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An $O(h^6)$ cubic spline interpolating procedure for harmonic functions

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ABSTRACT

An $O(h^6)$ method for the interpolation of harmonic functions in rectangular domains is described and analysed. The method is based on an earlier cubic spline technique [7], and makes use of recent results concerning the a posteriori correction of interpolatory cubic splines.

I. INTRODUCTION

This paper is concerned with a numerical method for the solution of harmonic boundary value problems in rectangular domains of the form

$$R := \{(x, y) : 0 < x < a \ , \quad 0 < y < b\}, \qquad a, b \in \mathbb{N} .$$

The method is based on an earlier cubic spline technique which was described, but not analyzed, by Papamichael and Whiteman [7] and which involves the following:

(a) Determining $O(h^6)$ finite-difference approximations to the solution of the harmonic problem at the grid points of a uniform mesh, of size h, covering the rectangle R.

(b) Determining approximations to the solution at any point in R by an interpolation procedure involving the construction of univariate cubic splines along lines parallel to the sides of R. This procedure of interpolation is characterised by the property that the parameters of the splines are given directly in terms of the finite-difference approximations at the grid points.

The main objectives of the present paper are as follows:

(i) To analyse the method of Papamichael and Whiteman [7] and to show that it gives $O(h^4)$ approximations to the solution of the harmonic boundary value problem at intermediate (non-nodal) points of R.

(ii) To describe a simple technique which can be used to "correct" the cubic splines of [7], thus providing $O(h^6)$ approximations at intermediate points of R.

(iii) To show that the above correction technique can also be used to provide accurate approximations to the derivatives of the harmonic function at any point of R.

We note that interpolation procedures of the type studied in the present paper have applications in connection with the numerical solution of elliptic boundary value problems by conformal transformation methods. In fact, the method of [7] was developed for use in conjunction with a conformal transformation method for the solution of harmonic boundary value problems defined in general simply- connected domains; see e.g. [4] and [6].

II. NOTATIONS AND PRELIMINARY RESULTS

The following notation will be used throughout the paper,

(i) R will denote a rectangle of the form

$$R := \{(x;,y) : 0 < x < a \ , \quad 0 < y < b\}$$

where a, b $\in$ IN .

(ii) Let n, m $\in$ IN be such that $a/n = b/m := h$. Then, Π_h will denote the square mesh

$$\Pi_h = \{(x,y) : x_i = ih;\ i = 0(1)n,\ y_j = jh;\ j = 0(1)m\},$$

covering $\overline{R} = R \cup \partial R$.

(iii) $\Pi_x^{\{j\}}$ and $\Pi_y^{\{i\}}$ will denote respectively the sets of mesh points on the j^{th} mesh row , $j = 0(1)m$, and i^{th} mesh column , $i = 0(1)n$, of Π_h .That is,

$$\Pi_x^{\{i\}} := \{\{x,y) : x = x_i;\ i = 0(1)n,\ y = y_i\},$$

and

$$\Pi_y^{\{i\}} : := \{\{x,y) : x = x_i ,y = y_j ; j = 0(1)m\}.$$

(iv) $PC\left(\Pi_x^{\{j\}}\right)$ will denote the class of all univariate piecewise cubic polynomials defined on the $j^{\underline{th}}$ mesh row , $j = 0(1)m$, of $\Box_h$ and having knots $\Pi_x^{\{j\}}$ Similarly, $PC\ \left(\Pi_y^{\{j\}}\right)$ will denote the corresponding class associated with the knots $\Pi_y^{\{j\}}$ of the $i^{\underline{th}}$ column , $i = 0(1)n$, of Π_h .

(v) $Spl\left(\Pi_x^{\{j\}}\right)$ will denote the class of all univariate smooth cubic splines defined on the $j^{\underline{th}}$ mesh row , $j = 0(1)m$, of Π_h and having knots $\Pi_x^{\{j\}}$. That is ,

$$Spl\left(\Pi_x^{\{j\}}\right) := PC\left(\Pi_x^{\{j\}}\right) | \cap C^2[0,a].$$

Similarly,

$$Spl\left(\Pi_y^{\{j\}}\right) \backslash := PC\left(\Pi_y^{\{j\}}\right) \ \cap C^2\,[0,b].$$

(vi) δ_x, δ_y will denote the usual central difference operators in the x,y directions,

(vii) Δ will denote the Laplacian operator

$$\Delta : = \partial^2 / \partial x^2 + \partial^2 / \partial y^2.$$

(viii) Δ_h will denote the well-known nine-point difference operator for Laplacian problems, i.e.

$$\begin{aligned} \Delta_h v(x,y) := {} & v(x - h, y + h) + 4v(x,y + h) + v(x + h,y + h) \\ & + 4v(x - h,y) - 20v(x,y) + 4v(x + h,y) \\ & + v\{x-h,y-h) + 4v(x,y - h) + v(x + h,y - h). \end{aligned} \tag{2.1}$$

(ix) C^N (R) will denote the class of all functions $v(x,y)$ whose derivatives $v^{(k,l)} := \partial^{k+1} v/\partial x^k \partial y^l$ are continuous in R for $0 \le k + l \le N$.

Lemma 2.1. Let the values $v_{i,j} := v(x_i,y_j)$, $(x_i,y_j) \in \Pi_h$, be given and, for any fixed j , $j = 1(1)m-1$, let $\sigma_j(x) \in PC\left(\Pi_x^{\{j\}}\right)$ be defined by the following conditions:

$$\sigma_i(\{x_i) = v_{,j} \; ; \quad i=0(1)n, \tag{2.2a}$$

and

$$\sigma_j^{(2)}(x_i) = -\frac{1}{h^2}\delta_v^2 v_{i,j} := \mu_{i,j} \; ; \; i = 0(1)n. \tag{2.2b}$$

Then, $\sigma_i(x) \in Spl\left(\Pi_x^{\{j\}}\right)$, provided that

$$\Delta_h v_{i,j} = 0 \quad ; \quad i = 1(1)n-1, \tag{2.3}$$

where Δ_h is the nine-point difference operator (2.1).

