
Initialization: Given s_max, set s = 0 ;
for s ≤ s_max do
    Calculate Ω_j(s) (j = 1, 2, ··· , N) ;
    Compute Σ_{j=1}^N Ω_j(s) x̂_j(s) ;
    Obtain Ẑ(s) by (27) ;
    Output Σ_{j=1}^N Ω_j(s) x̂_j(s) and ⟨0, Ẑ(s)⟩ ;
    Output Σ_{j=1}^N Ω_j(s) x̂_j(s) + |Ẑ(s)|1 and Σ_{j=1}^N Ω_j(s) x̂_j(s) − |Ẑ(s)|1;
```

Remark 4. *In Theorem 3 and Corollaries 1-2, under a zonotopes-based fusion framework, three distributed fusion estimators are designed by using the matrix-weighted method, the scalar-weighted method and the diagonal-matrix-weighted method, respectively. Resting on the obtained results, a distributed fusion algorithm (Algorithm 2) is then proposed. The computational complexity of Algorithm 2 with different fusion weights is given in TABLE 1. Note that the matrix-weighted, the scalar-weighted and the diagonal-matrix-weighted fusion methods have been well-studied in distributed fusion problems subject to stochastic noises, but have received little research attention yet as for UBB noises.*

Table 1: Computational complexity of Algorithm 2

Fusion weights $\Omega_j(s)$ $(j = 1, 2, \cdots, N)$	Number of flops
matrices given by (29)	$\mathcal{O}(p^2 M_j)$
scalars given by (30)	$\mathcal{O}(pM_j)$
diagonal matrices given by (31)	$\mathcal{O}(pM_j)$

In the following theorem, it is shown that with the above calculated fusion weights, the estimation accuracy of the fused estimate $\widehat{x}(s)$ is no worse than that of each local estimate.

Theorem 4. *In the case that the matrix-weighted/scalar-weighted/diagonal-matrix-weighted method is used, if the fusion weights $\Omega_j(s)$ $(j = 1, 2, \cdots, N)$ are chosen as a solution to the optimization problem (28), then one has*

$$\left\|\widehat{Z}(s)\right\|_F \leq \left\|\widehat{Z}_j(s)\right\|_F, \ \forall j \in \{1, 2, \cdots, N\}. \tag{32}$$

Proof. See Appendix Appendix I.

Remark 5. *In this paper, a systematic investigation has been launched on the distributed zonotopic fusion estimation issue for a class of nonlinear networked systems with both UBB noises and bit rate constraints. Compared with existing studies, this paper possesses the following distinguished features: 1) an allocation protocol is designed under a bit rate constraint such that the zonotope restraining the decoding error is minimized (in the sense of the F-radius); 2) the distributed fusion problem studied is new as the UBB noises and the bit rate constraints are considered, for the first time, for general nonlinear networked systems; 3) a novel method is proposed to deal with the nonlinear function in the system with hope to improve the estimation accuracy; and 4) three kinds of fusion weights (i.e., matrices, scalars and diagonal matrices) are designed under a zonotopes-based fusion framework.*

4. ILLUSTRATIVE EXAMPLE

Consider a nonlinear system in the form (1)-(2) with the following parameters:

$$g(x(s)) = \begin{bmatrix} 1.15x^{(1)}(s) + 0.2\cos(x^{(2)}(s)) \\ 0.2x^{(1)}(s) + 0.05\sin(x^{(2)}(s)) \\ 0.5x^{(2)}(s) + 0.8x^{(3)}(s) \end{bmatrix},$$

$$C_1(s) = \begin{bmatrix} 0.1 & 0.2 & 0 \\ 0.3 & 0.15 & 0.2 \end{bmatrix}, C_2(s) = \begin{bmatrix} 0.15 & 0 & 0.2 \\ 0.1 & 0.5 & 0.4 \end{bmatrix},$$

$$C_3(s) = \begin{bmatrix} 0.2 & 0.1 & 0 \\ 0.1 & 0 & 0.3 \end{bmatrix}, B(s) = 0.1I$$

where for $i = 1, 2, 3$, $x^{(i)}(s)$ represents the i-th component of $x(s)$.

The process and measurement noises are selected as

$$\begin{aligned} w(s) &= 0.2 \begin{bmatrix} \sin(s) & \cos(s) & -\sin(s) \end{bmatrix}^T, \\ v_1(s) &= 0.5 \begin{bmatrix} \cos(s) & \sin(s) \end{bmatrix}^T, \\ v_2(s) &= 0.3 \begin{bmatrix} \sin(s) & -\sin(s) \end{bmatrix}^T, \\ v_3(s) &= 0.2 \begin{bmatrix} \cos(s) & -\sin(s) \end{bmatrix}^T, \end{aligned}$$

which implies that

$$W(s) = 0.2I, V_1(s) = 0.5I, V_2(s) = 0.3I, V_3(s) = 0.2I.$$

The total bit rate is set to be 24 (i.e., $\vec{R} = 24$) and the positive scalars b_j $(j = 1, 2, 3)$ are chosen as $b_1 = 27$, $b_2 = 21$ and $b_3 = 16$. Then, the value of the bit rates $\vec{R}_j$ $(j = 1, 2, 3)$ can be determined by solving the optimization problem (20). By using the exhaustive method, it is found that $\vec{R}_1 = 9$, $\vec{R}_2 = 8$ and $\vec{R}_3 = 7$ is a solution to the optimization problem (20).

