# Largest nearest-neighbour link and connectivity threshold in a polytopal random sample ${ }^{\dagger}$ 

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#### Abstract

Let $X_{1}, X_{2}, \ldots$ be independent identically distributed random points in a convex polytopal domain $A \subset \mathbb{R}^{d}$. Define the largest nearest neighbour link $L_{n}$ to be the smallest $r$ such that every point of $\mathscr{X}_{n}:=\left\{X_{1}, \ldots, X_{n}\right\}$ has another such point within distance $r$. We obtain a strong law of large numbers for $L_{n}$ in the large- $n$ limit. A related threshold, the connectivity threshold $M_{n}$, is the smallest $r$ such that the random geometric graph $G\left(\mathscr{X}_{n}, r\right)$ is connected. We show that as $n \rightarrow \infty$, almost surely $n L_{n}^{d} / \log n$ tends to a limit that depends on the geometry of $A$, and $n M_{n}^{d} / \log n$ tends to the same limit.


## 1 Introduction

This paper is primarily concerned with the connectivity threshold and largest nearestneighbour link for a random sample $\mathscr{X}_{n}$ of $n$ points specified compact region $A$ in a $d$ dimensional Euclidean space.

The connectivity threshold, here denoted $M_{n}$, is defined to be the smallest $r$ such that the random geometric graph $G\left(\mathscr{X}_{n}, r\right)$ is connected. For any finite $\mathscr{X} \subset \mathbb{R}^{d}$ the graph $G(\mathscr{X}, r)$ is defined to have vertex set $\mathscr{X}$ with edges between those pairs of vertices $x, y$ such that $\|x-y\| \leq r$, where $\|\cdot\|$ is the Euclidean norm. More generally, for $k \in \mathbb{N}$, the $k$-connectivity threshold $M_{n, k}$ is the smallest $r$ such that $G\left(\mathscr{X}_{n}, r\right)$ is $k$-connected (see the definition in Section (2).

[^0]The largest nearest neighbour link, here denoted $L_{n}$, is defined to be the the smallest $r$ such that every vertex in $G\left(\mathscr{X}_{n}, r\right)$ has degree at least 1 . More generally, for $k \in \mathbb{N}$ with $k<n$, the largest $k$-nearest neighbour link $L_{n, k}$ is the smallest $r$ such that every vertex in $G\left(\mathscr{X}_{n}, r\right)$ has degree at least $k$. These thresholds are random variables, because the locations of the centres are random. We investigate their probabilistic behaviour as $n$ becomes large.

We shall derive strong laws of large numbers showing that that $n L_{n, k}^{d} / \log n$ converges almost surely ( as $n \rightarrow \infty$ ) to a finite positive limit, and establishing the value of the limit. Moreover we show that $n M_{n, k}^{d} / \log n$ converges to the same limit. These strong laws carry over to more general cases where $k$ may vary with $n$, and the distribution of points may be non-uniform. We give results of this type for $A$ a convex polytope.

Previous results of this type (both for $L_{n, k}$ and for $M_{n, k}$ ) were obtained for $A$ having a smooth boundary, and for $A$ a $d$-dimensional hypercube; see [5]. It is perhaps not obvious from the earlier results, however, how the limiting constant depends on the geometry of $\partial A$, the topological boundary of $A$, for general polytopal $A$, which is quite subtle.

It turns out, for example, that when $d=3$ and the points are uniformly distributed over a polyhedron, the limiting behaviour of $L_{n}$ is determined by the angle of the sharpest edge if this angle is less than $\pi / 2$. We believe (but do not formally prove here) that if this angle exceeds $\pi / 2$ then the point of $\mathscr{X}_{n}$ furthest from the rest of $\mathscr{X}_{n}$ is asymptotically uniformly distributed over $\partial A$, but if this angle is less than $\pi / 2$ the location of this point in is asymptotically uniformly distributed over the union of those edges which are sharpest.

Our motivation for this study is twofold. First, understanding the connectivity threshold in dimension two is vital in telecommunications, for example, in 5 G wireless network design, with the nodes of $\mathscr{X}_{n}$ representing mobile transceivers (see for example [1]). Second, detecting connectivity is a fundamental step for detecting all other higher dimensional topological features in modern topological data analysis (TDA), where the dimension of the ambient space may be very high. See [2, 3] for discussion of issues related to the one considered here, in relation to TDA. General motivation for considering random geometric graphs is discussed in [5].

While our main results are presented (in Section 22) in the concrete setting of a polytopal sample in $\mathbb{R}^{d}$, our proofs proceed via general lower and upper bounds (Propositions 3.2 and 3.6) that are presented in the more general setting of a random sample of points in a metric space satisfying certain regularity conditions. This could be useful in possible future work dealing with similar problems for random samples in, for example, a Riemannian manifold with boundary, a setting of importance in TDA.

## 2 Statement of results

Throughout this paper, we work within the following mathematical framework. Let $d \in \mathbb{N}$. Suppose we have the following ingredients:

- A finite compact convex polytope $A \subset \mathbb{R}^{d}$ (i.e., one with finitely many faces).
- A Borel probability measure $\mu$ on $A$ with probability density function $f$.
- On a common probability space $(\mathbb{S}, \mathscr{F}, \mathbb{P})$, a sequence $X_{1}, X_{2}, \ldots$ of independent identically distributed random $d$-vectors with common probability distribution $\mu$, and also a unit rate Poisson counting process ( $Z_{t}, t \geq 0$ ), independent of ( $X_{1}, X_{2}, \ldots$ ) (so $Z_{t}$ is Poisson distributed with mean $t$ for each $t>0$ ).

For $n \in \mathbb{N}, t>0$, let $\mathscr{X}_{n}:=\left\{X_{1}, \ldots, X_{n}\right\}$, and let $\mathscr{P}_{t}:=\left\{X_{1}, \ldots, X_{Z_{t}}\right\}$. These are the point processes that concern us here. Observe that $\mathscr{P}_{t}$ is a Poisson point process in $\mathbb{R}^{d}$ with intensity measure $t \mu$ (see e.g. [4]).

For $x \in \mathbb{R}^{d}$ and $r>0$ set $B(x, r):=\left\{y \in \mathbb{R}^{d}:\|y-x\| \leq r\right\}$. For $r>0$, let $A^{(r)}:=\{x \in$ $\left.A: B(x, r) \subset A^{o}\right\}$, the ' $r$-interior' of $A$.

For any point set $\mathscr{X} \subset \mathbb{R}^{d}$ and any $D \subset \mathbb{R}^{d}$ we write $\mathscr{X}(D)$ for the number of points of $\mathscr{X}$ in $D$, and we use below the convention $\inf (\varnothing):=+\infty$.

Given $n, k \in \mathbb{N}$, and $t \in(0, \infty)$, define the largest $k$-nearest neighbour $\operatorname{link} L_{n, k}$ by

$$
\begin{equation*}
L_{n, k}:=\inf \left(\left\{r>0: \mathscr{X}_{n}(B(x, r)) \geq k+1 \quad \forall x \in \mathscr{X}_{n}\right\}\right) \tag{2.1}
\end{equation*}
$$

Set $L_{n}:=L_{n, 1}$. Then $L_{n}$ is the largest nearest-neighbour link.
We are chiefly interested in the asymptotic behaviour of $L_{n}$ for large $n$. More generally, we consider $L_{n, k}$ where $k$ may vary with $n$.

Let $\theta_{d}:=\pi^{d / 2} / \Gamma(1+d / 2)$, the volume of the unit ball in $\mathbb{R}^{d}$. Given $x, y \in \mathbb{R}^{d}$, we denote by $[x, y]$ the line segment from $x$ to $y$, that is, the convex hull of the set $\{x, y\}$.

Given $m \in \mathbb{N}$ and functions $f: \mathbb{N} \cap[m, \infty) \rightarrow \mathbb{R}$ and $g: \mathbb{N} \cap[m, \infty) \rightarrow(0, \infty)$, we write $f(n)=O(g(n))$ as $n \rightarrow \infty$, if $\limsup _{n \rightarrow \infty}|f(n)| / g(n)<\infty$. We write $f(n)=\Omega(g(n))$ as $n \rightarrow \infty$ if $\liminf _{n \rightarrow \infty}(f(n) / g(n))>0$. Given $s>0$ and functions $f:(0, s) \rightarrow \mathbb{R}$ and $g:(0, s) \rightarrow(0, \infty)$, we write $f(r)=O(g(r))$ as $r \downarrow 0$ if $\limsup _{r \downarrow 0}|f(r)| / g(r)<\infty$. We write $f(r)=\Omega(g(r))$ as $r \downarrow 0$, if $\liminf _{r \downarrow 0}(f(r) / g(r))>0$.