Proof. For $x \in [x_i, x_{i+1}]$, $i = 0(1)n - 1$,

$$\begin{aligned}\sigma_j(x) &= \frac{1}{6h}\mu_{i,j}(x_{i+1}-x)^3 + \frac{1}{6h}\mu_{i+1,j}(x-x_i)^3 \\ &+ \frac{1}{h}\{v_{i,j} - \frac{h^2}{6}\mu_{i,j}\}(x_{i+1}-x) \\ &+ \frac{1}{h}\{v_{i+1+j} - \frac{h^2}{6}\mu_{i+1,j}\}(x-x_i) \\ &:= C_j\{x; x_i, v_{i,j}, \mu_{i,j}\}.\end{aligned} \tag{2.4}$$

Clearly, $\sigma_j,(x_i—) = \sigma_j(x_i+)$ and $\sigma_j^{(2)}(x_i—) = \sigma_j^{(2)} x_i+)$; $i = 1(1)n - 1$. Therefore, $\sigma_j(x) \in Spl\left(\Pi_y^{\{j\}}\right)$, provided that $\sigma_j^{(1)}(x_i—) = \sigma_j^{(1)}(x_i+)$; $i = 1(1)n — 1$, i.e. provided that

$$\mu_{i-1,j} + 4\mu_{i,j} + \mu_{i+1,j} = \frac{6}{h^2}\delta_x^2 v_{i,j} \; ; \; i = 1(1)n-1 \; ; \tag{2.5}$$

see e.g.[l:p.l0]. The elimination of the $\mu_{i,j}$ in (2.5) by means of (2.2b) then gives the finite-difference equations (2.3). □

Remark 2.1. By reversing the roles of x and y we have the following:

For any fixed i , i = 1(1)n — 1 , let $\tau_i(y) \in PC\left(\Pi_y^{\{j\}}\right)$ be defined by the following conditions :

$$\tau_i(y_j) = v_{i,j} \quad ; \quad j = 0(1)\,m,$$

$$\tau_i^{(2)}(y_j) = -\frac{1}{h^2}\delta_x^2 v_{i,j} \quad ; \quad j = 0(1)m.$$

Then, $\tau_i(y) \in \text{-}Spl\left(\Pi_y^{\{i\}}\right)$ provided that

$$\Delta_h v_{i,j} = 0 \quad ; \quad j = 1(1)m - 1. \square$$

Let u(x,y) denote the solution of a harmonic Dirichlet problem of the form

$$\Delta u(x,y) = 0 \quad , \quad \text{in } R, \tag{2.6a}$$

$$u(x,y) = f(x,y) \quad , \quad \text{on } \partial R, \tag{2.6b}$$

and assume that $f\{x,y)$ is sufficiently smooth so that $u(x,y) \in C^N(\overline{R})$ with $N \geq 6$. Also, for any fixed j , j = 1(1)m — 1, let $s_j(x)$ be the unique cubic spline satisfying the following conditions:

$$s_j(x_i) = u_{i,j} \quad ; \quad i = 0(1)n, \tag{2.7a}$$

and

$$S_j^{(2)}(x_i) = u_{i,j}^{(2,0)} - \frac{h^2}{12}u_{i,j}^{(4,0)} \; ; i = 0, n. \tag{2.7b}$$

That is, the cubic spline $S_j(x)$ interpolates the function $u(x,y_i)$ at the mesh points of the $j^{\underline{th}}$ row of Π_h and satisfies the two end-conditions (2.7b).

Lemma 2.2. Let $u\{x,y)$ be the solution of (2.6), and let $s_j(x) \in Spl\left(\Pi_x^{\{j\}}\right)$ be defined by the conditions (2.7). If $u(x,y) \in C^6(\overline{R})$ then, for any j, j=1(1)m-1,

$$\sup_{x\in|0,\alpha|} | s_j^{(k)}(x) - u^{(k,0)}(x, y_i) |= O(h^{4-k}) \; , \; 0 \le k \le 3, \tag{2.8}$$

$$\left.\begin{aligned} S_j^{(1)}(x_i) &= u_{i,j}^{(1,0)} - \frac{h^4}{180} u_{i.j}^{(5,0)} + O(h^5), \\ S_j^{(2)}(x_i) &= u_{i,j}^{(2,0)} - \frac{h^2}{12} u_{i,j}^{(4,0)} + O(h^4), \end{aligned}\right\} i = 0(1)n, \tag{2.9}$$

and

$$S_j^{(3)}(x_i \pm) = u_{i,j}^{(3,0)} \pm \frac{h}{2} u_{i,j}^{(4,0)} + \frac{h^2}{12} u_{i,j}^{(5,0)} + O(h^3), \tag{2.10}$$

where in (2.10) the subscript i ranges from 0 to n—1for x_i+, and from 1 to n for x_i—.

Proof. This follows from the results of [3] and [5], by observing that the end conditions (2.76) are of order p = 4 in the sense of Definition 2.1 of [5:p.491]. □

Lemma 2.3. Let u(x,y) and $s_i(x)$ be as in Lemma 2.2. If $u(x,y) \in C^6(\overline{R})$, then for any j , j = 1(1)n - 1,

$$s_j^{(2)}(x_i) = -\frac{1}{h^2} \delta_y^2 u_{i,j} + O(h^4) \quad ; \quad i = 1(1)n-1. \tag{2.11}$$

proof. By Taylor's series expansion,

$$\frac{1}{h^2}\delta_y^2 u_{i,j} = u_{i,j}^{(0,2)} + \frac{h^2}{12} u_{i,j}^{(0,4)} + O(h^4).$$

Therefore , form (2.9),

$$S_j^{(2)}(x_i) + \frac{1}{h^2}\delta_v^2 u_{i,j} = (u_{i,j}^{(2,0)} + u_{i,j}^{(0,2)}) - \frac{h^2}{12}(u_{i,j}^{(4,0)} - u_{i,j}^{(0,4)}) + O(h^4)$$

$$= O(h^4) \;\; ; \quad i = 1(1)n\text{-}1, j = 1(1)\, m - 1,$$

because $\Delta u := u^{(2,0)} + u^{(0,2)} = 0$ in R. □

Let $U_{i,j}$, i = 1(1)n -1 , j = 1(1)m -1, denote the finite-difference approximations to $u_{i,j} := u(x_i, y_j)$ obtained by applying the nine-point formula (2.3) at the interior mesh points of Π_h That is, the values $U_{i,j}$ are obtained by solving the linear system generated by the finite-difference equations

$$\Delta_h U_{i,j}=0 \quad ; \quad i = 1(1)n-1, j = 1(1)m-1, \tag{2.12a}$$

with

$$U_{o,j} = f_{o,j} \quad , \quad U_{n,i} = f_{n,j} \quad ; \quad j = 0(1)m, \tag{2.12b}$$

$$U_{i,o} = f_{i,o} \quad , \quad U_{i,m} = f_{i,m} \quad ; \quad i = 1(1)n-1. \tag{2.12c}$$

Then, the following result is well-known.

Lemma 2.4. If $u(x,y) \in C^{8}(\overline{R})$, then

$$U_{i,j} - u_{i,j} = 0(h^6) \quad ; \quad i = 1(1)n-1, j = 1(1)m-1. \tag{2.13}$$

Proof. See e.g. [8], □

III. CUBIC SPLINE INTERPOLATION ON THE MESH LINES OF π_h

In this and in the following sections we describe and analyse a simple method for computing $O(h^6)$ approximations to the solution $u(x,y)$ of (2.6) at any point (x, y) $\in R$. The method is based on constructing univariate cubic splines along lines parallel to the sides of R, and is characterized by the property that the parameters of these splines are given directly in terms of the finite-difference approximations $U_{i,j}$ corresponding to

(2.12). In this section we consider only the problem of approximating u(x,y) on the mesh lines of Π_h. The general approximating procedure is described in Section 4.