Set $s_{\max} = 30$ and $M_1 = M_2 = M_3 = 100$. With the initial value $x(0) = \begin{bmatrix} 0.1 & 0.7 & 0.9 \end{bmatrix}^T$ and $\langle \bar{x}(0), \bar{X}(0) \rangle = 0.5 \odot \langle \mathbf{1}, I \rangle$, the local estimates and the parameters of the local estimators are obtained via Algorithm 1. Then, by executing Algorithm 2, the fused estimate $\sum_{j=1}^{N} \Omega_j(s) \widehat{x}_j(s)$ and fused zonotope $\left\langle 0, \widehat{Z}(s) \right\rangle$ under different fusion weights (i.e., matrix weights, scalar weights and diagonal matrix weights) are obtained at each time instant. The simulation results are shown in Figs. 2-5. Figs. 2-4 depict the state variables $x_s^{(i)}$ $(i = 1, 2, 3)$, their estimates $\widehat{x}_s^{(i)}$ (i.e., the i-th component of $\sum_{j=1}^{N} \Omega_j(s) \widehat{x}_j(s)$) and their bounds calculated by using Algorithm 2 under different classes of fusion weights, which implies that the proposed fusion estimation method performs indeed well. Fig. 5 gives information about the F-radii of the local zonotopes $\langle 0, \widehat{Z}_j(s) \rangle$ $(j = 1, 2, 3)$ and the fused zonotopes (under different classes of fusion weights). It is clear from Fig. 5 that the fused estimates have better estimation accuracy than the local estimates, which shows the merits brought by the fusion under the established zonotopes-based fusion framework and the correctness of Theorem 4. Moreover, among the fused estimates with different fusion weights, the one with the matrix weights possesses the best estimation accuracy, and the one with the scalar weights has the worst estimation accuracy, which is consistent with the theoretical results.

5. CONCLUSION

This paper has addressed the distributed fusion for nonlinear networked systems subject to UBB noises and bit rate constraints. By solving an off-line optimization problem subject to the bit rate constraint, a bit rate allocation protocol has been designed to distribute the limited bandwidth resource of the communication network to each sensor node. Based on the received data from the network, several local extended-Kalman-type estimators have been designed such that the F-radii of the zonotopic sets confining the local estimation errors are minimized. By using the obtained local estimates and zonotopic sets, distributed fusion estimators weighted by matrices,

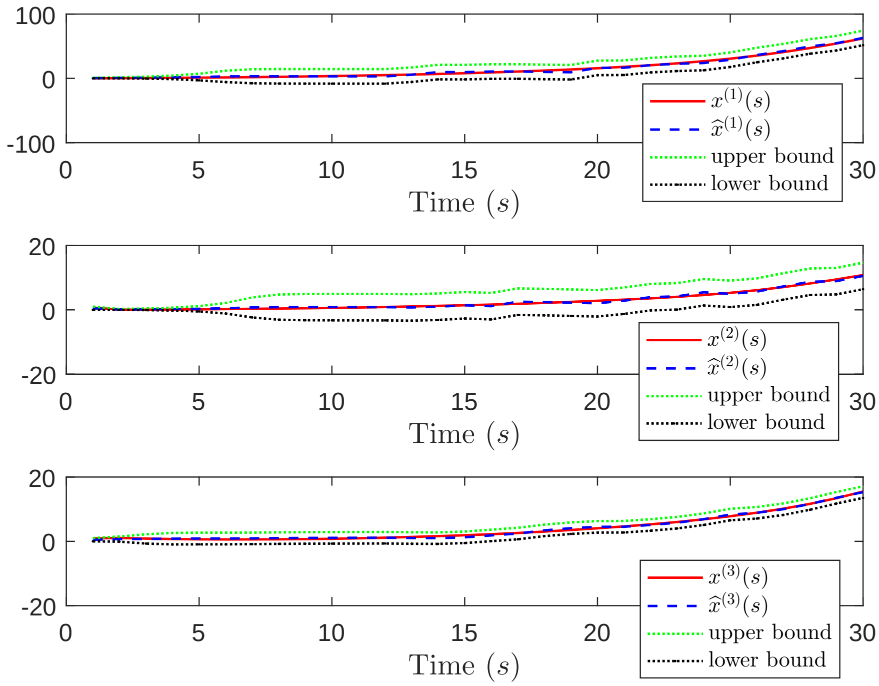

Figure 2: $x^{(i)}(s)$, their estimates $\widehat{x}^{(i)}(s)$ $(i = 1, 2, 3)$ and their bounds (with matrix weights).

scalars and diagonal matrices have been designed, respectively. In the end, the validity of the proposed distributed fusion method has been illustrated via a simulation example. Future research topics would include the extension of this paper to more complicated systems such as nonlinear time-delay systems [29, 8] and complex networks [50, 19, 17].

Appendix A. The Proof of Lemma 2

For presentation clarity, the proof is divided into the following two steps.

Step 1 Proof of $\left|\tilde{g}^{(i)}(\widehat{e}_j(s))\right| \leq \left\|\widehat{Z}_j(s)\right\|_\infty \underline{m}_j^{(i)}(s)$ with $\tilde{g}^{(i)}(\widehat{e}_j(s))$ being the i-th component of $\tilde{g}(\widehat{e}_j(s))$.