Throughout this section, assume we are given a constant $\beta \in[0, \infty]$ and a sequence $k: \mathbb{N} \rightarrow \mathbb{N}$ with

$$
\begin{equation*}
\lim _{n \rightarrow \infty}(k(n) / \log n)=\beta ; \quad \lim _{n \rightarrow \infty}(k(n) / n)=0 . \tag{2.2}
\end{equation*}
$$

We make use of the following notation throughout:

$$
\begin{array}{r}
f_{0}:=\operatorname{ess}_{\inf }^{x \in A} f_{1}(x) ; \quad f_{1}:=\inf _{x \in \partial A} f(x) ; \\
H(t):= \begin{cases}1-t+t \log t, & \text { if } t>0 \\
1, & \text { if } t=0 .\end{cases} \tag{2.4}
\end{array}
$$

Observe that $-H(\cdot)$ is unimodal with a maximum value of 0 at $t=1$. Given $a \in[0, \infty)$, we define the function $\hat{H}_{a}:[0, \infty) \rightarrow[a, \infty)$ by

$$
y=\hat{H}_{a}(x) \Longleftrightarrow y H(a / y)=x, y \geq a,
$$

with $\hat{H}_{0}(0):=0$. Note that $\hat{H}_{a}(x)$ is increasing in $x$, and that $\hat{H}_{0}(x)=x$ and $\hat{H}_{a}(0)=a$.
Throughout this paper, the phrase 'almost surely' or 'a.s.' means 'except on a set of $\mathbb{P}$-measure zero'. For $n \in \mathbb{N}$, we use $[n]$ to denote $\{1,2, \ldots, n\}$. We write $\left.f\right|_{A}$ for the restriction of $f$ to $A$.

Let $\Phi(A)$ denote the set of all faces of the polytope $A$ (of all dimensions up to $d-1$ ). Also, let $\Phi^{*}(A):=\Phi(A) \cup\{A\}$; it is sometimes useful for us to think of $A$ itself as a face, of dimension $d$.

Given a face $\varphi \in \Phi^{*}(A)$, denote the dimension of this face by $D(\varphi)$. Then $0 \leq D(\varphi) \leq$ $d$, and $\varphi$ is a $D(\varphi)$-dimensional polytope embedded in $\mathbb{R}^{d}$. Let $\varphi^{o}$ denote the relative interior of $\varphi$, and set $\partial \varphi:=\varphi \backslash \varphi^{o}$ (if $D(\varphi)=0$ we take $\varphi^{o}:=\varphi$ ). If $D(\varphi)<d$ then set $f_{\varphi}:=\inf _{x \in \varphi} f(x)$, and if $\varphi=A$ then set $f_{\varphi}:=f_{0}$.

Then there is a cone $\mathscr{K}_{\varphi}$ in $\mathbb{R}^{d}$ such that every $x \in \varphi^{o}$ has a neighbourhood $U_{x}$ such that $A \cap U_{x}=\left(x+\mathscr{K}_{\varphi}\right) \cap U_{x}$. Define the angular volume $\rho_{\varphi}$ of $\varphi$ to be the $d$-dimensional Lebesgue measure of $\mathscr{K}_{\varphi} \cap B(o, 1)$.

For example, if $\varphi=A$ then $\rho_{\varphi}=\theta_{d}$. If $D(\varphi)=d-1$ then $\rho_{\varphi}=\theta_{d} / 2$. If $D(\varphi)=0$ then $\varphi=\{\nu\}$ for some vertex $v \in \partial A$, and $\rho_{\varphi}$ equals the volume of $B(v, r) \cap A$, divided by $r^{d}$, for all sufficiently small $r$. If $d=2, D(\varphi)=0$ and $\omega_{\varphi}$ denotes the angle subtended by $A$ at the vertex $\varphi$, then $\rho_{\varphi}=\omega_{\varphi} / 2$. If $d=3$ and $D(\varphi)=1$, and $\alpha_{\varphi}$ denotes the angle subtended by $A$ at the edge $\varphi$ (which is the angle between the two boundary planes of $A$ meeting at $\varphi$ ), then $\rho_{\varphi}=2 \alpha_{\varphi} / 3$.

Theorem 2.1. Suppose $A$ is a compact convex finite polytope in $\mathbb{R}^{d}$. Assume that $\left.f\right|_{A}$ is continuous at $x$ for all $x \in \partial A$, and that $f_{0}>0$. Assume $k(\cdot)$ satisfies (2.2). Then, almost surely,

$$
\begin{array}{ll}
\lim _{n \rightarrow \infty} n L_{n, k(n)}^{d} / k(n)=\max _{\varphi \in \Phi^{*}(A)}\left(\frac{1}{f_{\varphi} \rho_{\varphi}}\right) & \text { if } \beta=\infty \\
\lim _{n \rightarrow \infty} n L_{n, k(n)}^{d} / \log n=\max _{\varphi \in \Phi^{*}(A)}\left(\frac{\hat{H}_{\beta}(D(\varphi) / d)}{f_{\varphi} \rho_{\varphi}}\right) & \text { if } \beta<\infty . \tag{2.6}
\end{array}
$$

In the next three results, we spell out some special cases of Theorem 2.1.
Corollary 2.2. Suppose that $d=2, A$ is a convex polygon and $\left.f\right|_{A}$ is continuous at $x$ for all $x \in \partial A$. Let $V$ denote the set of vertices of $A$, and for $v \in V$ let $\omega_{v}$ denote the angle subtended by $A$ at vertex $v$. Assume (2.2) holds with $\beta<\infty$. Then, almost surely,

$$
\begin{equation*}
\lim _{n \rightarrow \infty}\left(\frac{n L_{n, k(n)}^{2}}{\log n}\right)=\max \left(\frac{\hat{H}_{\beta}(1)}{\pi f_{0}}, \frac{2 \hat{H}_{\beta}(1 / 2)}{\pi f_{1}}, \max _{v \in V}\left(\frac{2 \beta}{\omega_{v} f(v)}\right)\right) . \tag{2.7}
\end{equation*}
$$

In particular, for any constant $k \in \mathbb{N}, \lim _{n \rightarrow \infty}\left(\frac{n \pi L_{n, k}^{2}}{\log n}\right)=\frac{1}{f_{0}}$.
Corollary 2.3. Suppose $d=3$ (so $\theta_{d}=4 \pi / 3$ ), A is a convex polyhedron and $\left.f\right|_{A}$ is continuous at $x$ for all $x \in \partial A$. Let $V$ denote the set of vertices of $A$, and $E$ the set of edges of $A$. For $e \in E$, let $\alpha_{e}$ denote the angle subtended by $A$ at edge $e$, and $f_{e}$ the infimum of $f$ over $e$. For $v \in V$ let $\rho_{v}$ denote the angular volume of vertex v. Suppose (2.2) holds with $\beta<\infty$. Then, almost surely,

$$
\lim _{n \rightarrow \infty}\left(\frac{n L_{n, k(n)}^{3}}{\log n}\right)=\max \left(\frac{\hat{H}_{\beta}(1)}{\theta_{3} f_{0}}, \frac{2 \hat{H}_{\beta}(2 / 3)}{\theta_{3} f_{1}}, \frac{3 \hat{H}_{\beta}(1 / 3)}{2 \min _{e \in E}\left(\alpha_{e} f_{e}\right)}, \max _{v \in V}\left(\frac{\beta}{\rho_{v} f(v)}\right)\right)
$$

In particular, if $\beta=0$ the above limit comes to $\max \left(\frac{3}{4 \pi f_{0}}, \frac{1}{\pi f_{1}}, \max _{e \in E}\left(\frac{1}{2 \alpha_{e} f_{e}}\right)\right)$.
Corollary $2.4([5])$. Suppose $A=[0,1]^{d}$, and $\left.f\right|_{A}$ is continuous at $x$ for all $x \in \partial A$. For $1 \leq j \leq d$ let $\partial_{j}$ denote the union of all $(d-j)$-dimensional faces of $A$, and let $f_{j}$ denote the infimum of $f$ over $\partial_{j}$. Assume (2.2) with $\beta<\infty$. Then

$$
\begin{equation*}
\lim _{n \rightarrow \infty}\left(\frac{n L_{n, k(n)}^{d}}{\log n}\right)=\max _{0 \leq j \leq d}\left(\frac{2^{j} \hat{H}_{\beta}(1-j / d)}{\theta_{d} f_{j}}\right), \quad \text { a.s. } \tag{2.8}
\end{equation*}
$$

It is perhaps worth spelling out what the preceding results mean in the special case where $\beta=0$ (for example, if $k(n)$ is a constant) and also $\mu$ is the uniform distribution on $A$ (i.e. $f(x) \equiv f_{0}$ on $A$ ). In this case, the right hand side of (2.6) comes to $\max _{\varphi \in \Phi^{*}(A)} \frac{D(\varphi)}{\left(d f_{0} \rho_{\varphi}\right)}$. The limit in (2.7) comes to $1 /\left(\pi f_{0}\right)$, while the limit in Corollary 2.3 comes to $f_{0}^{-1} \max \left[1 / \pi, \max _{e}\left(1 /\left(2 \alpha_{e}\right)\right)\right]$.