For any $j, j = 1(1)m - 1$, let $S_j(x) \in PC\left(\Pi_x^{\{j\}}\right)$ be defined by the following conditions:

$$S_i(x_i) = U_{i,j}\ ; \quad i = 0(1)n, \tag{3.1a}$$

$$S_j^{(2)}(x_i) = -\frac{1}{h^2}\delta_V^2 U_{i,j}\ ; \ i = 1(1)n-1, \tag{3.1b}$$

$$S_j^{(2)}(x_0) = \frac{6}{h^2}\delta_X^2 U_{1,j} - 4S_j^{(2)}(x_1) - S_j^{(2)}(x_2), \tag{3.1c}$$

and

$$S_j^{(2)}(x_n) = \frac{6}{h^2}\delta_X^2 U_{n-1,j} - 4S_j^{(2)}(x_{n-1}) - S_j^{(2)}(x_{n-2}). \tag{3.1d}$$

Thus is,

$$S_j(x) = C_j\{x\ ; x_i\ , U_{i,j}\ , M_{i,j}\} \quad , \quad x \in [x_i, x_{i+1}]\ ; \ i = 0(1)n - 1, \tag{3.2}$$

where the parameters $M_{i,j}$; $i = 1(1)n — 1$, $M_{0,i}$ and $M_{n,j}$ are given respectively by the expressions in the right hand sides of (3.16)-(3.1d); see Eq.(2.4). □

Lemma 3.1. For any j , $j = 1(1)m — 1$,

$$S_i(x) \in Spl\left(\Pi_x^{\{j\}}\right),$$

i.e. S_i (x) is a smooth cubic spline interpolating the values $U_{i,j}$ at the mesh points of the j^{th} row oh Π_h .

Proof. We need only show that $S_j^{(1)}(x_i—) = S_j^{(1)}(x_i+)$, $i = 1(1)n - 1$. For $i = 2(1)n - 2$ this follows from Lemma 2.1, and for i =1,n - 1 from (3.1)- (3.1d), because these two equations are the consistency relations that establish the continuity of $S_j^{(1)}$ at x_i and x_{n-1} . □

Remark 3.1. The equations (3.1) define an approximating procedure where, at each interior mesh point of Π_h, the x-derivative in the Laplace equation is approximated by the spline derivative $S_j^{(2)}($ (x_i) and the y-derivative by the usual central difference replacement. □

Lemma 3.2. Let u(x,*y)* be the solution of (2.6), and let $s_j(x), S_j(x) \in Spl\left(\Pi_x^{\{j\}}\right)$ be defined by the conditions (2.7) and (3.1). If $u(x,y) \in C^8(R)$, then for any j, j = 1(1)m - 1,

$$S_j^{(k)}(x_i) - S_j^{(k)}(x_i) = \mathcal{O}(h^{6-k}) \quad ; \quad i = 0(1)n, 0 \le k \le 2, \tag{3.3}$$

and

$$S_j^{(3)}(x_i \pm) - s_j^{(3)}(x_i \pm) = O(h^3), \tag{3.4}$$

where in (3.4) the subscript i ranges from 0 to n — 1 for x_i+, and from 1 ton for x_i —.

Proof. The conditions (3.16) and the results of Lemmas 2.3, 2.4 imply that

$$S_j^{(2)}(x_i) - s_j^{(2)}(x_i) = -\frac{1}{h^2}\delta_y^2(U_{i,j} - U_{i,j}) + O(h)^4 = O(h^4) \;\; ; \; i = 1(1)n - 1,$$

and these together with (3.1c)-(3.1d) and the two corresponding consistency relations for the cubic spline s_j (x) give that

$$S_j^{(2)}(x_0) - s_j^{(2)}(x_0) = O(h^4),$$

$$S_j^{(2)}(x_n) - s_j^{(2)}(x_n) = O(h^4).$$

Thus $S_j^{(2)}(x_i) - s_j^{(2)}(x_i) = \mathcal{O}(h^4)$, i =0(1)n. The other results follow easily from standard cubic spline identities. □

Lemma 3.3. Let *u{x,y)* and $S_i(x)$ be as in Lemma 3.2. If $u(x,y) \in C^8(R)(\overline{R})$, then for any j , j = 1(1)m - 1 ,

$$\sup_{x\in|0,\alpha|} |s_j^{(k)}(x) - u^{(k,0)}(x,y_j)| = \mathcal{O}(h^{4-k}), \quad 0 \le k \le 3, \tag{3.5}$$

$$\left.\begin{aligned} S_j^{(1)}(x_i) &= u_{i,j}^{(1,0)} - \frac{h^4}{180} u_{i,j}^{(5,0)} + \mathcal{O}(h^5), \\ S_j^{(2)}(x_i) &= u_{i,j}^{(2,0)} - \frac{h^2}{12} u_{i,j}^{(4,0)} + \mathcal{O}(h^4), \end{aligned}\right\} i = 0(1)n, \tag{3.6}$$

and

$$S_j^{(3)}(x_i \pm) = u_{i,j}^{(3,0)} \pm \frac{h}{2} u_{i,j}^{(4,0)} + \frac{h^2}{12} u_{i,j}^{(5,0)} + O(h^3), \tag{3.7}$$

where in (3.7) the subscript i ranges from 0 to n — 1 for x_i+, and from 1 ton for x_i — .

Proof. At once from (2.8) – (2.10) and (3.3) - (3.4). □

Once the finite-difference approximations $U_{i,j}$ are computed, the cubic splines $S_j(x)$, j — 1(1)m — 1, can be used to approximate u(x, y) at intermediate points on the mesh rows of Π_h,. Unfortunately, whilst the values $U_{i,j}$ are $O(h^4)$ approximations, the intermediate solutions given by the splines $S_i(x)$ are only $O(h^4)$; see (3.5). However, because of the asymptotic expansions (3.6)-(3.7), it is possible to "*correct*" the splines $S_j(x)$ and thus obtain $O(h^6)$ approximations to *u(x,y)* at any point on a mesh row of Π_h, This can be done by using the "*a posteriori correction technique*" of Lucas [3], in which the corrected approximations are given by the piecewise quintic polynomial defined by Eqs (3.8)-(3.10) below; see also [5].