According to the mean value theorem, one has

$$g(x(s)) = g(\widehat{x}_j(s)) + \begin{bmatrix} G^{(1)}(s) \\ G^{(2)}(s) \\ \vdots \\ G^{(p)}(s) \end{bmatrix} \widehat{e}_j(s) \tag{A.1}$$

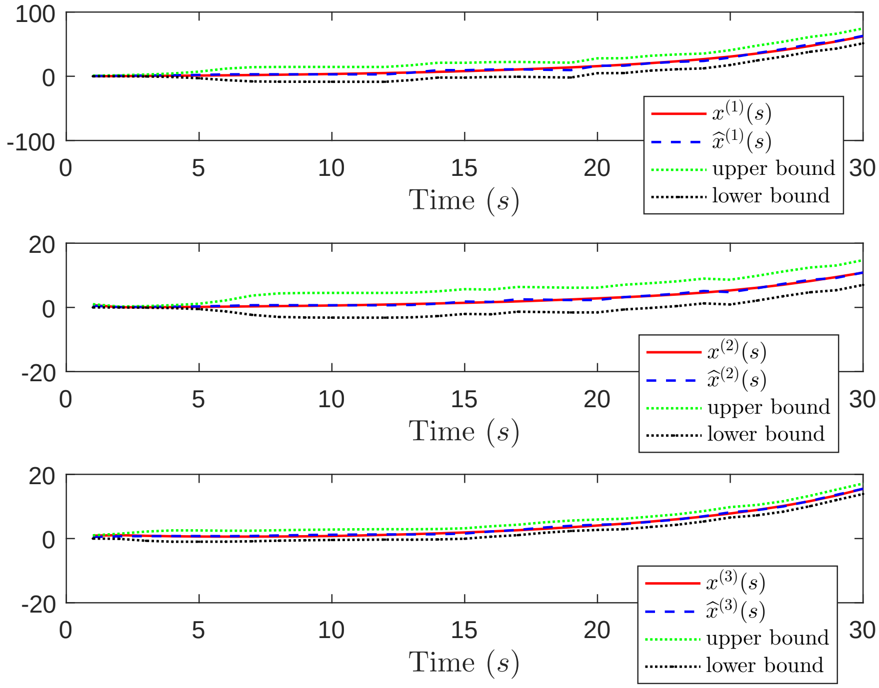

Figure 3: $x^{(i)}(s)$, their estimates $\widehat{x}^{(i)}(s)$ $(i = 1, 2, 3)$ and their bounds (with scalar weights).

where for $i = 1, 2, \cdots, p$,

$$G^{(i)}(s) \triangleq \frac{\partial g^{(i)}(x(s))}{\partial x(s)}\Big|_{x(s)=\widehat{x}_j(s)+\rho^{(i)}(s)\widehat{e}_j(s)}$$

with $\rho^{(i)}(s) \in (0, 1)$ is a scalar.

In view of (15), it follows from $\widehat{e}_j(s) = x(s) - \widehat{x}_j(s)$ and Definition 1 that there exists a vector $z_j(s)$ satisfying

$$\|z_j(s)\|_\infty \leq 1 \tag{A.2}$$

such that

$$\widehat{e}_j(s) = \widehat{Z}_j(s)z_j(s). \tag{A.3}$$

From (A.3), one has

$$\begin{aligned} &\widehat{x}_j(s) + \rho^{(i)}(s)\widehat{e}_j(s) \\ = &\widehat{x}_j(s) + \widehat{Z}_j(s)\rho^{(i)}(s)z_j(s) \end{aligned}$$

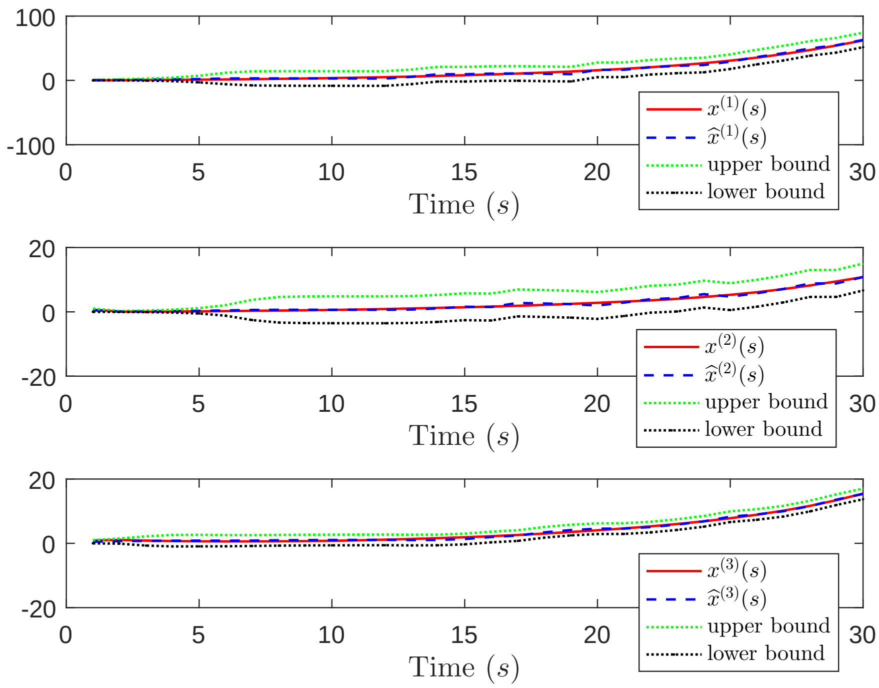

Figure 4: $x^{(i)}(s)$, their estimates $\widehat{x}^{(i)}(s)$ $(i = 1, 2, 3)$ and their bounds (with diagonal matrix weights).