So far we have only presented results for the largest $k$-nearest neighbor link. A closely related threshold is the $k$-connectivity threshold defined by

$$
M_{n, k}:=\inf \left\{r>0: G\left(\mathscr{X}_{n}, r\right) \text { is } k \text {-connected }\right\},
$$

where a graph $G$ of order $n$ is said to be $k$-connected $(k<n)$ if $G$ cannot be disconnected by the removal of at most $k-1$ vertices. Set $M_{n, 1}=M_{n}$. Then $M_{n}$ is the connectivity threshold.

Notice that for all $k, n$ with $k<n$ we have

$$
\begin{equation*}
L_{n, k} \leq M_{n, k} \tag{2.9}
\end{equation*}
$$

Indeed, if $r<L_{n, k}$, then there exists $i \in[n]$ such that $\operatorname{deg} X_{i}<k$ in $G\left(\mathscr{X}_{n}, r\right)$. Then the removal of all vertices adjacent to $X_{i}$ disconnects $G\left(\mathscr{X}_{n}, r\right)$, implying that $r<M_{n, k}$. This proves the claim.

Our second main result shows that $\left(M_{n, k} / L_{n, k}\right) \rightarrow 1$ almost surely as $n \rightarrow \infty$. For this result we need $d \geq 2$.

Theorem 2.5. Suppose $d \geq 2$. Suppose $A$ is a compact convex finite polytope in $\mathbb{R}^{d}$. Assume that $\left.f\right|_{A}$ is continuous at $x$ for all $x \in \partial A$, and that $f_{0}>0$. Assume $k(\cdot)$ satisfies (2.2) Then, almost surely,

$$
\begin{array}{ll}
\lim _{n \rightarrow \infty} n M_{n, k(n)}^{d} / k(n)=\max _{\varphi \in \Phi^{*}(A)}\left(\frac{1}{f_{\varphi} \rho_{\varphi}}\right) & \text { if } \beta=\infty \\
\lim _{n \rightarrow \infty} n M_{n, k(n)}^{d} / \log n=\max _{\varphi \in \Phi^{*}(A)}\left(\frac{\hat{H}_{\beta}(D(\varphi) / d)}{f_{\varphi} \rho_{\varphi}}\right) & \text { if } \beta<\infty . \tag{2.11}
\end{array}
$$

Remark 2.6. One can spell out consequences of Theorem 2.5 in dimensions $d=2,3$ and the case of $[0,1]^{d}$ with exactly the same statement as in Corollaries $2.2+2.4$,

Remark 2.7. Theorems 2.1] and 2.5] extend earlier work found in [5] on the case where $A$ is the unit cube, to more general polytopal regions. The case where $A$ has a smooth boundary is also considered in [5] (in this case with also $k(n)=$ const., the result was first given in [6] for $L_{n, k}$ and in [7] for $M_{n, k}$ ).

Remark 2.8. In [8], similar results are given for the $k$-coverage threshold $R_{n, k}$, which is given by

$$
\begin{equation*}
R_{n, k}:=\inf \left\{r>0: \mathscr{X}_{n}(B(x, r)) \geq k \quad \forall x \in A\right\} ; \quad n, k \in \mathbb{N} \tag{2.12}
\end{equation*}
$$

Our results here, together with [8, Theorem 4.2], show that both $L_{n, k(n)}$ and $M_{n, k(n)}$ are asymptotic to $R_{n, k(n)}$ almost surely, as $n \rightarrow \infty$.

## 3 Proofs

In this section we prove the results stated in Section 2. Throughout this section we are assuming we are given a constant $\beta \in[0, \infty]$ and a sequence $(k(n))_{n \in \mathbb{N}}$ satisfying (2.2).

Recall that $\mu$ denotes the distribution of $X_{1}$, and this has a density $f$ with support $A$, and that $L_{n, k}$ is defined at (2.1). Recall that $\hat{H}_{\beta}(x)$ is defined to be the $y \geq \beta$ such that $y H(\beta / y)=x$, where $H(\cdot)$ was defined at (2.4).

For $n \in \mathbb{N}$ and $p \in[0,1]$ let $\operatorname{Bin}(n, p)$ denote a binomial random variable with parameters $n, p$. Recall that $H(\cdot)$ was defined at (2.4), and $Z_{t}$ is a $\operatorname{Poisson}(t)$ variable for $t>0$. The proofs in this section rely heavily on the following lemma.

Lemma 3.1 (Chernoff bounds). Suppose $n \in \mathbb{N}, p \in(0,1), t>0$ and $0 \leq k<n$.
(a) If $k \geq n p$ then $\mathbb{P}[\operatorname{Bin}(n, p) \geq k] \leq \exp (-n p H(k /(n p)))$.
(b) If $k \leq n p$ then $\mathbb{P}[\operatorname{Bin}(n, p) \leq k] \leq \exp (-n p H(k /(n p)))$.
(c) If $k \geq e^{2} n p$ then $\mathbb{P}[\operatorname{Bin}(n, p) \geq k] \leq \exp (-(k / 2) \log (k /(n p))) \leq e^{-k}$.
(d) If $k<t$ then $\mathbb{P}\left[Z_{t} \leq k\right] \leq \exp (-t H(k / t))$.
(e) If $k \in \mathbb{N}$ then $\mathbb{P}\left[Z_{t}=k\right] \geq(2 \pi k)^{-1 / 2} e^{-1 /(12 k)} \exp (-t H(k / t))$.

Proof. See e.g. [5, Lemmas 1.1, 1.2 and 1.3].

### 3.1 A general lower bound

In this subsection we present an asymptotic lower bound on $L_{n, k(n)}$, not requiring any extra assumptions on $A$. In fact, $A$ here can be any metric space endowed with a Borel probability measure $\mu$ which satisfies the following for some $\varepsilon^{\prime}>0$ and some $d>0$ :

$$
\begin{equation*}
\mu(B(x, r)) \geq \varepsilon^{\prime} r^{d}, \quad \forall r \in(0,1), x \in A . \tag{3.1}
\end{equation*}
$$

The definition of $L_{n, k}$ at (2.1) carries over in an obvious way to this general setting.
Later, we shall derive the results stated in Section2by applying the results of this subsection to the different regions within $A$ (namely interior, boundary, and lower-dimensional faces).

Given $r>0, a>0$, define the 'packing number' $v(r, a)$ be the largest number $m$ such that there exists a collection of $m$ disjoint closed balls of radius $r$ centred on points of $A$, each with $\mu$-measure at most $a$.

Proposition 3.2 (General lower bound). Assume (3.1) with $d, \varepsilon^{\prime}>0$. Let $a>0, b \geq$ 0 . Suppose $v\left(r, a r^{d}\right)=\Omega\left(r^{-b}\right)$ as $r \downarrow 0$. Assume (2.2). Then almost surely, if $\beta=\infty$ then $\liminf _{n \rightarrow \infty}\left(n L_{n, k(n)}^{d} / k(n)\right) \geq 1 / a$. If $\beta<\infty$ then $\liminf _{n \rightarrow \infty}\left(n L_{n, k(n)}^{d} / \log n\right) \geq$ $a^{-1} \hat{H}_{\beta}(b / d)$, almost surely.

Proof. First suppose $\beta=\infty$. Let $u \in(0,1 / a)$. Set $r_{n}:=(u k(n) / n)^{1 / d}, n \in \mathbb{N}$. By (2.2), $r_{n} \rightarrow 0$ as $n \rightarrow \infty$. Then, given $n$ sufficiently large, we have $v\left(r_{n}, a r_{n}^{d}\right)>0$ so we can find $y_{n} \in A$ such that $\mu\left(B\left(y_{n}, r_{n}\right)\right) \leq a r_{n}^{d}$, and hence $n \mu\left(B\left(y_{n}, r_{n}\right)\right) \leq \operatorname{auk}(n)$. If
$k(n) \leq e^{2} n \mu\left(B\left(y_{n}, r_{n}\right)\right)$ (and hence $n \mu\left(B\left(y_{n}, r_{n}\right)\right) \geq e^{-2} k(n)$ ), then since $\mathscr{X}_{n}\left(B\left(y_{n}, r_{n}\right)\right)$ is binomial with parameters $n$ and $\mu\left(B\left(y_{n}, r_{n}\right)\right)$, by Lemma 3.1(a) we have that

$$
\begin{aligned}
\mathbb{P}\left[\mathscr{X}_{n}\left(B\left(y_{n}, r_{n}\right)\right) \geq k(n)\right] & \leq \exp \left(-n \mu\left(B\left(y_{n}, r_{n}\right)\right) H\left(\frac{k(n)}{n \mu\left(B\left(y_{n}, r_{n}\right)\right)}\right)\right) \\
& \leq \exp \left(-e^{-2} k(n) H\left((a u)^{-1}\right)\right)
\end{aligned}
$$

while if $k(n)>e^{2} n \mu\left(B\left(y_{n}, r_{n}\right)\right)$ then by Lemma3.1(c), $\mathbb{P}\left[\mathscr{X}_{n}\left(B\left(y_{n}, r_{n}\right)\right) \geq k(n)\right] \leq e^{-k(n)}$. Therefore $\mathbb{P}\left[\mathscr{X}_{n}\left(B\left(y_{n}, r_{n}\right)\right) \geq k(n)\right]$ is summable in $n$ because $k(n) / \log n \rightarrow \infty$ as $n \rightarrow \infty$ by (2.2).