For any j, $j = 1(1)m - 1$, let $S_j(x) \in Spl\left(\Pi_x^{\{j\}}\right)$ be defined by the conditions (3.1), and let $\hat{S}_j(x)$ denote the piecewise defined quintic polynomial

$$\hat{S}_j(x) := S_j(x) + \frac{h^4}{4!} P_0\{(x - x_i)/h\} U_{i,j}^{(4,0)} + \frac{h^5}{5!} P_1\{(x - x_i)/h\} U_{i,j}^{(5,0)},$$

$$x \in [x_i, x_i + 1] \quad ; i = 0(1)n - 1, \tag{3.8}$$

where P_0, P_1, are the polynomials

$$P_0(\xi) := \xi^4 - 2\xi^3 + \xi^2 \qquad P_1(\xi) := \xi^5 - \frac{5}{3}\xi^3 + \frac{2}{3}\xi \ , \tag{3.9}$$

and $U_{i,j}^{(4,0)}$, $U_{i,j}^{(5,0)}$ and approximations to the derivatives $U_{i,j}^{(4,0)}$ and, $U_{i,j}^{(5,0)}$ of $u(x,y)$ These approximations are given by linear combinations of the spline derivatives $S_j^{(2)}(x_i)$ as follows:

$$U_{i,j}^{(4,0)} := \frac{1}{h^2}\{S_j^{(2)}(x_{i-1}) - 2S_j^{(2)}(x_i) + S_j^{(2)}(x_{i+1})\} ;$$
$$i = 2(1)n - 2, \tag{3.10a}$$

$$U_{i,j}^{(4,0)} := \frac{1}{h^2}\{2S_j^{(2)}(x_i) - 5S_j^{(2)}(x_{i+1}) + 4S_j^{(2)}(x_{i+2}) - S_j^{(2)}(x_{i+3})\};$$
$$i = 0,1, \tag{3.10b}$$

$$U_{n-1,j}^{(4,0)} := \frac{1}{h^2}\{-S_j^{(2)}(X_{n-4}) + 4S_j^{(2)}(x_{n-3}) - 5S_j^{(2)}(xn - 2) + 2S_j^{(2)}(x_{n-1})\}, \tag{3.10c}$$

and

$$U_{i,j}^{(5,0)} := \frac{1}{h}\{U_{i+1+j}^{(4,0)} - U_{i,j}^{(4,0)}\} \ ; \ i = 0(1)n - 2, \tag{3.10d}$$

$$U_{n-1,j}^{(5,0)} := \frac{1}{h}\{U_{n-1,j}^{(4,0)} - U_{n-2,j}^{(4,0)}\} . \tag{3.10e}$$

Lemma 3.4 below concerns the quality of the derivative approximations (3.10).

Lemma 3.4. Let u(x,y) and $S_j(x)$ be as in Lemma 3.2, and let $U_{i,j}^{(4,0)}.U_{i,j}^{(5,0)}$ be given by (3.10). If $u(x,y) \in C^8(\overline{R})$ then for any j, $j = 1(1)m - 1$,

$$\left.\begin{aligned} u_{i,j}^{(4,0)} &= U_{i,j}^{(4,0)} + O(h^2), \\ u_{i,j}^{(5,0)} &= U_{i,j}^{(5,0)} + O(h), \end{aligned}\right\} \quad i = 0(1)n. \tag{3.11}$$

Proof . Since

$$S_j^{(3)}(x_i+) - S_j^{(3)}(x_i-) = \frac{1}{h}\{S_j^{(2)}(x_{i-1}) - 2S_j^{(2)}(x_i) + S_j^{(2)}(x_{i+1})\},$$

(3.7) implies that

$$u_{i,j}^{(4,0)} = U_{i,j}^{(5,0)} + O(h^2) \;\; ; \; i = 2(1)n - 2.$$

The other results in (3.11) then follow by elementary finite-difference arguments; see e.g. [2], [3] and [5]. □

We can now state the main result of this section as follows.

Theorem 3.1. Let $u(x,y)$ be the solution of (2.6), let $S_j(x) \in Spl\left(\Pi_x^{\{j\}}\right)$ be defined by the conditions (3.1), and let $\hat{S}_j(x)$ denote the piecewise defined quintic polynomial (3.8)-(3.10). If $u(x,y) \in C^8(\overline{R})$,, then for any j ,j = 1(1)m - 1,

$$\sup_{x \in |0,a|} |\hat{S}_j^{(k)}(x) - u^{(k,o)}(x, y_i)| = O(h^{6-k}) \;,\; 0 \le k \le 3. \tag{3.12}$$

Proof. Let $x = x_t + \mu h$, $0 \le \mu, \le 1$, and in the Taylor series expansion of $u^{(k,0)}(x_i + \mu h, y_j) - S_j^{(k)}(x_i + \mu h)$ use (3.6) and (3.7) to express the derivatives

$u_{i,j}^{(r,0)} - S_j^{(r)}(x_i)$; r = 1,2,3, in terms of $u_{i,j}^{(4,0)}$ and $u_{i,j}^{(5,0)}$ The result then follows

from (3.11),by replacing the derivatives $u_{i,j}^{(4,0)}$ and $u_{i,j}^{(5,0)}$ by the approximations $U_{i,j}^{(4,0)}$

and $U_{i,j}^{(5,0)}$.□

Remark 3.2. By reversing the roles of x and y we have the following:

For any i , i = 1(1)m — 1, let $T_i(y)$ denote the y-counterpart of $S_i(x)$. That is,

$$Ti(y) := C_i \{y ; y_i , U_{i,j} , N_{i,j}\} ,$$

$$y \in [y_i, y_{i+1}] ; j = 0(1)m - 1, \tag{3.13}$$

where the parameters $N_{i,j} := T_i^{(2)}(y_j)$ are given by

$$N_{i,j} := \frac{1}{h^2}\delta_x^2 U_{i,j} \; ; \; j = 1(1)m - 1, \tag{3.14a}$$

$$N_{i,0} := \frac{6}{h^2}\delta_y^2 U_{i,1} - 4N_{i,1} - N_{i,2}, \tag{3.14b}$$

and

$$N_{i,m} := \frac{6}{h^2}\delta_y^2 U_{i,m-1} - 4N_{i,m-1} - N_{i,m-2} \; ; \tag{3,14c}$$

see Eqs (2.4), (3.1)and(3.2). Then,$T_i(y) \in Spl\left(\Pi_y^{\{i\}}\right)$and for $u[x,y) \in C^8(\overline{R})$

$$\sup_{y \in |0,b|} | T_i^{(l)}(y) - u^{(0,l)}(x_i, y) | = O(h^{4-l}) \; , \; 0 \le l \le 3. \tag{3.15}$$

Also, let $\hat{T}_i$ (y) denote the y-counterpart of $\hat{S}_j$ *(x)*. That is,

$$\hat{T}_i(y) := T_i(y) + \frac{h^4}{4!} P_0\{(y - y_i)/h\} U_{i,j}^{(0,4)}$$

$$+ \frac{h^5}{5!} P_1\{(y - y_j)/h\} U_{i,j}^{(0,5)} \tag{3.16}$$

where P_0, P_1 are the polynomials (3.9), and

$$U_{i,j}^{(0,4)} := \frac{1}{h^2}\{T_i^{(2)}(y_{i-1}) - 2T_i^{(2)}(y_i) + T_i^{(2)}(y_{i+1})\};$$

$$j=2(1)m-2, \tag{3.17}$$

e.t.c. ; see Eqs (3.8)-(3.10). Then, for $u(x,y) \in C^8(\overline{R})$

$$\sup_{y \in |0,b|} | \hat{T}_i^{(l)}(y) - u^{(0,l)}(x_i, y) | = O(h^{6-l}) \; , \; 0 \le l \le 3. \; \square \tag{3.18}$$

