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EXACT AND APPROXIMATE BOUNDARY DATA INTERPOLATION IN THE FINITE ELEMENT METHOD

by

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1 . Introduction

Matching boundary data exactly in an elliptic problem avoids one of Strang's "variational crimes". (Strang and Fix (1973)). Supporting numerical evidence for this procedure is given by Marshall and Mitchell (1973), who considered the solution of Laplace's equation with Dirichlet boundary data by bilinear elements over squares and measured the errors in the L_2 norm. Then Marshall and Mitchell (1978) obtained some surprising results: for certain triangular elements, matching the boundary data exactly produced worse results than the usual procedure of interpolating the boundary data.

This paper provides a partial analysis of these unusual results. We find that the best approximation properties of the Rayleigh-Ritz-Galerkin approximation in the energy norm are true for both the exact and discretized schemes. We analyze the interpolation remainders for certain rectangular and triangular schemes and we find no direct connection between these and the finite element remainders. A new triangular element that improves the interpolation remainder theory is given. Numerical examples are also included.

2. Finite Element and Interpolation Remainder Theory

We consider Poisson's equation $-\Delta u = f$ on polygonal domain D with Dirichlet boundary data $u = g$ on ∂D. The corresponding weak problem is to find a function u such that

$$\begin{aligned} &a(u,v) = (f,v) \quad \text{for all} \quad V \in \overset{0}{H}_1(D) \\ &u - g \in \overset{0}{H}_1(D) \end{aligned} \tag{2.1}$$

where a(.,.) is the usual semi-definite energy $(\overset{0}{H}_1)$ inner product and {.,.) is the L_2 inner product. The exact boundary data matching Rayleigh-Ritz-Galerkin (RRG) approximation U_E satisfies

$$a(U_E, V) = (f,v) \quad \text{for all} \quad v \in \overset{0}{S} = \langle \phi_i \rangle \subset \overset{0}{H}_1(D) \tag{2.2}$$

Where $\quad U_E = G_E + \sum U_E^i \phi_i$

and $\quad G_E = g$ on ∂D.

Here we use local elements, e.g., the functions Φ_i. can be piecewise bilinear elements with the corresponding "transfinite" function G_E (see Sections 3 and 4 for specific examples). The corresponding discrete boundary data matching RRG approximation U_D satisfies

$$a(U_D, V) = (f,v) \quad \text{for all} \quad v \in \overset{0}{S} \tag{2.3}$$

Where $\quad U_D = G_D + \sum U_D^i \phi_i$

and $\quad G_D$ approximates g on ∂D.

Let $|\,.\,| = [a(.,.)]^{1/2}$ be the usual energy semi-norm. Then the following best approximation properties hold:

$$|u - U_E| \leq |u - \tilde{u}_E|$$

$$|u - U_D| \leq |u - \tilde{u}_D| \qquad (2.4)$$

where $\tilde{u}_E, = G_E + \Sigma\, a_i\, \Phi_i.$ and

$\tilde{u}_D = G_D + \Sigma\, b_i\, \Phi_i.$ for arbitrary $\{a_i\}$ and $\{b_i\}$.

Boundary layer effect: Since $G_E = g$ on ∂D, then U_D can be considered as a RRG approximation to U_E, which implies the best approximation property

$$|U_E - U_D| \leq |\, U_E - \tilde{u}_D| \quad . \qquad (2.5)$$

The possibility $\tilde{u}_D = G_D + \Sigma\, U_E^i \varphi_i$ leads to

$$|U_E - U_D| \leq |G_E - G_D| \qquad (2.6)$$

which is an upper bound on the difference between the exact and discrete RRG approximations, but of course gives no guide as to which of U_E and U_D is the better approximation to u. For the local schemes considered in this paper, the right hand side of (2.6) involves an integral over the "boundary layer" of elements next to the boundary.

Interpolation remainder theory is linked to RRG approximations by the best approximation property. More specifically, $\tilde{u}_E$ and $\tilde{u}_D$ can be chosen to be the interpolants to u from the appropriate approximating sets that contain U_E and U_D , respectively. For this paper we obtain bounds on $|u - \tilde{u}_E|$ and $|u - \tilde{u}_D|$ for various local interpolation schemes. In addition, we give numerical results for $|u - U_E|$ and $|u - U_D|$ for some model problems.

Theoretically, the result

$$|u - \tilde{u}_E| \leq |u - \tilde{u}_D| \tag{2.7}$$

does not guarantee

$$|u - U_E| \leq |u - U_D| \tag{2.8}$$

However, our numerical results indicate that (2.7) does imply (2,8). More surprisingly, even in one case where $|u - \tilde{u}_D| \leq |u - \tilde{u}_E|$ we find that (2.8) holds.