Let $\delta_{0} \in(0,1)$. By (3.1) $\mu\left(B\left(y_{n}, \delta_{0} r_{n}\right) \geq \varepsilon^{\prime} \delta_{0}^{d} u k(n) / n\right.$. Therefore by Lemma 3.1(b), $\mathbb{P}\left[\mathscr{X}_{n}\left(B\left(y_{n}, \delta_{0} r_{n}\right)\right)=0\right] \leq \exp \left(-\varepsilon^{\prime} \delta_{0}^{d} u k(n)\right)$, which is summable in $n$.

Thus by the Borel-Cantelli lemma, almost surely event $F_{n}:=\left\{\mathscr{X}_{n}\left(B\left(y_{n}, r_{n}\right)\right)<k(n)\right\} \cap$ $\left\{\mathscr{X}_{n}\left(B\left(y_{n}, \delta_{0} r_{n}\right)\right)>0\right\}$ occurs for all but finitely many $n$. But if $F_{n}$ occurs then $L_{n, k(n)} \geq$ $\left(1-\delta_{0}\right) r_{n}$ so that $n L_{n, k(n)}^{d} / k(n) \geq\left(1-\delta_{0}\right)^{d} u$. This gives the result for $\beta=\infty$.

Now suppose instead that $\beta<\infty$. Suppose first that $b=0$, so that $\hat{H}_{\beta}(b / d)=\beta$. Assume that $\beta>0$ (otherwise the result is trivial). Choose $\beta^{\prime} \in(0, \beta)$. Let $\delta>0$ with $\beta^{\prime}<\beta-2 \delta$ and with $\beta^{\prime} H\left(\frac{\beta-2 \delta}{\beta^{\prime}}\right)>\delta$. This is possible because $H\left(\beta / \beta^{\prime}\right)>0$ and $H(\cdot)$ is continuous. For $n \in \mathbb{N}$, set $r_{n}:=\left(\left(\beta^{\prime} \log n\right) /(a n)\right)^{1 / d}$. Also set $k^{\prime}(n)=\lceil(\beta-\delta) \log n\rceil$, and $k^{\prime \prime}(n)=\lceil(\beta-2 \delta) \log n\rceil$. By assumption $v\left(r_{n}, a r_{n}^{d}\right)=\Omega(1)$, so for all $n$ large enough, we can (and do) choose $x_{n} \in A$ such that $n \mu\left(B\left(x_{n}, r_{n}\right)\right) \leq n a r_{n}^{d}=\beta^{\prime} \log n$. Then by a simple coupling, and Lemma 3.1(a),

$$
\begin{aligned}
\mathbb{P}\left[\mathscr{X}_{n}\left(B\left(x_{n}, r_{n}\right)\right) \geq k^{\prime \prime}(n)\right] & \left.\leq \mathbb{P}\left[\operatorname{Bin}\left(n,\left(\beta^{\prime} \log n\right) / n\right)\right) \geq k^{\prime \prime}(n)\right] \\
& \leq \exp \left(-\left(\beta^{\prime} \log n\right) H\left(\frac{\beta-2 \delta}{\beta^{\prime}}\right)\right) \leq n^{-\delta} .
\end{aligned}
$$

Let $\delta^{\prime} \in(0,1)$. By (3.1), for $n$ large enough and all $x \in A$,

$$
n \mu\left(B\left(x, \delta^{\prime} r_{n}\right)\right) \geq n \varepsilon^{\prime}\left(\delta^{\prime} r_{n}\right)^{d}=\varepsilon^{\prime}\left(\delta^{\prime}\right)^{d}\left(\beta^{\prime} / a\right) \log n
$$

so that by Lemma3.1 (b), $\mathbb{P}\left[\mathscr{X}_{n}\left(B\left(x, \delta^{\prime} r_{n}\right)\right)=0\right] \leq n^{-\varepsilon^{\prime}\left(\delta^{\prime}\right)^{d} \beta^{\prime} / a}$.
Now choose $K \in \mathbb{N}$ such that $\delta K>1$ and $K \varepsilon^{\prime}\left(\delta^{\prime}\right)^{d} \beta^{\prime} / a>1$. For $n \in \mathbb{N}$ set $z(n):=n^{K}$. For all large enough $n$ we have $k^{\prime}(z(n)) \geq k^{\prime \prime}(z(n+1))$, so by the preceding estimates,

$$
\begin{aligned}
& \mathbb{P}\left[\mathscr{X}_{z(n+1)}\left(B\left(x_{z(n+1)}, r_{z(n+1)}\right)\right) \geq k^{\prime}(z(n))\right] \\
& \quad \leq \mathbb{P}\left[\mathscr{X}_{z(n+1)}\left(B\left(x_{z(n+1)}, r_{z(n+1)}\right)\right) \geq k^{\prime \prime}(z(n+1))\right] \leq(n+1)^{-\delta K}
\end{aligned}
$$

and since $x_{z(n+1)} \in A$, also $\mathbb{P}\left[\mathscr{X}_{z(n)}\left(B\left(x_{z(n+1)}, \delta^{\prime} r_{z(n)}\right)\right)=0\right] \leq n^{-\varepsilon^{\prime}\left(\delta^{\prime}\right)^{d} \beta^{\prime} K / a}$. Both of these upper bounds are summable in $n$, so by the Borel-Cantelli lemma, almost surely for all
large enough $n$ we have the event

$$
\left\{\mathscr{X}_{z(n+1)}\left(B\left(x_{z(n+1)}, r_{z(n+1)}\right)\right)<k^{\prime}(z(n))\right\} \cap\left\{\mathscr{X}_{z(n)}\left(B\left(x_{z(n+1)}, \delta^{\prime} r_{z(n)}\right)\right)>0\right\} .
$$

Suppose the above event occurs and suppose $m \in \mathbb{N}$ with $z(n) \leq m \leq z(n+1)$. Note that $r_{z(n+1)} / r_{z(n)} \rightarrow 1$ as $n \rightarrow \infty$. Then, provided $n$ is large enough,

$$
L_{m, k^{\prime}(z(n))} \geq r_{z(n+1)}-\delta^{\prime} r_{z(n)} \geq\left(1-\delta^{\prime}\right)^{2} r_{m}
$$

and moreover $k^{\prime}(z(n)) \leq k(m)$ so that $L_{m, k(m)} \geq\left(1-\delta^{\prime}\right)^{2} r_{m}$. Hence it is almost surely the case that

$$
\liminf _{m \rightarrow \infty}\left(m L_{m, k(m)}^{d} / \log m\right) \geq\left(1-\delta^{\prime}\right)^{2 d} \liminf _{m \rightarrow \infty}\left(m r_{m}^{d} / \log m\right)=\left(1-\delta^{\prime}\right)^{2 d} a^{-1} \beta^{\prime}
$$

and this yields the result for this case.
Now suppose instead that $\beta<\infty$ and $b>0$. Let $u \in\left(a^{-1} \beta, a^{-1} \hat{H}_{\beta}(b / d)\right)$; note that this implies $u a H(\beta /(u a))<b / d$. Choose $\varepsilon>0$ such that $(1+\varepsilon) u a H(\beta /(u a))<(b / d)-$ $9 \varepsilon$. Also let $\delta^{\prime} \in(0,1)$.

For each $n \in \mathbb{N}$ set $r_{n}=(u(\log n) / n)^{1 / d}$. Let $m_{n}:=v\left(r_{n}, a r_{n}^{d}\right)$, and choose $x_{n, 1}, \ldots$, $x_{n, m_{n}} \in A$ such that the balls $B\left(x_{n, 1}, r_{n}\right), \ldots, B\left(x_{n, m_{n}}, r_{n}\right)$ are pairwise disjoint and each have $\mu$-measure at most $a r_{n}^{d}$.