We end this section by observing that the corrected approximations given by $\hat{S}_j(x)$ and $\hat{T}_i(y)$ satisfy the nine-point formula (2.3). The precise result is as follows:

Theorem 3.2. Let $\hat{S}_j(x)$ and $\hat{T}_i(y)$ denote the piecewise defined quintic polynomials (3.8) and (3.16), and for $0 \le \mu, v \le 1$ let:

$$U_{i+\mu,j} := \hat{S}_j(x_i + \mu h) \quad ; \quad i = 0(1)n-1, j = 1(1)m-1, \tag{3.19a}$$

and

$$U_{i,j+v} := \hat{T}_i(y_i + vh) \quad ; \quad i = 1(1)n-1, j = 0(1)m-1. \tag{3.19b}$$

Then,

$$\Delta_h U_{i+\mu,v} = 0 \quad ; \quad i = 1(1)n-2, j = 2(1)m-2, \tag{3.20a}$$

and

$$\Delta_h U_{i,j+v} = 0 \quad ; \quad i = 2(1)n-2, j = 1(1)m-2. \tag{3.20b}$$

Proof. This follows easily from the formulae giving $\hat{S}_j(x)$ and $\hat{T}_i(y)$, by recalling (2.12a) and observing that

$$\Delta_h S_j^{(2)}(x_i) = -\frac{1}{h^2}\{\delta_y^2 U_{i,j}\} = 0 \quad ; \quad i = 1(1)n-1,$$

$$j = 2(1)m-2,$$

and

$$\Delta_h T_i^{(2)}(y_i) = -\frac{1}{h^2}\Delta_h\{\delta_x^2 U_{i,j}\} = 0 \ ; \ i = 2(1)n-2,$$

$$j=1(1)m-1. \square$$

IV. CUBIC SPLINE INTERPOLATION IN *R*

Let $0 \leq \mu, v \leq 1$, and let $\Pi_x^{\{j=v\}}$ and $\Pi_y^{\{i+\mu\}}$ denote respectively the uniform partitions of the lines $y = y_i + vh$ and $x = x_i + \mu h$, corresponding to the partitions $\Pi_x^{\{j\}}$ and $\Pi_y^{\{j\}}$. That is, for any j, $j = 0(1)m — 1$,

$$\Pi_x^{\{j+v\}} := \{\{x,y) : x = x_i ; \quad i = 0(1)n, y = y_i + vh\}, \tag{4.1a}$$

and for any i , i = 0(1)n — 1,

$$\Pi_y^{\{i+\mu\}} := \{(x,y) : x = x_i + \mu h , y = y_j ; j = 0(1) m\}. \tag{4.16}$$

Also, as in Theorem 3.2, let

$$U_{i+\mu,j} := \hat{S}_j(x_i + \mu h) \quad ; \quad i = 0(1)n-1, y = 1(1)m-1, \tag{4.2.a}$$

$$U_{i,j+v} := \hat{T}_i(y_i + vh) \quad ; \quad i = 1(1)n-1, j = 0(1)m-1, \tag{4.2b}$$

and for any j, j = 1(1)m — 1, let $S_{j+v}(x) \in PC\Pi_x^{\{j+v\}}$: be defined by the conditions (3.1) with the subscript j replaced by $j + v$. That is,

$$S_{j+v}(x) := C_{j+v}\{x : x_i, Ui, j+v, Mi,j+v\},$$

$$x \in [x_i, x_{i+1}] ; i = 0(1)n-1, \tag{4.3}$$

where the parameters $M_{i,j+v}$ are given by the right hand sides of (3.16)-(3.1d) with the subscript j replaced by $j + v$. Similarly, for any i , i = 1(1)n — 1, let $T_{i+\mu}(y) \in PC\,\Pi_y^{\{i+\mu\}}$: be defined by

$$T_{i+\mu}(y) := C_{i+\mu}\{y ; y_i, U_{i+j}, N_{i+\mu,j}\},$$

$$y \in [y_i, y_{i+1}] ; y = 0(1)m-1, \tag{4.4}$$

where the parameters $N_{i+\mu,j}$ are given by (3.14) with the subscript t replaced by $i + \mu$. Then, because of Theorem 3.2, a trivial modification of the proof of Lemma 3.1 shows that

$$S_{i+v}\{x) \in Spl\ \Pi_x^{\{j+v\}} :; \quad j = 1(1)m-2,$$

and

$$T_{i+\mu}(y) \in Spl\ \Pi_y^{\{i+\mu\}} : \ ; \quad i = 1(1)n-2.$$

In other words, for any j, $j = 1(1)m — 2$, $S_{i+v}(x)$ is a smooth cubic spline defined on the line $y = y_i + vh$ and interpolating the corrected approximations $U_{i,j+v}$ at the knots (4.1a). Similarly, for any i, $i = 1(1)n — 2$, $T_{i+\mu}(y)$ is a smooth cubic spline defined on $x — x_i + \mu h$ and interpolating the values $U_{i+\mu,j}$ at the knots (4.16). Since, for $u(x,y) \in C^8(\overline{R})$,

$$U_{i+\mu,j} - u_{i+\mu,j} = O(h^6) \quad ; \quad i = 0(1)\ n-1,\ j = 1(1)\ m-1,$$

and

$$U_{i,j+v} - u_{i,j+v} = O(h^6) \quad ; \quad i = 1(1)\ n-1,\ j = 0(1)m-1,$$

the theorem below can be established easily, by modifying in an obvious manner the analysis of Section 2.

Theorem 4.1. Let $u(x,y)$ be the solution of (2.6) and, for $0 \leq \mu, v \leq 1$, let $S_{i+v}\{x) \in Spl\left(\Pi_x^{\{j+v\}}\right)$ and $T_{i+\mu}(y) \in Spl\ \Pi_y^{\{i+\mu\}}$ be given, as described above, by (4.3) and (4.4). Also, let $\hat{S}_{j+v}(x)$ and $\hat{T}_{i+\mu}(y)$ denote the corrected piecewise quintic polynomials corresponding to $S_{i+v}(x)$ and $T_{i+\mu}(y)$. (That is, $\hat{S}_{j+v}(x)$ and $\hat{T}_{i+\mu}(y)$ are given respectively by (3.8)-(3.10) with the subscript j replaced by $j + v$, and by (3.16), (3.17) with the subscript i replaced by $i + \mu$.) If $u(x,y) \in C^3(\overline{R})$, then:

(i) For $0 \le k \le 3$ and any j, j = 1(1)m-2,

$$\sup_{x\in|0,\alpha|} |S^{(k)}_{j+v}(x) - u^{(k,0)}(x, yj + vh)| = O(h^{4-k}), \qquad (4.5a)$$

and

$$\sup_{x\in|0,\alpha|} |\hat{S}^{(k)}_{j+v}(x) - u^{(k,0)}(x, y_{i+vh})| = O(h^{6-k}) \qquad (4.5b)$$

(ii) For $0 \le 1 \le 3$ and any i , i =1(1)n-2,

$$\sup_{y\in|0,b|} |\hat{T}^{(l)}_{i+\mu}(y) - u^{(0,1)}(xi + \mu h, y)| = O(h^{4-1}). \qquad (4.6a)$$

and

$$\sup_{x\in|0,\alpha|} |\hat{T}^{(k)}_{j+\mu}(y) - u^{(0,1)}(x_i + \mu h, y)| = O(h^{6-1}) \quad \square \qquad (4.6b)$$

Assume that the finite-difference approximations $U_{i,j}$, i = 1(1)n—1, j = 1(1)m—1, have been computed by solving the linear system (2.12). Then, because of (4.5b) and (4.66), the corrected piecewise quintic polynomials $\hat{S}_{i+v}(x)$ and $\hat{T}_{i+\mu}(y)$ can be used to compute $O(h^6)$ approximations $U_{i+\mu, j+v}$ to $u_{i+\mu, j+v} := u(x_i + \mu h, y_i + vh)$ at any point

$$(x,y) := (x_i + \mu h, y_i + vh) \quad , \quad (x_i, y_i) \in \Pi_h \quad , \quad 0 \le m, v \le 1,$$

in R. This can be done as follows:

(I) If $i \in \{0,1,..., n-1\}$ and $j \in \{1,2,... ,m-2\}$, then $U_{i+\mu,j+v}$ can be computed from $\hat{S}_{j+v}(x)$, i.e. by taking

$$U_{i+\mu,j+v} := \hat{S}_{j+v}(x_i + \mu h). \qquad (4.7)$$

(II) If $i \in \{1,2,..., n-2\}$ and $j \in \{0, m — 1\}$, then $U_{i+\mu,j+v}$ can be computed from $\hat{T}_{i+\mu}\{y)$, i.e. by taking

$$U_{i+\mu,j+v} := \hat{T}_{i+\mu}(y_i + vh). \qquad (4.8)$$

(More generally, (4.8) can be used for $i \in \{1, 2, ..., n-2\}$ and $j \in \{0, 1, ..., m-1\}$.)

(III) If $i \in \{0, n-1\}$ and $j \in \{0, m-1\}$, let $p := i + \mu$ and $q := j + \nu$. Then, by using the nine-point formula, $U_{p,q} := U_{i+\mu, j+\nu}$, can be computed from

$$U_{p,q} := 20U_{p+\alpha, q+\beta} - (U_{p+2\alpha, q} + U_{p,q+2\beta} + U_{p+2\alpha, q+2\beta})$$

$$-4(U_{p+\alpha, q} + U_{p,q+\beta} + U_{p+2\alpha, q+\beta} + U_{p+\alpha, q+2\beta}), \tag{4.9}$$

where $\alpha = 1$ when $i = 0$, $\alpha = -1$ when $i = n-1$, $\beta = 1$ when $j = 0$, $\beta = -1$ when $j = m-1$, and where the approximations in the right hand side are given by (4.7) and (4.8); see Remark 4.1 below.

Remark 4.1. With reference to (III), it is easy to see that $U_{p,q} - u_{p,q} = O(h^6)$. This follows because $\Delta_h U_{p+\alpha, q+\beta} = 0(h^6)$, and the values $U_{p+\alpha, q+\beta} U_{p+2\alpha, q}$ e.t.c. in the right hand side of (4.9) are $0(h^6)$ approximations to $u_{p+\alpha, q+\beta}, u_{p+2\alpha, q}$, e.t.c. □

Remark 4.2. The interpolation procedures (I)-(III) provide approximations to the solution of (2.6) at intermediate points in R directly in terms of the finite-difference approximations $U_{i,j}$. That is, once the linear system (2.12) is solved, the procedure does not require the solution of any other linear system. □

Remark 4.3. The interpolation procedures used in [7], differ from those described above mainly because the cubic splines of [7] are not corrected. That is, in [7] the approximations at intermediate points are computed by means of cubic splines $\tilde{S}_{j+\nu}(x)$ and $\tilde{T}_{i+\mu}(y)$ which interpolate the values

$$\tilde{U}_{i+\mu, j} := S_j(x_i + \mu h) \quad ; \quad i = 0(1)n-1, \ j = 1(1)m-1,$$

and

$$\tilde{U}_{i,j+v} := T_i(y_i + vh) \quad ; \quad I = 1(1)n - 1 , j = 0(1) m - 1,$$

rather than the corrected values (4.2). Thus, the procedures of [7] can only produce $O(h^4)$ approximations at intermediate ponts of R; see (3.5), (3.15), (4.5a) and (4.6a). □

Remark 4.4. Let $\tilde{S}_{j+v}(x)$ be the cubic spline of [7], referred to in Remark 4.3. Then, by writing $y = y_j + vh$, we can identify $S_{j+v}(x)$ as a piecewise bicubic polynomial $\tilde{Q}_i(x, y)$ defined in

$$\overline{R}_1 := \{ x, y) : 0 \le x \le a , h \le y \le b - h \}.$$

More precisely, with

$$T_o(y) := f(0,y) \quad , \quad T_n(y) := f(a, y),$$

$$M_i(y) := \frac{1}{h^2}(T_i(y-h) - 2T_i(y) + T_i(y+h) \; ; i = 1(1)n-1,$$

$$M_o(y) := \frac{6}{h^2}(T_o(y) - 2T_1(y) + TL(y)) - 4M_1(y) - M_2(y),$$

and

$$M_n(y) := \frac{6}{h^2}(T_{n-2}(y) - 2T_{n-1}(y) + T_n(y)) - 4M_{n-1}(y) - M_{n-2}(y),$$

we have that

$$\begin{aligned}\tilde{S}_{j+v}(x) &:= M_i(y)\frac{(x_{i+1}-x)^3}{6h} + M_{i+1}(y)\frac{(x-x_i)^3}{6h} \\ &+ (T_i(y) - \frac{h^2}{6}M_i(y))\frac{(x_{i+1}-x)}{h} \\ &+ (T_{i+1}(y) - \frac{h^2}{6}M_{i+1}(y)\frac{(x-x_i)}{h}, \\ &:= \tilde{Q}_1(x,y) \; ; \quad x_i \le x \le x_{i+1}, y_j \le y \le y_{i+1}.\end{aligned}$$