The variational crime of not matching the boundary data exactly provokes a question of norms. The energy semi-norm $|.| = [a(.,.)^{½}$ is not a norm on H_1 . By application of the Sobolev Imbedding Theorems, it can be made into a norm by the definition

$$\| \, . \, \|^2 = a(.,.) + \int_{\partial D} |.|^2 \, ds \, . \tag{2.9}$$

For this norm,

$$\left\| u - U_E \right\|^2 = a(u - U_E, u \sim U_E) + \int_{\partial D} \left| u - U_E \right|^2 ds \, .$$

$$= a(u-U_E, u-U_E) \tag{2.10}$$

$$\left\| u - U_E \right\|^2 = a(u - U_D, u \sim U_D) + \int_{\partial D} \left| u - U_D \right|^2 ds \, .$$

$$= a(u - U_D, u \sim U_D) + \int_{\partial D} \left| g - G_D \right|^2 ds \, .$$

The best approximation properties still hold in this new norm.

3. Rectangular Elements

We consider piecewise bilinear interpolation over a union of squares. Let Bu be the usual bilinear interpolant over $[0,1] \times [0,1]$ and Ju be the semi-discretized interpolant over $[0,1] \times [0,1]$ with the x-axis being part of the boundary of the region, that is,

$$Ju = (1 - y)\, u(x,0) + (1 - x)y\, u(0,1) + xy\, u(1,1)$$

$$= Bu + (1 - y)\, [u(x,0) - (1 - x)\, u(0,0) - x\, u(1,0)] \qquad (3.1)$$

So the identifications with the general $\tilde{u}_E$ and $\tilde{u}_D$ are $\tilde{u}_E = Ju$ and $\tilde{u}_D = Bu$ in this Section. From (3.1) we obtain

$$\left[\frac{\partial[u - Ju]}{\partial x}\right]^2 = \left[\frac{\partial[u - Bu]}{\partial x}\right]^2 + \alpha \qquad (3.2)$$

$$\left[\frac{\partial[u - Ju]}{\partial y}\right]^2 = \left[\frac{\partial[u - Bu]}{\partial y}\right]^2 + \beta \qquad (3.3)$$

For suitable functions α and β. We show that

$$\int_0^1 \int_0^1 (\alpha + \beta)\, dx\, dy < 0 \qquad (3.4)$$

and hence that

$$|u - Ju| \leq |u - Bu| \qquad (3.5)$$

Marshall (1975) used a truncated Taylor expansion to estimate the functions α and β. In order to compute α and β exactly, we first used a Sard kernal analysis similar to that of Barnhill and Gregory (1976). However, a simpler analysis can be used in some cases. If we show that (3.4) is true whenever u(x,y) is a monomial $x^m y^n$, $m, n \geq 0$, then the continuity of the energy norm suffices to prove (3.4) for all relevant u, from which the desired result (3.5) follows.

Notice that

$$B(x^m y^n) - J(x^m y^n) = x^m y^n \qquad \text{for } 0 \le m, n \le 1$$

and

$$B(x^m y^n) = J(x^m y^n) \not\equiv x^m y^n \qquad \text{for } m > 1 \text{ and } n > 0.$$

Hence $B(x^m y^n)$ and $J(x^m y^n)$ disagree only for $m > 1$ and $n = 0$. Careful use of the definitions of α and β, (3.2) and (3.3) respectively, applied to $u(x,y) = x^m$, eventually yields the following:

$$\int_0^1 \int_0^1 \alpha = -\frac{2}{3}\frac{(m-1)^2}{2m-1}$$

$$\int_0^1 \int_0^1 \beta = -\frac{2}{3}\frac{(m-1)^2}{(m+2)\,(2m+1)}$$

so that

$$\int_0^1 \int_0^1 (\alpha+\beta) = -\frac{2}{3}\frac{(m-1)^2\,(2m^2+3m+3)}{(m+2)(2m-1)(2m+1)} \qquad (3.7)$$

which is negative for $m \ge 2$.

As mentioned earlier, numerical verification of this theoretical result was given by Marshall and Mitchell (1973)and by Marshall (1975). (See also Mitchell and Wait (1977).) It should be remembered of course that these authors measured errors in the L_2 norm and not in the energy semi-norm used in the proofs in this paper.

4. Triangular Elements

Marshall and Mitchell's (1978) results, that matching the boundary data exactly in a triangulated region produces worse results than interpolating the boundary data, were quite surprising. They considered two C^o triangular schemes: the "side vertex" method and the "Nielson" method. (The side-vertex method is also called the "radial" method and the Nielson scheme is an instance of a polynomial blended triangular interpolant. See Barnhill (1977), p. 101 f. for additional information about such triangular interpolants.) Here we give an analysis of the Nielson scheme.