Set $\lambda(n):=n+n^{3 / 4}$ and $\lambda^{-}(n):=n-n^{3 / 4}$. For $1 \leq i \leq m_{n}$, if $k(n) \geq 1$ then by a simple coupling, and Lemma3.1(e),

$$
\begin{aligned}
& \mathbb{P}\left[\mathscr{P}_{\lambda(n)}\left(B\left(x_{n, i}, r_{n}\right)\right) \leq k(n)\right] \geq \mathbb{P}\left[Z_{\lambda(n) a r_{n}^{d}} \leq k(n)\right] \\
\geq & \left(\frac{e^{-1 /(12 k(n))}}{\sqrt{2 \pi k(n)}}\right) \exp \left(-\lambda(n) a r_{n}^{d} H\left(\frac{k(n)}{\lambda(n) a r_{n}^{d}}\right)\right) .
\end{aligned}
$$

Now $\lambda(n) r_{n}^{d} / \log n \rightarrow u$ so by (2.2), $k(n) /\left(\lambda(n) a r_{n}^{d}\right) \rightarrow \beta /(u a)$ as $n \rightarrow \infty$. Thus by the continuity of $H(\cdot)$, provided $n$ is large enough, for $1 \leq i \leq m_{n}$,

$$
\begin{gathered}
\mathbb{P}\left[\mathscr{P}_{\lambda(n)}\left(B\left(x_{n, i}, r_{n}\right)\right) \leq k(n)\right] \\
\geq\left(\frac{e^{-1 / 12}}{\sqrt{2 \pi(\beta+1) \log n}}\right) \exp \left(-(1+\varepsilon) a u H\left(\frac{\beta}{a u}\right) \log n\right) .
\end{gathered}
$$

Hence, by our choice of $\varepsilon$, there is a constant $c>0$ such that for all large enough $n$ and all $i \in\left[m_{n}\right]$ we have

$$
\begin{equation*}
\mathbb{P}\left[\mathscr{P}_{\lambda(n)}\left(B\left(x_{n, i}, r_{n}\right)\right) \leq k(n)\right] \geq c(\log n)^{-1 / 2} n^{9 \varepsilon-b / d} \geq n^{8 \varepsilon-b / d} \tag{3.2}
\end{equation*}
$$

Since $x_{n, i} \in A$, by (3.1), for $n$ large enough and $1 \leq i \leq m_{n}$ we have $\mu\left(B\left(x_{n, i}, \delta^{\prime} r_{n}\right)\right) \geq$ $\varepsilon^{\prime}\left(\delta^{\prime} r_{n}\right)^{d}$ (as well as $\left.\mu\left(B\left(x_{n, i}, r_{n}\right)\right) \leq a r_{n}^{d}\right)$. Thus, given the value of $\mathscr{P}_{\lambda(n)}\left(B\left(x_{n, i}, r_{n}\right)\right)$,
the value of $\mathscr{P}_{\lambda-(n)}\left(B\left(x_{n, i}, \delta^{\prime} r_{n}\right)\right)$ is binomially distributed with probability parameter bounded away from zero. Also $\max _{1 \leq i \leq m_{n}} \mathbb{E}\left[\mathscr{P}_{\lambda(n)}\left(B\left(x_{n, i}, r_{n}\right)\right)\right]$ tends to infinity as $n \rightarrow$ $\infty$. Therefore there exists $\eta>0$ such that for all large enough $n$, defining the event

$$
E_{n, i}:=\left\{\mathscr{P}_{\lambda(n)}\left(B\left(x_{n, i}, r_{n}\right)\right) \leq k(n)\right\} \cap\left\{\mathscr{P}_{\lambda^{-}(n)}\left(B\left(x_{n, i}, \delta^{\prime} r_{n}\right) \geq 1\right\}\right.
$$

we have for all large enough $n$ that

$$
\inf _{1 \leq i \leq m_{n}} \mathbb{P}\left[E_{n, i} \mid \mathscr{P}_{\lambda(n)}\left(B\left(x_{n, i}, r_{n}\right)\right) \leq k(n)\right] \geq \eta
$$

Hence, setting $E_{n}:=\cup_{i=1}^{m_{n}} E_{n, i}$, for all large enough $n$ we have

$$
\mathbb{P}\left[E_{n}^{c}\right] \leq\left(1-\eta n^{8 \varepsilon-b / d}\right)^{m_{n}} \leq \exp \left(-\eta m_{n} n^{8 \varepsilon-b / d}\right)
$$

By assumption $m_{n}=v\left(r_{n}, a r_{n}^{d}\right)=\Omega\left(r_{n}^{-b}\right)$ so that for large enough $n$ we have $m_{n} \geq$ $n^{(b / d)-\varepsilon}$, and therefore $\mathbb{P}\left[E_{n}^{c}\right]$ is is summable in $n$.

By Lemma 3.1(d), and Taylor expansion of $H(x)$ about $x=1$ (see the print version of [5, Lemma 1.4] for details; there may be a typo in the electronic version), for all $n$ large enough $\mathbb{P}\left[Z_{\lambda(n)}<n\right] \leq \exp \left(-\frac{1}{9} n^{1 / 2}\right)$. Similarly $\mathbb{P}\left[Z_{\lambda-(n)}>n\right] \leq \exp \left(-\frac{1}{9} n^{1 / 2}\right)$. If $E_{n}$ occurs, and $Z_{\lambda-(n)} \leq n$, and $Z_{\lambda(n)} \geq n$, then for some $i \leq m_{n}$ there is at least one point of $\mathscr{X}_{n}$ in $B\left(x_{n, i}, \delta^{\prime} r_{n}\right)$ and at most $k(n)$ points of $\mathscr{X}_{n}$ in $B\left(x_{n, i}, r_{n}\right)$, and hence $L_{n, k(n)}>$ $\left(1-\delta^{\prime}\right) r_{n}$. Hence by the union bound

$$
\mathbb{P}\left[L_{n, k(n)} \leq r_{n}\left(1-\delta^{\prime}\right)\right] \leq \mathbb{P}\left[E_{n}^{c}\right]+\mathbb{P}\left[Z_{\lambda(n)}<n\right]+\mathbb{P}\left[Z_{\lambda-(n)}>n\right]
$$

which is summable in $n$ by the preceding estimates. Therefore by the Borel-Cantelli lemma,

$$
\mathbb{P}\left[\liminf \left(n L_{n, k(n)}^{d} / \log n\right) \geq u\left(1-\delta^{\prime}\right)^{d}\right]=1, \quad u<a^{-1} \hat{H}_{\beta}(b / d), \delta^{\prime} \in(0,1)
$$

so the result follows for this case too.

### 3.2 Proof of Theorem 2.1

In this subsection we assume, as in Theorem 2.1, that $A$ is a compact convex finite polytope in $\mathbb{R}^{d}$. We also assume that the probability measure $\mu$ has density $f$ with respect to Lebesgue measure on $\mathbb{R}^{d}$, and that $\left.f\right|_{A}$ is continuous at $x$ for all $x \in \partial A$, and that $f_{0}>0$, recalling from (2.3) that $f_{0}:=\operatorname{ess}_{\inf }^{x \in A} f f(x)$. Also we let $k(n)$ satisfy (2.2) for some $\beta \in[0, \infty]$. Let Vol denote $d$-dimensional Lebesgue measure

Lemma 3.3. There exists $\varepsilon^{\prime}>0$ depending only on $f_{0}$ and $A$, such that (3.1) holds.

Proof. Let $B_{0}$ be a (fixed) ball contained in $A$, and let $b$ denote the radius of $B_{0}$. For $x \in A$, let $S_{x}$ denote the convex hull of $B_{0} \cup\{x\}$. Then $S_{x} \subset A$ since $A$ is convex. If $x \notin B_{0}$, then for $r<b$ the set $B(x, r) \cap S_{x}$ is the intersection of $B(x, r)$ with a cone having vertex $x$, and since $A$ is bounded the angular volume of this cone is bounded away from zero, uniformly over $x \in A \backslash B_{0}$. Therefore $r^{-d} \operatorname{Vol}(B(x, r) \cap A)$ is bounded away from zero uniformly over $r \in(0, b)$ and $x \in A \backslash B_{0}$ (and hence over $x \in A$ ). Since we assume $f_{0}>0$, 3.1) follows.

Recall that $v(r, a)$ was defined just before Proposition 3.2. Recall that for each face $\varphi \in \Phi^{*}(A)$ we denote the angular volume of $A$ at $\varphi$ by $\rho_{\varphi}$, and set $f_{\varphi}:=\inf _{\varphi} f(\cdot)$ (if $\varphi \in \Phi(A)$ ) or $f_{\varphi}=f_{0}$ (if $\varphi=A$ ).