$$\in \left\langle \widehat{x}_j(s), \widehat{Z}_j(s) \right\rangle. \tag{A.4}$$

Utilizing (A.3) again and taking (A.4) into account, one has

$$\begin{aligned} &\left\| G^{(i)}(s)\widehat{e}_j(s) \right\|_\infty \\ = &\left\| G^{(i)}(s)\widehat{Z}_j(s)z_j(s) \right\|_\infty \\ \leq &\left\| G^{(i)}(s) \right\|_\infty \left\| \widehat{Z}_j(s) \right\|_\infty \|z_j(s)\|_\infty \\ \leq &\, \underline{m}_j^{(i)}(s) \left\| \widehat{Z}_j(s) \right\|_\infty . \end{aligned} \tag{A.5}$$

Here, the existence of $\underline{m}_j^{(i)}(s)$ is guaranteed by the assumption that $g(x(s))$ is twice continuously differentiable.

According to (A.5), one can see that there exists a scalar $\underline{z}_j^{(i)}(s) \in [-1, 1]$ such that

$$\begin{aligned} &G^{(i)}(s)\widehat{e}_j(s) \\ = &\, \underline{m}_j^{(i)}(s) \left\| \widehat{Z}_j(s) \right\|_\infty \underline{z}_j^{(i)}(s). \end{aligned} \tag{A.6}$$

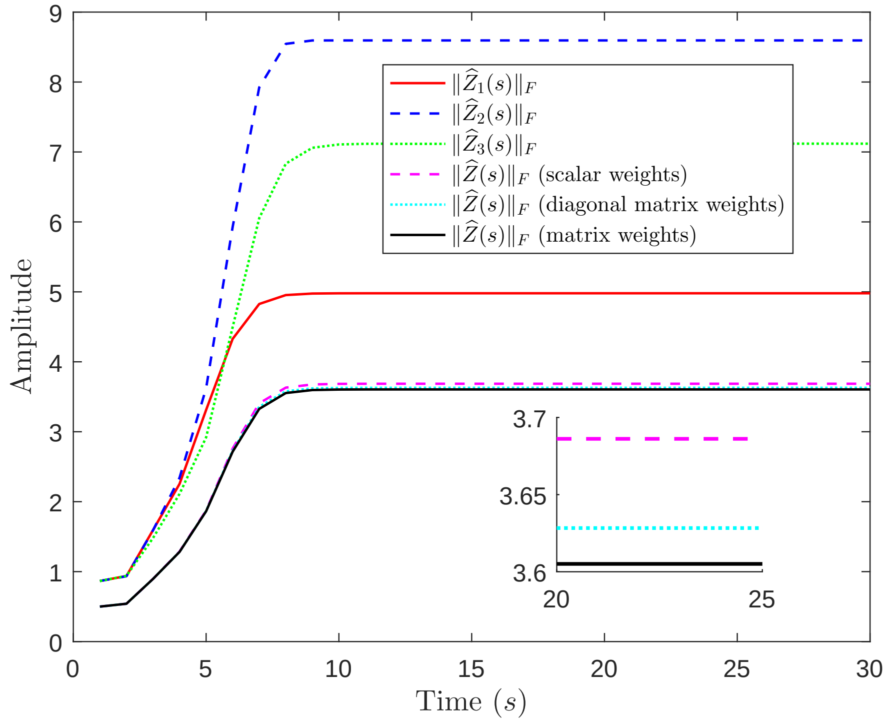

Figure 5: $\|\widehat{Z}_j(s)\|_F$ $(j = 1, 2, 3)$ and $\|\widehat{Z}(s)\|_F$ with three different classes of fusion weights .

From (A.1) and (A.6), it is easy to see from $\tilde{g}(\widehat{e}_j(s)) = g(x(s)) - g(\widehat{x}_j(s))$ that

$$\left|\tilde{g}^{(i)}(\widehat{e}_j(s))\right| \leq \underline{m}_j^{(i)}(s) \left\|\widehat{Z}_j(s)\right\|_\infty . \tag{A.7}$$

<u>*Step 2*</u> Proof of $|\tilde{g}^{(i)}(\widehat{e}_j(s))| \leq \left| \frac{\partial g^{(i)}(x(s))}{\partial x(s)} \Big|_{x(s)=\widehat{x}_j(s)} \widehat{Z}_j(s) \right| \mathbf{1} + \frac{p}{2} \left\|\widehat{Z}_j(s)\right\|_\infty^2 \bar{m}_j^{(i)}(s)$.
Applying the first-order Taylor-series expansion to $g(x(s))$ yields

$$\begin{aligned} & g(x(s)) \\ & = g(\widehat{x}_j(s)) + \mathcal{G}(s)\widehat{e}_j(s) \\ & \quad + \frac{1}{2} \begin{bmatrix} \widehat{e}_j^T(s)\mathscr{G}^{(1)}(s) \\ \widehat{e}_j^T(s)\mathscr{G}^{(2)}(s) \\ \vdots \\ \widehat{e}_j^T(s)\mathscr{G}^{(p)}(s) \end{bmatrix} \widehat{e}_j(s) \end{aligned} \tag{A.8}$$

where

$$\mathcal{G}(s) \triangleq \frac{\partial g(x(s))}{\partial x(s)} \Big|_{x(s)=\widehat{x}_j(s)},$$

$$\mathscr{G}^{(i)}(s) \triangleq \frac{\partial^2 g^{(i)}(x(s))}{\partial x^2(s)}\Big|_{x(s)=\widehat{x}_j(s)+\varrho^{(i)}(s)\widehat{e}_j(s)}$$

and $\varrho^{(i)}(s) \in (0,1)$ is a scalar for $i = 1, 2, \cdots, p$.