Furthermore, as was observed in [7], $\widetilde{Q}_1(x, y) \in C^{(2,2)}(R_1)$[1], i.e. the bicubic polynomial $\widetilde{Q}_1(x, y)$ is a bicubic spline. This follows easily from the continuity properties of the splines $T_i(y)$ and the fact that the values $\alpha_{i,j+v} := T_i^{(1)}(y)$ and $\beta_{i,j+v} := T_i^{(2)}(y)$ satisfy the nine-point formula (2.3). In exactly the same manner it can be shown that

$$\widetilde{Q}_2(x, y) := \widetilde{T}_{i+\mu}(y) \;;\; x := x_i + \mu h,$$

is a bicubic spline in

$$\overline{R}_2 = \{(x, y) : h \le x \le a - h,\ 0 \le y \le b\}. \square$$

Remark 4.5. We can of course also identify the functions $\hat{S}_{j+v}(x)$ and $\hat{T}_{i+\mu}(y)$, that give the corrected approximations, as piecewise biquintic polynomials $\hat{Q}_1(x, y) := \hat{S}_{j+v}(x)$; $x = x_i + \mu h$, and $\hat{Q}_2(x, y) := \hat{T}_{i+\mu}(y)$; $y = y_i + vh$, in the rectangles $\overline{R}_1$ and $\overline{R}_2$ respectively. Regarding continuity, we now only have that $\hat{Q}_p(x, y) \in C(\overline{R}_p)$, $p = 1,2$. It is however easy to show that the derivative jump discontinuities of $\hat{Q}_p(x, y)$ are *" small"* in the sense that

$$\hat{Q}_1^{(k,l)}(x, y_j+) - \hat{Q}_1^{(k,l)}(x, y_j) = O(h^{6-(k+l)}),$$
$$\hat{Q}_2^{(k,l)}(x_i+, y) - \hat{Q}_2^{(k,l)}(x_i-, y) = O(h^{6-(k+l)});$$
$$0 \le k, l \le 3. \square$$

Remark 4.6. From Theorem 4.1 we know that for any $(x,y) \in \overline{R}_1$,

$$|\hat{Q}_1^{(k,0)}(x, y) - u^{(k,0)}(x, y)| = O(h^{6-k}) \;;\; 0 \le k \le 3,$$

and, for any $(x, y) \in \overline{R}_2$,

[1] *$C^{(p,q)}(R)$ denotes the set of all functions defined on R for which the derivatives $\partial^{k+l}/\partial^k x \partial^l y$, $0 \le k \le p$, $0 \le l \le q$, exist and are continuous.*

$$|\hat{Q}_2^{(0,l)}(x,y) - u^{(0,l)}(x,y)| = O(h^{6-l}) \quad ; 0 \le l \le 3,$$

More generally , it is easy to show that for $(x,y) \in \overline{R}_1$,

$$|\hat{Q}_1^{(k,l)}(x,y) - u^{(k,l)}(x,y)| = O(h^{6-(k+l)}) \ ; \ 0 \le k,l \le 3,$$

and for $(x, y) \in \overline{R}_2$,

$$|\hat{Q}_2^{(k,l)}(x,y) - u^{(k,l)}(x,y)| = O(h^{6-(k+l)}) \ ; 0 \le k,l \le 3.$$

That is, $\hat{Q}_1^{(k,l)}(x,y)$, $\hat{Q}_2^{(k,l)}(x,y)$; $0 \le k,l \le 3$ can be used to provide $O(h^{6-(k+l)})$ approximations to $u^{(k,l)}(x,y)$ at any point $(x,y) \in \overline{R}_1 \cup \overline{R}_2$ For points $(x,y) \in R \setminus \{\overline{R}_1 \cup \overline{R}_2\}$ i.e. , for points $x :— x_i + \mu h$, $y := y_i + vh,$ $i \in \{0,n-1\}$, $j \in \{0,m-1\}$, $O(h^{6-(k+l)})$ approximations to $u^{(k,l)}(x,y)$ can be btained by modifying in an obvious manner the interpolation procedure (III) above, i,e. by applying the nine –point formula to values computed from $\hat{Q}_1^{(k,l)}(x,y)$ and $\hat{Q}_2^{(k,l)}(x,y)$.□

V. MIXED BOUNDARY VALUE PROBLEMS

In this section we illustrate the application of the interpolation technique of Section 4 to the solution of mixed boundary value problems and, more specifically, to the solution of boundary value problems satisfying a Dirichlet boundary condition on at least one side of the rectangle *R* and Neumann conditions on the remaining sides. We do this by considering a harmonic problem of the form

$$\Delta u(x,y) = 0, \qquad \text{in R,} \qquad (5.1a)$$

$$u^{(1,0)}(x,y) = g(y), \qquad \text{on } L := \{(0,y) : 0 < y < b\}, \qquad (5.1b)$$

$$u(x,y) = f(x,y), \qquad \text{on } \partial R \setminus L, \qquad (5.1c)$$

where g(y) and f(x,y) are sufficiently smooth functions defined on L and $\partial R \backslash L$, respectively. Then, as suggested by Rosser [8], it is possible to obtain $0(h^6)$ approximations $U_{i,j}$ to $u_{i,j}$ by solving the linear system generated by the finite-difference equations

$$\Delta_h U_{i,j} = 0 \;\; ; \quad I = 1(1)n-1, j = 1(1)m-1, \qquad (5.2a)$$

and

$$\frac{-137}{60} U_{0,j} + 5U_{1,j} - 5U_{2,j} + \frac{10}{3} U_{3,j} - \frac{5}{4} U_{4,j} + \frac{1}{5} U_{5,j} = hg(jh);$$

$$j = 1(1)m-1, \qquad (5.2b)$$

with

$$U_{n,j} = f_{n,j} \;\; ; \quad j = 1(1)m-1, \qquad (5.2c)$$

$$U_{i,0} = f_{i,o} \;\; ; \quad U_{i,n} = f_{i,n} \;\; ; \quad i = 0(1)n. \qquad (5.2d)$$

Once the finite-difference approximations $U_{i,j}$, i = 0(1)m - 1, j = 1(1)m - 1, are computed, the three procedures of Section 4 can be applied directly to provide $O(h^6)$ approximations $U(x,y)$ to u(x,y) at any point (x,y) $\in R$, except at points lying in the rectangle $\tau_h := \{(x,y) : 0 < x < h , 0 < y < b\}$. For such points and, more generally, for points $(x, y) \in r_h \cup L$ the procedures must be modified, because the values

$U(0, y)$, $0 < y < b$, are not available when $y \notin \Pi_y^{\{0\}}$. However, these values can be found as follows:

Let $\hat{Q}_1$ (x, y) denote the biquintic polynomial of Remark 4.5, and approximate the Neumann condition (5.1b) by

$$\hat{Q}_1^{(1,0)}(O, y) = g(y) \tag{5.3}$$

The equation (5.3) in full is

$$-\frac{h}{180}\{62M_0(y) - 23M_1(y) - 9M_2(y) + 5M_3(y) - M_4(y)\} - \frac{1}{h}U(0, y) + \frac{1}{h}\hat{T}_1(y) = g(y), \tag{5.4}$$

where

$$M_i(y) = \frac{1}{h^2}(\hat{T}_i(y-h) - 2\hat{T}_i(y) + \hat{T}_i(y+h)); \quad i = 1(1)4, \tag{5.5a}$$

and

$$M_o(y) = \frac{6}{h^2}(U(0, y) - 2\hat{T}_1(y) + \hat{T}_2(y)) - 4M_1(y) - M_2(y). \tag{5.5b}$$

Therefore, by substituting the expression for M_0 (y) in (5.4) and rearranging , we find that

$$U(0, y) = \frac{1}{552}\{225h^2M_1(y) + 53h^2M_2(y) + 5h^2M_3(y) - h^2M_4(y) + 924\hat{T}_1(y) - 372\hat{T}_2(y) - 180hg(y)\} \tag{5.6}$$

As might be expected, the value *U(0,y)* given by (5.6) is an $O(h^6)$ approximation to u(0, y). This can be verified by expanding the right hand side of (5.6) about the point (x, y) and recalling that $\Delta u = 0$ in *R*. □

VI. NUMERICAL RESULTS

In this section we present the results of two numerical examples illustrating the theory of previous sections. These results were computed on an ECLIPSE MV/15000 , using programs written in double-precision Fortran, i.e. a precision of between 16 and 17 significant figures.

As before, let $\hat{S}_{j+\nu}(x)$ and $\hat{S}_{j+\nu}(x)$ denote respectively the corrected piecewise quintic polynomial of Theorem 4.1, and the cubic spline of [7] referred to in Remarks 4.3 and 4.4. Then, the numerical results listed are estimates of

$$\sup_{x\in|0,\alpha|} |\hat{S}j+\nu(x) - u^{(k,0)}(x, y_i + \nu h)|$$

and

$$\sup_{x\in|0,\alpha|} |\hat{S}_{j+\nu}(x) - u^{(k,0)}(x, y_i, +\nu h)|,$$

obtained by fixing j and v and sampling the errors at a set of 200 equally spaced points on [0, a]. We denote the error estimates corresponding to a mesh of size h by $\widetilde{E}_{j+\nu}(h)$ and $\widetilde{E}_{j+\nu}(h)$ and in the tables we also list the computed values

$$\tilde{r}_{j+\nu}(h) := \log_2\{\widetilde{E}^{(k)}_{j+\nu}(h)/\widetilde{E}^{(k)}_{j+\nu}(h/2)\}$$

and

$$\hat{r}j+\nu(h) := \log\{\hat{E}^{(k)}_{j+\nu}(h)/\hat{E}_{j+\nu}(h/2)\}$$

which give the observed rates of convergence.

Example 6.1.

$$\Delta u(x,y) = 0, \qquad \text{in } R, \tag{6.1a}$$

$$u(x, y) = y/\{(2+x)^2 + y^2\}, \quad \text{on } \partial R, \tag{6.1b}$$

where R is the unit square $R = \{(x, y) : 0 < x < 1 , 0 < y < 1\}$.

Exact solution : $u(x,y) = y/\{(2 + x)^2 + y^2\}$.

Numerical results : The numerical results, obtained by using a mesh of size $h = 0.05$ and taking $j = 6$ and $v = 2/3$ are given in Table 6.1. □

Example 6.2.

$$\Delta u(x,y) = 0, \qquad \text{in R,} \tag{6.2a}$$

$$u^{(1,0)}(0,y) = \cosh y, \quad \text{on } L := \{(0,y) : 0 < y < b\} \tag{6.2b}$$

$$u^{(x,y)} = \sin x \cosh y, \quad \text{on } \partial R \setminus L, \tag{6.2c}$$

where, as in Example 6.1, R is the unit square.

Exact solution : $u(x,y) = \sin x\cosh y$.

Numerical results : The numerical results, obtained by using a mesh of size $h = 0.05$, and taking $j = 6$ and $v = 2/3$ are given in Table 6.2. □

TABLE 6.1.

$h = 0.05, \quad j = 6, \quad \nu = 2/3$

	$k = 0$	$k = 1$	$k = 2$	$k = 3$
$\widetilde{E}^{(k)}_{j+\nu}$	1.30E-8	7.03E-7	9.03E-5	1.18E-2
$\widetilde{r}^{(k)}_{j+\nu}$	4.0	3.0	1.9	1.0
$\hat{E}^{(k)}_{j+\nu}$	1.26E-10	8.77E-9	1.75E-6	2.06E-4
$\hat{r}^{(k)}_{j+\nu}$	5.8	4.8	3.9	2.8

Theoretical rates : $\widetilde{r}^{(k)}_{j+\nu} = 4 - k \;;\; \hat{r}^{(k)}_{j+\nu} = 6 - k.$

TABLE 6.2.

$h = 0.05, \quad j = 6, \quad \nu = 2/3$

	$k = 0$	$k = 1$	$k = 2$	$k = 3$
$\widetilde{E}^{(k)}_{j+\nu}$	2.48E-8	1.31E-6	1.79E-4	2.22E-2
$\widetilde{r}^{(k)}_{j+\nu}$	4.0	3.0	2.0	1.0
$\hat{E}^{(k)}_{j+\nu}$	2.02E-9	1.99E-8	1.93E-6	1.84E-4
$\hat{r}^{(k)}_{j+\nu}$	5.9	4.9	3.9	2.9

Theoretical rates $\widetilde{r}^{(k)}_{j+\nu} = 4 - k \;;\; \hat{r}^{(k)}_{j+\nu} = 6 - k.$

VII. DISCUSSION

We make the following general comments regarding the applicability of the method:

(i) The numerical results of Section 6 confirm the theory and illustrate the high accuracy that can be achieved by the interpolation technique of Section 4. We note, however, that the theory of the method requires that $u(x,y) \in C^8(\bar{R})$ and that the solutions of our numerical examples were chosen to satisfy this requirement.

(ii) In a practical application it is unlikely that the solution of the harmonic boundary value problem will satisfy the high continuity required by the theory. It is however well-known that if $u(x, y)$ does not have serious boundary singularities (e.g. low derivatives becoming unbounded at some point or points on the boundary ∂R), then the nine-point formula will give accurate approximations to u(x,y). We note, in particular, that if the problem involves only jump discontinuities in the boundary conditions, then the damaging effect of these discontinuities can be removed easily by using the technique of Rosser [8]. (See e.g. [6], where the cubic spline technique of [7] has been applied successfully to harmonic problems involving such discontinuities.)

(iii) The application of the method requires that the aspect ratio b/a of the rectangle R is a rational number. However, the method can also be used in conjunction with a special finite-difference method due to Rosser [8], even when the aspect ratio of R is irrational. (The technique of [8] involves the solution of two simple harmonic problems defined respectively in two overlapping sub rectangles with rational aspect ratios.)

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