Lemma 3.4. Let $\varphi \in \Phi^{*}(A)$. Assume $\left.f\right|_{A}$ is continuous at $x$ for all $x \in \varphi$. Then, almost surely:

$$
\begin{align*}
\liminf _{n \rightarrow \infty}\left(n L_{n, k(n)}^{d} / k(n)\right) \geq\left(\rho_{\varphi} f_{\varphi}\right)^{-1} & & \text { if } \beta=\infty  \tag{3.3}\\
\liminf _{n \rightarrow \infty}\left(n L_{n, k(n)}^{d} / \log n\right) \geq\left(\rho_{\varphi} f_{\varphi}\right)^{-1} \hat{H}_{\beta}(D(\varphi) / d) & & \text { if } \beta<\infty \tag{3.4}
\end{align*}
$$

Proof. Let $a>f_{\varphi}$. Take $x_{0} \in \varphi$ such that $f\left(x_{0}\right)<a$. If $D(\varphi)>0$, assume also that $x_{0} \in \varphi^{o}$. By the assumed continuity of $\left.f\right|_{A}$ at $x_{0}$, for all small enough $r>0$ we have $\mu\left(B\left(x_{0}, r\right)\right) \leq a \rho_{\varphi} r^{d}$, so that $v\left(r, a \rho_{\varphi} r^{d}\right)=\Omega(1)$ as $r \downarrow 0$. Hence, by Proposition 3.2 (taking $b=0$ ), if $\beta=\infty$ then almost surely $\liminf _{n \rightarrow \infty} n L_{n, k(n)}^{d} / k(n) \geq 1 /\left(a \rho_{\varphi}\right)$, and (3.3) follows.

If $\beta<\infty$ and if $D(\varphi)=0$, then by Proposition 3.2 (with $b=0$ ), almost surely $\liminf _{n \rightarrow \infty}\left(n L_{n, k(n)}^{d} / \log n\right) \geq \hat{H}_{\beta}(0) /\left(a \rho_{\varphi}\right)$, and hence (3.4) in this case.

Now suppose $\beta<\infty$ and $D(\varphi)>0$. Take $\delta>0$ such that $f(x)<a$ for all $x \in$ $B\left(x_{0}, 2 \delta\right) \cap A$, and such that moreover $B\left(x_{0}, 2 \delta\right) \cap A=B\left(x_{0}, 2 \delta\right) \cap\left(x_{0}+\mathscr{K}_{\varphi}\right)$ (the cone $\mathscr{K}_{\varphi}$ was defined in Section 2). Then for all $x \in B\left(x_{0}, \delta\right) \cap \varphi$ and all $r \in(0, \delta)$, we have $\mu(B(x, r)) \leq a \rho_{\varphi} r^{d}$.

There is a constant $c>0$ such that for small enough $r>0$ we can find at least $c r^{-D(\varphi)}$ points $x_{i} \in B\left(x_{0}, \delta\right) \cap \varphi$ that are all at a distance more than $2 r$ from each other, and therefore $v\left(r, a \rho_{\varphi} r^{d}\right)=\Omega\left(r^{-D(\varphi)}\right)$ as $r \downarrow 0$. Thus by Proposition 3.2 we have

$$
\liminf _{n \rightarrow \infty}\left(n L_{n, k(n)}^{d} / k(n)\right) \geq\left(a \rho_{\varphi}\right)^{-1} \hat{H}_{\beta}(D(\varphi) / d)
$$

almost surely, and (3.4) follows.
If we assumed $\left.f\right|_{A}$ to be continuous on all of $A$, we would not need the next lemma because we could instead use Lemma 3.4 for $\varphi=A$ as well as for lower-dimensional faces. However, in Theorem 2.1 we make the weaker assumption that $\left.f\right|_{A}$ is continuous at $x$ only for $x \in \partial A$. In this situation, we also require the following lemma to deal with $\varphi=A$.

Lemma 3.5. It is the case that

$$
\begin{align*}
\mathbb{P}\left[\liminf \left(n L_{n, k(n)}^{d} / k(n)\right) \geq 1 /\left(\theta_{d} f_{0}\right)\right]=1 & & \text { if } \beta=\infty ;  \tag{3.5}\\
\mathbb{P}\left[\liminf _{n \rightarrow \infty}^{d}\left(n L_{n, k(n)}^{d} / \log n\right) \geq \hat{H}_{\beta}(1) /\left(\theta_{d} f_{0}\right)\right]=1 & & \text { if } \beta<\infty . \tag{3.6}
\end{align*}
$$

Proof. Let $\alpha>f_{0}$. Then by taking $B=A$ in [8, Lemma 6.4],

$$
\begin{equation*}
\liminf _{r \downarrow 0} r^{d} v\left(r, \alpha \theta_{d} r^{d}\right)>0 . \tag{3.7}
\end{equation*}
$$

Set $r_{n}:=\left(k(n) /\left(n \theta_{d} \alpha\right)\right)^{1 / d}$ if $\beta=\infty$, and set $r_{n}:=\left(\hat{H}_{\beta}(1)(\log n) /\left(n \theta_{d} \alpha\right)\right)^{1 / d}$ if $\beta<\infty$.
If $\beta=\infty$, then by (3.7) we can apply Proposition 3.2 (taking $a=\alpha \theta_{d}$ and $b=0$ ) to deduce that $\liminf _{n \rightarrow \infty} n L_{n, k(n)}^{d} / k(n) \geq\left(\theta_{d} \alpha\right)^{-1}$, almost surely, and (3.5) follows.

Suppose instead that $\beta<\infty$. By (3.7), $v\left(r, \alpha \theta_{d} r^{d}\right)=\Omega\left(r^{-d}\right)$ as $r \downarrow 0$. Hence by Proposition 3.2, almost surely $\liminf _{n \rightarrow \infty}\left(n L_{n, k(n)}^{d} / \log n\right) \geq\left(\alpha \theta_{d}\right)^{-1} \hat{H}_{\beta}(1)$. The result follows by letting $\alpha \downarrow f_{0}$.

Proof of Theorem 2.1] First suppose $\beta<\infty$. It is clear from (2.1) and (2.12) that $L_{n, k} \leq$ $R_{n, k+1}$ for all $n, k$. Also by (2.2) we have $(k(n)+1) / \log n \rightarrow \beta$ as $n \rightarrow \infty$. Therefore using [8, Theorem 4.2] for the second inequality below, we obtain almost surely that

$$
\begin{equation*}
\limsup _{n \rightarrow \infty}\left(\frac{n L_{n, k(n)}^{d}}{\log n}\right) \leq \limsup _{n \rightarrow \infty}\left(\frac{n R_{n, k(n)+1}^{d}}{\log n}\right) \leq \max _{\varphi \in \Phi^{*}(A)}\left(\frac{\hat{H}_{\beta}(D(\varphi) / d)}{f_{\varphi} \rho_{\varphi}}\right) \tag{3.8}
\end{equation*}
$$

Alternatively, this upper bound could be derived using (2.9) and the asymptotic upper bound on $M_{n}$ that we shall derive in the next section for the proof of Theorem 2.5,

By Lemmas 3.5 and 3.4, we have a.s. that

$$
\begin{equation*}
\liminf _{n \rightarrow \infty}\left(n L_{n, k(n)}^{d} / \log n\right) \geq \max _{\varphi \in \Phi^{*}(A)}\left(\frac{\hat{H}_{\beta}(D(\varphi) / d)}{f_{\varphi} \rho_{\varphi}}\right) \tag{3.9}
\end{equation*}
$$

and combining this with (3.8) yields (2.6).
Now suppose $\beta=\infty$. In this case, again using the inequality $L_{n, k} \leq R_{n, k+1}$ and [ 8 , Theorem 4.2], we obtain instead of (3.8) that a.s.

$$
\begin{equation*}
\limsup _{n \rightarrow \infty}\left(n L_{n, k(n)}^{d} / k(n)\right) \leq \max _{\varphi \in \Phi^{*}(A)}\left(\frac{1}{f_{\varphi} \rho_{\varphi}}\right) . \tag{3.10}
\end{equation*}
$$

Also by Lemmas 3.5 and 3.4, instead of (3.9) we have a.s. that

$$
\liminf _{n \rightarrow \infty}\left(n L_{n, k(n)}^{d} / k(n)\right) \geq \max _{\varphi \in \Phi^{*}(A)}\left(\frac{1}{f_{\varphi} \rho_{\varphi}}\right)
$$

and combining this with (3.10) yields (2.5).