Recalling (A.2) and (A.3), one has

$$\|\widehat{e}_j(s)\|_\infty = \left\|\widehat{Z}_j(s) z_j(s)\right\|_\infty \leq \left\|\widehat{Z}_j(s)\right\|_\infty . \tag{A.9}$$

In accordance with (A.9), it is obvious that there exist a vector $\eta_j(s) \in \mathbb{R}^p$ with $\|\eta_j(s)\|_\infty \leq 1$ such that

$$\widehat{e}_j(s) = \left\|\widehat{Z}_j(s)\right\|_\infty \eta_j(s). \tag{A.10}$$

Substituting (A.10) into $\widehat{e}_j^T(s)\mathscr{G}^{(i)}(s)\widehat{e}_j(s)$ gives

$$\begin{aligned} &\widehat{e}_j^T(s)\mathscr{G}^{(i)}(s)\widehat{e}_j(s) \\ = &\left\|\widehat{Z}_j(s)\right\|_\infty^2 \eta_j^T(s)\mathscr{G}^{(i)}(s)\eta_j(s). \end{aligned} \tag{A.11}$$

Similar to (A.4), one has

$$\begin{aligned} &\widehat{x}_j(s) + \varrho^{(i)}(s)\widehat{e}_j(s) \\ =& \widehat{x}_j(s) + \widehat{Z}_j(s)\varrho^{(i)}(s) z_j(s) \\ \in& \left\langle \widehat{x}_j(s), \widehat{Z}_j(s)\right\rangle . \end{aligned} \tag{A.12}$$

According to (A.11) and (A.12), one can see that

$$\begin{aligned} &\left\|\widehat{e}_j^T(s)\mathscr{G}^{(i)}(s)\widehat{e}_j(s)\right\|_\infty \\ =& \left\|\widehat{Z}_j(s)\right\|_\infty^2 \left\|\eta_j^T(s)\mathscr{G}^{(i)}(s)\eta_j(s)\right\|_\infty \\ \leq& \left\|\widehat{Z}_j(s)\right\|_\infty^2 \left\|\eta_j^T(s)\right\|_\infty \left\|\mathscr{G}^{(i)}(s)\right\|_\infty \|\eta_j(s)\|_\infty \\ \leq& p\bar{m}_j^{(i)}(s) \left\|\widehat{Z}_j(s)\right\|_\infty^2 \end{aligned} \tag{A.13}$$

where the existence of $\bar{m}_j^{(i)}(s)$ is also guaranteed by the assumption that $g(x(s))$ is twice continuously differentiable.

With (A.13), one can see that there exists a scalar $\bar{z}_j^{(i)}(s) \in [-1, 1]$ such that

$$\begin{aligned} &\widehat{e}_j^T(s)\mathscr{G}^{(i)}(s)\widehat{e}_j(s) \\ =& p\bar{m}_j^{(i)}(s) \left\|\widehat{Z}_j(s)\right\|_\infty^2 \bar{z}_j^{(i)}(s). \end{aligned} \tag{A.14}$$

From (A.3), (A.8) and (A.14), one can easily obtain that

$$\left|\tilde{g}^{(i)}(\widehat{e}_j(s))\right|$$

$$
\begin{aligned}
&= \left|\mathcal{G}^{(i)}(s)\widehat{Z}_j(s)z_j(s) + \frac{p}{2}\left\|\widehat{Z}_j(s)\right\|_\infty^2 \bar{m}_j^{(i)}(s)\bar{z}_j^{(i)}(s)\right| \\
&= \left|\mathcal{G}^{(i)}(s)\widehat{Z}_j(s)z_j(s)\right| + \left|\frac{p}{2}\left\|\widehat{Z}_j(s)\right\|_\infty^2 \bar{m}_j^{(i)}(s)\bar{z}_j^{(i)}(s)\right| \\
&\leq \left|\mathcal{G}^{(i)}(s)\widehat{Z}_j(s)\right|\mathbf{1} + \frac{p}{2}\left\|\widehat{Z}_j(s)\right\|_\infty^2 \bar{m}_j^{(i)}(s) \qquad \text{(A.15)}
\end{aligned}
$$

where $\mathcal{G}^{(i)}(s)$ represents the i-th row of the matrix $\mathcal{G}(s)$.

Aggregation of Step 1 and Step 2

With (A.7) and (A.15), one can easily find that $\tilde{g}^{(i)}(\widehat{e}_j(s)) \leq \widehat{\mathcal{G}}_j^{(i)}(s)$, which together with Definition 1 implies that

$$
\tilde{g}^{(i)}(\widehat{e}_j(s) \in \left\langle 0, \widehat{\mathcal{G}}_j^{(i)}(s)\right\rangle. \qquad \text{(A.16)}
$$

Based on (A.16), one can easily obtain (16). The proof is now complete.