Nielson Scheme

Let Nu be the semi-discretized Nielson interpolant on the standard triangle T with vertices (1,0), (0,1) and (0,0). This interpolant picks up the function values at (0,1) and along the line segment $y = 0,\ 0 \le x \le 1$, and is given by

$$Nu = xu(1,0) + yu(0,1) + (1-y)\,u(x,0) - xu(1-y,0)\ . \qquad (4.1)$$

Its complete discretization is the Courant linear interpolant

$$Lu = xu(1,0) + yu(0,1) + (1-x-y)\,u(0,0)\ . \qquad (4.2)$$

(So $\tilde{u}_E = Nu$ and $\tilde{u}_D = Lu$ here). By reasoning as in Section 3, we consider the equation

$$|u - Nu|^2 = |u - Lu|^2 + \iint_T (\alpha + \beta)\,dx\,dy \qquad (4.3)$$

for those monomials for which Nu and Lu disagree, namely $u = x^m$, $m \ge 2$.

We eventually obtain

$$\iint_T \alpha = -\frac{(m-1)^2\,(2m+3)}{(2m+2)\,(2m+2)\,(m+2)}$$

and

$$\iint_T \beta = -\frac{(m-1)^2\,(2m+9m+6)}{3(2m+1)\,(2m+2)\,(m+2)}$$

and

$$\iint_T (\alpha+\beta)\,dx\;dy = -\frac{2(m-1)^2\,(m+1)\,(2m^2+3m-12)}{3(2m+1)\,(2m+2)\,(m+2)(2m-1)} \qquad (4.4)$$

which is positive for all $m \geq 2$.

This seems to indicate that the exact matching technique is worse in this case, a result confirmed numerically by Marshall and Mitchell using the L_2 norm. However, we obtain the surprising numerical result (Table 4.1) that, in the energy norm,

$$|u - U_E| < |u - U_D| \qquad (4.5)$$

for the problems computed.

The Barnhill, Birkhoff, Gordon (BBG) triangular element was also considered. The interpolation remainder theory produced no definite conclusions. The numerical results in Table 4.1 indicate that exact matching is usually better for this scheme.

Table 4.1 RRG Errors in $(\overset{0}{H}_1)$ Semi-Norm

$$-\Delta u = 0 \text{ on } [0,1] \times [0,1] = R$$

$$u = g \text{ on } \partial R$$

Problem 1: $u = \sin 2x\, e^{-2y}$

h	Courant Linear	Nielson	BBG
$\frac{1}{2}$	0.4544	0.3965	0.4070
$\frac{1}{4}$	0.2364	0.2098	0.2124
$\frac{1}{8}$	0. 1195	0.1115	0. 1121
$\frac{1}{16}$	0.05989	0.05776	0.05791

Problem 2: $u = \ln r - 2$

where $r^2 = (x-0.437)^2 + (y + 0.3)^2$

h	Courant Linear	Nielson	BBG
$\frac{1}{2}$	0.53861	0.51558	0.51962
$\frac{1}{4}$	0.30157	0.27193	0.27575
$\frac{1}{8}$	0.15732	0. 14858	0.14956
$\frac{1}{16}$	0.07 966	0.07737	0.07761

Problem 3: $u = e^x \cos y$

h	Courant Linear	Nielson	BBG
$\frac{1}{2}$	0.4055	0.3324	0.3427
$\frac{1}{4}$	0.1815	0.1567	0.1595
$\frac{1}{8}$	0.0714	0.0625	0.0634
$\frac{1}{16}$	0.0266	0.0235	0.0237

Problem 4: $u = \sin 4x\, e^{-4y}$

h	Courant Linear	Nielson	BBG
$\frac{1}{2}$	1.2088	1.1938	1.2149
$\frac{1}{4}$	0.6649	0.6337	0.6416
$\frac{1}{8}$	0.3053	0.2873	0.2897
$\frac{1}{16}$	0.1244	0.1175	0.1182

Improved Nielson

We now create an "improved" Nielson element

$$N^* u = P_1 u + P_2 u - P_1 P_2 u \tag{4.6}$$

where

$$P_1 u = yu(x,1-x) + xu(1-y,y)$$

$$P_2 u = u(x,0) + u(0,y) - u(0,0) + 4xy\{u(\tfrac{1}{2},\tfrac{1}{2}) - u(\tfrac{1}{2},0) - u(0,\tfrac{1}{2}) + u(0,0)\}$$

$$\begin{aligned} P_1 P_2 u = y\{ & u(x,0) + u(0,1-x) - u(0,0) \\ & + 4x(1-x)Cu(\tfrac{1}{2},\tfrac{1}{2}) - u(\tfrac{1}{2},0) - u(0,\tfrac{1}{2}) + u(0,0)]\} \\ & + x\{u(1-y,0) + u(0,y) - u(0,0) \\ & + 4y(1-y)[u(\tfrac{1}{2},\tfrac{1}{2}) - u(\tfrac{1}{2},0) - u(0,\tfrac{1}{2}) + u(0,0)]\} \end{aligned}$$