### 3.3 A general upper bound

In this subsection we present an asymptotic upper bound for $M_{n, k(n)}$. As we did for the lower bound in Section 3.1, we shall give our result (Proposition 3.6 below) in a more general setting; we assume that $A$ is a general metric space endowed with two Borel measures $\mu$ and $\mu_{*}$ (possibly the same measure, possibly not). Assume that $\mu$ is a probability measure and that $\mu_{*}$ is a doubling measure, meaning that there is a constant $c_{*}$ (called a doubling constant for $\left.\mu_{*}\right)$ such that $\mu_{*}(B(x, 2 r)) \leq c_{*} \mu_{*}(B(x, r))$ for all $x \in A$ and $r>0$. We shall require further conditions on $A$ : an ordering condition (O), a condition on balls (B), a topological condition (T) and a geometrical condition $(\mathrm{G})$ as follows:
( O ) There is a total ordering of the elements of $A$.
(B) For all $x \in A$ and $r>0$, the ball $B(x, r)$ is connected.
(T) The space $A$ is unicoherent (see [5] Section 9.1]), and also connected.
(G) There exists $\delta_{1}>0$, and $K_{0} \in(1, \infty)$, such that for all $r<\delta_{1}$ and any $x \in A$, the number of components of $A \backslash B(x, r)$ is at most two, and if there are two components, at least one of these components has diameter at most $K_{0} r$.

Given $D \subset A$ and $r>0$, we write $D_{r}$ for $\{y \in A: \operatorname{dist}(y, D) \leq r\}$. Also, let $\kappa(D, r)$ be the $r$-covering number of $D$, that is, the minimal $m \in \mathbb{N}$ such that $D$ can be covered by $m$ balls centred in $D$ with radius $r$.

As before, given $\mu$ we assume $X_{1}, X_{2}, \ldots$ to be independent $\mu$-distributed random elements of $A$ with the $k$-connectivity threshold $M_{n, k}$ defined to be the minimal $r$ such that $G\left(\mathscr{X}_{n}, r\right)$ is $k$-connected, with $\mathscr{X}_{n}:=\left\{X_{1}, \ldots, X_{n}\right\}$.
Proposition 3.6 (General upper bound). Suppose that $\left(A, \mu, \mu_{*}\right)$ are as described above and A satisfies conditions $(O),(B),(T),(G)$. Let $\ell \in \mathbb{N}$ and let $d>0$. For each $j \in[\ell]$ let $a_{j}>0, b_{j} \geq 0$. Suppose that for each $K \in \mathbb{N}$, there exists $r_{0}(K)>0$ such that for all $r \in\left(0, r_{0}(K)\right)$, there is a partition $\{T(j, K, r), j \in[\ell]\}$ of $A$ with the following two properties. Firstly for each fixed $K \in \mathbb{N}, j \in[\ell]$, we have

$$
\begin{equation*}
\kappa(T(j, K, r), r)=O\left(r^{-b_{j}}\right) \quad \text { as } r \downarrow 0, \tag{3.11}
\end{equation*}
$$

and secondly, for all $K \in \mathbb{N}, j \in[\ell], r \in\left(0, r_{0}(K)\right)$ and any $G \subset A$ intersecting $T(j, K, r)$ with $\operatorname{diam}(G) \leq K r$, we have

$$
\begin{equation*}
\mu\left(G_{r} \backslash G\right) \geq a_{j} r^{d} . \tag{3.12}
\end{equation*}
$$

Assume (2.2). Then, almost surely,

$$
\begin{array}{ll}
\limsup _{n \rightarrow \infty}\left(n M_{n, k(n)}^{d} / k(n)\right) \leq \max _{j \in[\ell]}\left(a_{j}^{-1}\right) & \text { if } \beta=\infty \\
\limsup _{n \rightarrow \infty}^{d}\left(n M_{n, k(n)}^{d} / \log n\right) \leq \max _{j \in[\ell]}\left(a_{j}^{-1} \hat{H}_{\beta}\left(b_{j} / d\right)\right) & \text { if } \beta<\infty
\end{array}
$$

Later we shall use Proposition 3.6 in the case where $A$ is a convex polytope in $\mathbb{R}^{d}$ to prove Theorem [2.5, taking $\mu$ to be the measure with density $f$ and taking $\mu_{*}$ to be the restriction of Lebesgue measure to $A$ (in fact, if $f$ is bounded above then we could take $\mu_{*}=\mu$ instead). The sets in the partition each represent a region near to a particular face $\varphi \in \Phi^{*}(A)$ (if $\varphi=A$ the corresponding set in the partition is an interior region). In this case, coefficients $a_{j}$ in the measure lower bound (3.12) depend heavily on the geometry of the determining cone near a particular face.

As a first step towards proving Proposition 3.6, we spell out some useful consequences of the measure doubling property. In this result (and again later) we use $|\cdot|$ to denote the cardinality (number of elements) of a set.

Lemma 3.7. Let $\mu_{*}$ be a doubling measure on the metric space $A$, with doubling constant $c_{*}$. We have the following.
(i) For any $\varepsilon \in(0,1)$, there exists $\rho(\varepsilon) \in \mathbb{N}$ such that $\kappa(B(x, r), \varepsilon r) \leq \rho(\varepsilon)$ for all $x \in A, r \in(0, \infty)$.
(ii) For all $r \in(0,1)$ and all $D \subset A$, we can find $\mathscr{L} \subset D$ with $|\mathscr{L}| \leq \kappa(D, r / 5)$, such that $D \subset \cup_{x \in \mathscr{L}} B(x, r)$, and moreover the balls $B(x, r / 5), x \in \mathscr{L}$, are disjoint.

Proof. To prove (i), let $x \in A, r>0$. By the Vitali covering lemma, we can find a set $\mathscr{U} \subset B(x, r)$ such that balls $B(y, \varepsilon r / 5), y \in \mathscr{U}$ are disjoint and that $B(x, r) \subset \cup_{y \in \mathscr{U}} B(y, \varepsilon r)$. Set $\rho(\varepsilon):=\left\lceil c_{*}^{\left[\log _{2}(15 / \varepsilon)\right\rceil}\right\rceil$. Then by using the doubling property of $\mu_{*}$ repeatedly, we have $\mu_{*}(B(y, 3 r)) \leq \rho(\varepsilon) \mu_{*}(B(y, r / 5))$ for all $y \in A$. Moreover $B(x, 2 r) \subset B(y, 3 r)$ for all $y \in \mathscr{U}$. Also $\cup_{y \in \mathscr{U}} B(y, \varepsilon r / 5) \subset B(x, 2 r)$ and the union is disjoint. Thus

$$
|\mathscr{U}| \mu_{*}(B(x, 2 r)) \leq \sum_{y \in \mathscr{U}} \mu_{*}(B(y, 3 r)) \leq \rho(\varepsilon) \sum_{y \in \mathscr{U}} \mu_{*}(B(y, \varepsilon r / 5)) \leq \rho(\varepsilon) \mu_{*}(B(x, 2 r)),
$$

and therefore $|\mathscr{U}| \leq \rho(\varepsilon)$; the claim about $\kappa(B(x, r), \varepsilon r)$ follows.
Now we prove (ii). Let $\mathscr{L}^{0} \subset D$ with $\left|\mathscr{L}^{0}\right|=\kappa(D, r / 5)$ and with $B \subset \cup_{x \in \mathscr{L}} B(x, r / 5)$. By the Vitali covering lemma, we can find $\mathscr{L} \subset \mathscr{L}^{0}$ such that $D \subset \cup_{x \in \mathscr{L} B} B(x, r)$ and the balls $B(x, r / 5), x \in \mathscr{L}$, are disjoint, and (ii) follows.

Given countable $\sigma \subset A, r>0$ and $k \in \mathbb{N}$, we say that $\sigma$ is $(r, k)$-connected if the geometric graph $G(\sigma, r)$ is $k$-connected. Assuming condition (B) holds, we see that $\sigma$ is $(r, 1)$ connected if and only if $\sigma_{r / 2}$ is a connected subset of $A$.

Lemma 3.8 (Peierls argument). Assume ( $O$ ). Let $\ell \in \mathbb{N}, a \in[1, \infty)$. Let $r \in(0,1 / a)$ and $n \in \mathbb{N}$. Let $\mathscr{L} \subset A$ with the property that $|\mathscr{L} \cap B(x, r)| \leq \ell$ for all $x \in A$, and let $x_{0} \in \mathscr{L}_{r}$. Then the number of (ar,1)-connected subsets of $\mathscr{L}$ containing $x_{0}$ with cardinality $n$ is at most $c^{n}$, where $c$ depends only on $\ell, a$ and $c_{*}$.

Proof. First we claim that $|\mathscr{L} \cap B(x, a r)| \leq \ell \rho(1 / a)$ for all $x \in A$, where $\rho(1 / a)$ is as given in Lemma 3.7-(i). Indeed, we can cover $B(x, a r)$ by $\rho(1 / a)$ balls of radius $r$, and each of these balls contains at most $\ell$ points of $\mathscr{L}$.