Appendix B. The Proof of Lemma 3

From (6), it is easy to see that

$$
\left(\frac{b_j}{\lfloor 2^{\vec{R}_j/q_j}\rfloor}\right)^{-1} \varsigma_j^{(l)}(s) \in \langle 0, 1\rangle. \qquad \text{(B.1)}
$$

Then, it follows from (B.1) and Lemma 1 that

$$
\begin{aligned}
\varsigma_j^{(l)}(s) &= \frac{b_j}{\lfloor 2^{\vec{R}_j/q_j}\rfloor}\left(\frac{b_j}{\lfloor 2^{\vec{R}_j/q_j}\rfloor}\right)^{-1}\varsigma_j^{(l)}(s) \\
&\in \left\langle 0, \frac{b_j}{\lfloor 2^{\vec{R}_j/q_j}\rfloor}\right\rangle, \qquad \text{(B.2)}
\end{aligned}
$$

which further leads to

$$
\varsigma_j(s) = \left[\varsigma_j^{(1)}(s) \quad \cdots \quad \varsigma_j^{(q_j)}(s)\right]^T \in \left\langle 0, \frac{b_j}{\lfloor 2^{\vec{R}_j/q_j}\rfloor} I\right\rangle. \qquad \text{(B.3)}
$$

The proof is now complete.

Appendix C. The Proof of Theorem 1

Recalling the one-step prediction error $\check{e}_j(s+1)$ given in (9), one has from Lemma 1 and Lemma 2 that

$$
\begin{aligned}
\check{e}_j(s+1) &= \tilde{g}(\widehat{e}_j(s)) + B(s)w(s) \\
&\in \langle 0, \widehat{\mathcal{G}}_j(s)\rangle \oplus B(s) \odot \langle 0, W(s)\rangle
\end{aligned}
$$

$$
\begin{aligned}
&= \left\langle 0, \begin{bmatrix} \widehat{\mathcal{G}}_j(s) & B(s)W(s) \end{bmatrix} \right\rangle \\
&= \left\langle 0, \check{Z}_j(s+1) \right\rangle .
\end{aligned} \tag{C.1}
$$

Then, it follows from (C.1) and Lemma 3 that

$$
\begin{aligned}
&\widehat{e}_j(s+1) \\
&= \Xi_j(s+1)\check{e}_j(s+1) - F_j(s+1)\varsigma_j(s+1) \\
&\quad - F_j(s+1)v_j(s+1) \\
&\in \Xi_j(s+1) \odot \check{Z}_j(s+1) \oplus (-F_j(s+1)) \odot \left\langle 0, \frac{b_j}{\lfloor 2^{\vec{R}_j/q_j} \rfloor} I \right\rangle \\
&\quad \oplus (-F_j(s+1)) \odot \langle 0, V_j(s+1) \rangle,
\end{aligned}
$$

which together with Lemma 1 gives

$$
\widehat{e}_j(s+1) = \left\langle 0, \widehat{Z}_j(s+1) \right\rangle .
$$

The proof is now complete.

Appendix D. The Proof of Theorem 2

We first prove that the local estimation error $\widehat{e}_j(s)$ satisfies $\widehat{e}_j(s) \in \langle 0, \widehat{Z}_j(s) \rangle$ ($s \in \mathbb{N}$) by resorting to mathematical induction.

When $s = 0$, it follows from Assumption 1 and the initial value of the estimator (8) that $\widehat{e}_j(0) \in \left\langle 0, \widehat{Z}_j(0) \right\rangle$.

Assume that $\widehat{e}_j(s) \in \left\langle 0, \widehat{Z}_j(s) \right\rangle$ holds. From Theorem 1, one has $\widehat{e}_j(s+1) \in \left\langle 0, \widehat{Z}_j(s+1) \right\rangle$. Hence, $\widehat{e}_j(s) \in \left\langle 0, \widehat{Z}_j(s) \right\rangle$ holds for $\forall s \in \mathbb{N}$.

It remains to prove that the estimator parameter given in (24) can minimize the F-radius of $\left\langle 0, \widehat{Z}_j(s+1) \right\rangle$.

With (22), one has

$$
\begin{aligned}
&\left\| \widehat{Z}_j(s+1) \right\|_F^2 \\
&= \operatorname{tr}\left\{ \widehat{Z}_j(s+1)\widehat{Z}_j^T(s+1) \right\} \\
&= \operatorname{tr}\Big\{ \Xi_j(s+1)\check{Z}_j(s+1)\check{Z}_j^T(s+1)\Xi_j^T(s+1) \\
&\qquad + \frac{b_j^2}{\left(\lfloor 2^{\vec{R}_j/q_j} \rfloor \right)^2} F_j(s+1)F_j^T(s+1) \\
&\qquad + F_j(s+1)V_j(s+1)V_j^T(s+1)F_j^T(s+1) \Big\} \\
&= \operatorname{tr}\Big\{ F_j(s+1)\Theta_j(s+1)F_j^T(s+1) - F_j(s+1)\Upsilon_j(s+1)
\end{aligned}
$$

$$
\begin{aligned}
&\quad - \Upsilon_j^T(s+1)F_j^T(s+1) + \check{Z}_j(s+1)\check{Z}_j^T(s+1)\Big\} \\
&= \mathrm{tr}\Big\{ \left(F_j^T(s+1) - \Theta_j^{-1}(s+1)\Upsilon_j(s+1)\right)^T \Theta_j(s+1) \\
&\quad \times \left(F_j^T(s+1) - \Theta_j^{-1}(s+1)\Upsilon_j(s+1)\right) \\
&\quad + \check{Z}_j(s+1)\check{Z}_j^T(s+1) \\
&\quad - \Upsilon_j^T(s+1)\Theta_j^{-1}(s+1)\Upsilon_j(s+1)\Big\}.