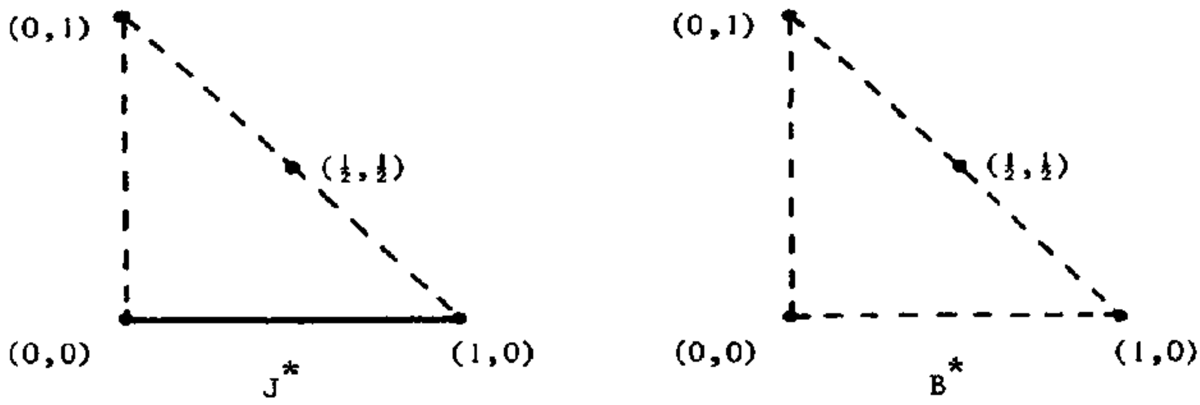
N is exact for the function xy and N is not

We discretize N*u linearly along x = 0 and quadratically along y = 1 - x to obtain

$$\begin{aligned} J^* u = {} & (1-y)\, u(x,0) - xu(1-y,0) + 2xy(x+y-1)\, u(0,0) \\ & + y(1-2x)\, u(0,1) + x\{y(2x-1) + (1-2y)(1-y)\}\, u(1,0) \\ & + 4xy(1-x-y)\, u(\tfrac{1}{2},0) + 4xyu(\tfrac{1}{2},\tfrac{1}{2}) \end{aligned} \qquad (4,7)$$

The fully discretized N* is

$$B^* u = (1-x-y)\, u(0,0) + y(1-2x)\, u(0,1) + x(1-2y)\, u(1,0) + 4xyu(\tfrac{1}{2},\tfrac{1}{2}) \tag{4.8}$$



<u>Figure 4. 1</u> Stencils for $J^* u^*$ and $B^* u^*$

The monomials for which $J^* u$ and $B^* u$ disagree are x^m, $m \geq 2$. In the case $m = 2$, we obtain $\iint (\alpha + \beta)\, dx\, dy < 0$, which agrees with the results obtained from the truncated Taylor expansion analysis. Supporting numerical evidence is given in Table 4.2, However, for large m, $\iint \alpha = -0(1)$ and $\iint \beta = 0(m)$ so that the analysis becomes inconclusive.

Table 4,2 RRG Errors in $\overset{0}{H}_1$ Semi-Norm

J*u is the exact bilinear corresponding to (4.7) and B*u its discretization in (4.8). The same four model problems are used.

Problem 1 / Problem 2

h	B*u	J*u	B*u	J*u
$\frac{1}{2}$	0.31372	0.20391	0.36198	0.27907
$\frac{1}{4}$	0.15616	0.12665	0.19811	0.16065
$\frac{1}{8}$	0.078001	0.070180	0.09938	0.08793
$\frac{1}{16}$	0.038991	0.036953	0.04972	0.04649

Problem 3 / Problem 4

h	B*u	J* u	B*u	J* u
$\frac{1}{2}$	0.31187	0.21452	0.73338	0.50324
$\frac{1}{4}$	0.15566	0.13115	0.38091	0.28547
$\frac{1}{8}$	0.07780	0.07146	0.19098	0.16192
$\frac{1}{16}$	0.03889	0.03726	0.09552	0.08752

5. Conclusions

(i) Exact matching usually produces better numerical results in the energy semi-norm. If this semi-norm is made into a norm, the exact matching technique would be even better.

(ii) Exact matching need not be better, in the L_2 norm.

(iii) Interpolation remainder theory analysis in the energy semi—norm is suggestive but not conclusive.

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