\end{aligned}
\tag{D.1}
$$

From (D.1), it is easy to see that the estimator parameter given in (24) can minimize the F-radius of $\left\langle 0, \widehat{Z}_j(s+1)\right\rangle$ and the minimum F-radius satisfies (26), which ends the proof.

Appendix E. The Proof of Lemma 4

This lemma is proven by induction.

When $s = 0$, one can see from Assumption 1 and $\widehat{Z}_j(0) = \bar{X}(0)$ that $\widehat{Z}_j(0)\widehat{Z}_j^T(0)$ is nonsingular. Assume that $\widehat{Z}_j(s)\widehat{Z}_j^T(s)$ is nonsingular. It remains to show that $\widehat{Z}_j(s+1)\widehat{Z}_j^T(s+1)$ is also nonsingular.

Recall that $\widehat{Z}_j(s)\widehat{Z}_j^T(s)$ is nonsingular. Then, in accordance with the assumption that $\frac{\partial g(x(s))}{\partial x(s)}$ is nonsingular, it is easy to see from the expression of $\widehat{\mathcal{G}}_j(s)$ (given in (16)) that $\widehat{\mathcal{G}}_j(s)$ is positive-definite, which together with (23) gives $\check{Z}_j(s+1)\check{Z}_j^T(s+1) > 0$. Then, it follows the expression of $\widehat{Z}_j(s+1)$ (given in (22)) that

$$
\begin{aligned}
&\widehat{Z}_j(s+1)\widehat{Z}_j^T(s+1) \\
=&\Bigg(\left(\check{Z}_j(s+1)\check{Z}_j^T(s+1)\right)^{-1} + C_j^T(s+1) \\
&\times \left(\left(b_j^2 / \left(\lfloor 2^{\vec{R}_j/q_j} \rfloor\right)^2\right) I + V_j(s+1)V_j^T(s+1)\right)^{-1} \\
&\times C_j(s+1)\Bigg) \\
>&0,
\end{aligned}
$$

which implies that $\widehat{Z}_j(s+1)\widehat{Z}_j^T(s+1)$ is nonsingular. The proof is now complete.

Appendix F. The Proof of Theorem 3

Utilizing the Lagrange multiplier method, the following auxiliary function is introduced:

$$
\begin{aligned}
\mathcal{J}(s) \triangleq& \sum_{i=1}^{p} \left(\alpha^{(i)}(s)\right)^T \left(I_p - \sum_{j=1}^{N}\Omega_j(s)\right)\beta^{(i)}(s) \\
&+ \left\|\widehat{Z}(s)\right\|_F^2
\end{aligned}
\tag{F.1}
$$

where $\left(\alpha^{(i)}(s)\right)^T = \begin{bmatrix}\alpha^{(i,1)}(s) & \alpha^{(i,2)}(s) & \cdots & \alpha^{(i,p)}(s)\end{bmatrix} \in \mathbb{R}^{1\times p}$ is a row vector whose components are the Lagrange multipliers and $\beta^{(i)}(s) \triangleq \Bigg[\underbrace{0\cdots 0}_{i-1} \quad 1 \quad \underbrace{0\cdots 0}_{p-i}\Bigg]^T$.

According to (27), $\mathcal{J}(s)$ can be rewritten as

$$\mathcal{J}(s) = \mathrm{tr}\Bigg\{\sum_{j=1}^{N}\Omega_j(s)\widehat{Z}_j(s)\widehat{Z}_j^T(s)\Omega_j^T(s) + \sum_{i=1}^{p}\left(\alpha^{(i)}(s)\right)^T\left(I_p - \sum_{j=1}^{N}\Omega_j(s)\right)\beta^{(i)}(s)\Bigg\}. \tag{F.2}$$

Then, taking the partial derivative of $\mathcal{J}(s)$ with respect to $\Omega_j(s)$ gives

$$\frac{\partial\mathcal{J}(s)}{\partial\Omega_j(s)} = 2\Omega_j(s)\widehat{Z}_j(s)\widehat{Z}_j^T(s) - \sum_{i=1}^{p}\alpha^{(i)}(s)\left(\beta^{(i)}(s)\right)^T. \tag{F.3}$$

Letting the derivative in (F.3) be zero, one obtains

$$2\Omega_j(s)\widehat{Z}_j(s)\widehat{Z}_j^T(s) = \alpha(s)\beta(s) \tag{F.4}$$

where

$$\alpha(s) \triangleq \begin{bmatrix}\alpha^{(1)}(s) & \alpha^{(2)}(s) & \cdots & \alpha^{(p)}(s)\end{bmatrix},$$
$$\beta(s) \triangleq \begin{bmatrix}\beta^{(1)}(s) & \beta^{(2)}(s) & \cdots & \beta^{(p)}(s)\end{bmatrix}^T.$$

It follows from (F.4), Lemma 4 and $\beta(s) = I_p$ that

$$2\Omega_j(s) = \alpha(s)\left(\widehat{Z}_j(s)\widehat{Z}_j^T(s)\right)^{-1}. \tag{F.5}$$

Summing up (F.5) on both sides from 1 to N with respect to j results in

$$2\sum_{j=1}^{N}\Omega_j(s) = \alpha(s)\sum_{j=1}^{N}\left(\widehat{Z}_j(s)\widehat{Z}_j^T(s)\right)^{-1}. \tag{F.6}$$

Combining (F.6) with the restriction $\sum_{j=1}^{N}\Omega_j(s) = I_p$ yields

$$\alpha(s) = 2\left(\sum_{j=1}^{N}(\widehat{Z}_j(s)\widehat{Z}_j^T(s))^{-1}\right)^{-1}. \tag{F.7}$$

Substituting (F.7) into (F.5) and taking

$$\frac{\partial^2\mathcal{J}(s)}{\partial\Omega_j^2(s)} = 2\widehat{Z}_j(s)\widehat{Z}_j^T(s) > 0$$

into account, it is easy to see that under the constraint $\sum_{j=1}^{N}\Omega_j(s) = I_p$, $\left\|\widehat{Z}(s)\right\|_F^2$ is minimized by the matrix weights given in (29). The proof is now complete.

Appendix G. The Proof of Corollary 1

By means of the Lagrange multiplier method, the following auxiliary function is introduced:

$$\mathcal{J}_1(s) \triangleq \text{tr}\left\{\sum_{j=1}^{N}\omega_j^2(s)\widehat{Z}_j(s)\widehat{Z}_j^T(s)\right\} + \lambda(s)(\omega^T(s)\mathbf{1} - 1) \tag{G.1}$$

where $\lambda(s) \in \mathbb{R}$ denotes the Lagrange multiplier and $\omega(s) \triangleq \begin{bmatrix}\omega_1(s) & \omega_2(s) & \cdots & \omega_N(s)\end{bmatrix}^T$.

Taking the partial derivative of $\mathcal{J}_1(s)$ with respect to $\omega_j(s)$ and letting the calculated derivative be zero, one has

$$\frac{\partial \mathcal{J}_1(s)}{\partial \omega_j(s)} = 2\omega_j(s)\text{tr}\left\{\widehat{Z}_j(s)\widehat{Z}_j^T(s)\right\} + \lambda(s) = 0, \tag{G.2}$$

which gives

$$\lambda(s)\Big(\text{tr}\left\{\widehat{Z}_j(s)\widehat{Z}_j^T(s)\right\}\Big)^{-1} = -2\omega_j(s). \tag{G.3}$$

Here, the invertibility of $\text{tr}\left\{\widehat{Z}_j(s)\widehat{Z}_j^T(s)\right\}$ is affirmed by Lemma 4.

Summing up (G.3) on both sides from 1 to N with respect to j yields

$$\lambda(s) = -2\Bigg(\sum_{j=1}^{N}\Big(\text{tr}\left\{\widehat{Z}_j(s)\widehat{Z}_j^T(s)\right\}\Big)^{-1}\Bigg)^{-1}. \tag{G.4}$$

Substituting (G.4) into (G.2) obtains (30). Noticing

$$\frac{\partial^2 \mathcal{J}_1(s)}{\partial \omega_j^2(s)} = 2\text{tr}\Big\{\widehat{Z}_j(s)\widehat{Z}_j^T(s)\Big\} > 0,$$

one can easily see that the weights given by (30) are the solution to the optimization problem (28).

Appendix H. The Proof of Corollary 2

With $\Omega_j(s) = \text{diag}\left\{\varpi_j^{(1)}(s), \cdots, \varpi_j^{(p)}(s)\right\}$, one has

$$\left\|\widehat{Z}(s)\right\|_F^2 = \sum_{i=1}^{p}\sum_{j=1}^{N}\left(\varpi_j^{(i)}(s)\right)^2 \widehat{Z}_j^{(i)}(s)\left(\widehat{Z}_j^{(i)}(s)\right)^T. \tag{H.1}$$

By introducing the following Lagrangian function

$$\begin{aligned}\mathcal{J}_2(s) \triangleq & \sum_{i=1}^{p}\sum_{j=1}^{N}\left(\varpi_j^{(i)}(s)\right)^2 \widehat{Z}_j^{(i)}(s)\left(\widehat{Z}_j^{(i)}(s)\right)^T \\ & + \sum_{i=1}^{p}\bar{\lambda}^{(i)}(s)\left(\sum_{j=1}^{N}\varpi_j^{(i)}(s) - 1\right)\end{aligned} \tag{H.2}$$

where $\bar{\lambda}^{(i)}(s) \in \mathbb{R}$ $(i = 1, 2, \cdots, p)$ are the Lagrange multipliers, the proof is similar to that of Corollary 1 and is thus omitted here.

Appendix I. The Proof of Theorem 4

Since the fusion weights $\Omega_j(s)$ $(j = 1, 2, \cdots, N)$ are chosen as a solution to the optimization problem (28), one can easily obtain that

$$\begin{aligned}
&\min_{\Omega_1(s),\Omega_2(s),\cdots,\Omega_N(s)} \left\|\widehat{Z}(s)\right\|_F^2 \\
=&\min_{\Omega_1(s),\Omega_2(s),\cdots,\Omega_N(s)} \operatorname{tr}\left\{\sum_{j=1}^{N}\Omega_j(s)\widehat{Z}_j(s)\widehat{Z}_j^T(s)\Omega_j^T(s)\right\} \\
\le&\ \operatorname{tr}\left\{\widehat{Z}_j(s)\widehat{Z}_j^T(s)\right\} \\
=&\left\|\widehat{Z}_j(s)\right\|_F^2, \qquad (I.1)
\end{aligned}$$

holds for all $j \in \{1, 2, \cdots, N\}$, which ends the proof.

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