There is a standard algorithm (of constructing a non-decreasing sequence of lists) for counting the connected sets of $\mathbb{Z}^{d}$; see [5, Lemma 9.3] for details of the algorithm.

The algorithm remains valid in this general setting, with the lexicographical ordering replaced by the total ordering of $A$ (using assumption (O)). This algorithm has to stop at time $n$ (cardinality of the set), and at each step the number of possibilities for the set of the added elements is bounded by $2^{\ell \rho(1 / a)}$ (all possible subsets of the set of points of $\mathscr{L}$ within distance ar from a fixed point); hence the number of $a r$-connected sets of cardinality $n$ is at most $2^{\ell \rho(1 / a) n}$.

Preparing for a proof of Proposition 3.6, we recall a condition that is equivalent to $k$-connectedness of a graph $G$. We say that non-empty sets $U, W \subset V$ in a graph $G$ with vertex set $V$ form a $k$-separating pair if (i) the subgraph of $G$ induced by $U$ is connected, and likewise for $W$; (ii) no element of $U$ is adjacent to any element of $W$; (iii) the number of vertices of $V \backslash(U \cup W)$ lying adjacent to $U \cup W$ is at most $k$. We say that $U$ is a $k$ separating set for $G$ if (i) the subgraph of $G$ induced by $U$ is connected, and (ii) at most $k$ vertices of $V \backslash U$ lie adjacent to $U$. The relevance of these definitions is presented in the following lemma.

Lemma 3.9. [5] Lemma 13.1] Let $G$ be a graph with more than $k+1$ vertices. Then $G$ is either $(k+1)$-connected, or it has $k$ separating pair, but not both.

By Lemma 3.9, to prove Proposition 3.6 it suffices to prove, for arbitrary $u>$ $\max _{j} a_{j}^{-1} \hat{H}_{\beta}\left(b_{j} / d\right)$, the non-existence of $(k(n)-1)$-separating pairs in $G\left(\mathscr{X}_{n}, r_{n}\right)$ with $r_{n}=(u \log n / n)^{1 / d}$, as $n \rightarrow \infty$. Notice that, for any fixed $K \in \mathbb{N}$, if $(U, W)$ is a $(k-1)$ separating pair, then either both $U$ and $W$ have diameter at least $K r_{n}$, or one of them, say $U$, is a $(k-1)$-separating set of diameter at most $K r_{n}$. Here by the diameter of a a non-empty set $U \subset A$ we mean the number $\operatorname{diam}(U):=\sup _{u, v \in U} \operatorname{dist}(u, v)$.

The goal is to prove that neither outcome is possible when $n \rightarrow \infty$. Let us first eliminate the existence of a small separating set.

Lemma 3.10. Suppose the assumptions of Proposition 3.6 hold. If $\beta=\infty$, let $u>$ $\max _{j} a_{j}^{-1}$ and for $n \in \mathbb{N}$, set $r_{n}=(u k(n) / n)^{1 / d}$. If $\beta<\infty$, let $u>\max _{j \in[\ell]} a_{j}^{-1} \hat{H}_{\beta}\left(b_{j} / d\right)$, and for $n \in \mathbb{N}$ set $r_{n}=(u(\log n) / n)^{1 / d}$. For $K \in \mathbb{N}$, let $E_{n}(K, u)$ be the event that there exists a $(k(n)-1)$-separating set for $G\left(\mathscr{X}_{n}, r_{n}\right)$ of diameter at most $K r_{n}$. Then, given any $K \in \mathbb{N}$, almost surely $E_{n}(K, u)$ occurs for only finitely many $n$.

Proof. First assume $\beta<\infty$. The condition on $u$ implies that $u a_{j}>\beta$ and $u a_{j} H\left(\beta /\left(u a_{j}\right)\right)>$ $b_{j} / d$, for each $j \in[\ell]$. Then we can and do choose $\beta^{\prime}>\beta$ and $\varepsilon \in(0,1 / 4)$ such that for
each $j \in[\ell],(1-3 \varepsilon)^{d} u a_{j}>\beta^{\prime}$ and

$$
(1-3 \varepsilon)^{d} u a_{j} H\left(\frac{\beta^{\prime}}{(1-3 \varepsilon)^{d} u a_{j}}\right)>\frac{b_{j}}{d}+\varepsilon .
$$

For $n \in \mathbb{N}$ define $k^{\prime}(n)=\left\lceil\beta^{\prime} \log n\right\rceil$.
Let $K \in \mathbb{N}$, and for $r \in\left(0, r_{0}(K)\right)$ let $T(j, K, r)$ be as in the assumptions of Proposition 3.6. For $j \in[\ell]$, we claim that $\kappa\left(T\left(j, K, r_{n}\right), \varepsilon r_{n} / 5\right)=O\left(r_{n}^{-b_{j}}\right)$ as $n \rightarrow \infty$. Indeed,

$$
\kappa\left(T\left(j, K, r_{n}\right), \varepsilon r_{n} / 5\right) \leq \kappa\left(T\left(j, K, r_{n}\right), r_{n}\right) \sup _{x \in A} \kappa\left(B\left(x, r_{n}\right), \varepsilon r_{n} / 5\right) \leq \rho \kappa\left(T\left(j, K, r_{n}\right), r_{n}\right),
$$

where $\rho=\rho(\varepsilon / 5)$ is the constant in Lemma3.7-(i). The claim follows from the assumption (3.11).

Choose $n_{0} \in \mathbb{N}$ such that $r_{n}<r_{0}(k)$ for all $n \in \mathbb{N}$ with $n \geq n_{0}$. By Lemma3.7-(i), for each $j \in[\ell]$ and $n \in \mathbb{N}$ we can find a set $\mathscr{L}_{n}^{j} \subset T\left(j, K, r_{n}\right)$, with $\left|\mathscr{L}_{n}^{j}\right| \leq \kappa\left(T\left(j, K, r_{n}\right), \varepsilon r_{n} / 5\right)=$ $O\left(r_{n}^{-b_{j}}\right)$, such that $T\left(j, K, r_{n}\right) \subset \cup_{x \in \mathscr{L}_{n}^{j}} B\left(x, \varepsilon r_{n}\right)$ and that the balls $B\left(x, r_{n} \varepsilon / 5\right), x \in \mathscr{L}_{n}^{j}$, are disjoint. Set

$$
\begin{equation*}
\mathscr{L}_{n}:=\cup_{i=1}^{\ell} \mathscr{L}_{n}^{j} . \tag{3.13}
\end{equation*}
$$

For $n \geq n_{0}, j \in[\ell]$ let $\mathscr{T}_{n}^{j}=\left\{\sigma \subset \mathscr{L}_{n}: \operatorname{diam}(\sigma) \leq 2 K r_{n}, \sigma \cap T\left(j, K, r_{n}\right) \neq \varnothing\right\}$. We claim that the cardinality of $\mathscr{T}_{n}^{j}$ is $O\left(\left|\mathscr{L}_{n}^{j}\right|\right)=O\left(r_{n}^{-b_{j}}\right)$. Indeed, $\sigma \cap T\left(j, K, r_{n}\right) \neq \varnothing$ means $\sigma \cap \mathscr{L}_{n}^{j} \neq \varnothing$. Moreover, as explained below,

$$
\begin{equation*}
\limsup _{n \rightarrow \infty} \sup _{x \in \mathscr{L}_{n}}\left|B\left(x, 2 K r_{n}\right) \cap \mathscr{L}_{n}\right|<\infty, \tag{3.14}
\end{equation*}
$$

and $\operatorname{diam}(\sigma) \leq 2 K r_{n}$. The claim about cardinality follows from this.
Now we show (3.14). By Lemma 3.7-(i), for $n$ large and for all $x \in A$, we can cover $B\left(x, 2 K r_{n}\right)$ by $\rho(\varepsilon /(10 K))$ balls of radius $r_{n} \varepsilon / 5$, and each of these balls contains at most $\ell$ points of $\mathscr{L}_{n}$.

For $n \geq n_{0}$ and $\sigma \subset \mathscr{L}_{n}$, set

$$
\begin{equation*}
D_{\sigma, n}:=\sigma_{(1-2 \varepsilon) r_{n}} \backslash \sigma_{\varepsilon r_{n}} \tag{3.15}
\end{equation*}
$$

Let $J \in \mathbb{N}$ with $J>1 / \varepsilon$. For $m \in \mathbb{N}$, define $z(m):=m^{J}$. For $\sigma \subset \mathscr{L}_{z(m)}$, define

$$
F_{m}(\sigma)=\left\{\mathscr{X}_{z(m)}\left(D_{\sigma, z(m)}\right)<k^{\prime}(z(m))\right\} .
$$

Now let $n \in \mathbb{N}$ and choose $m=m(n)$ such that $z(m) \leq n<z(m+1)$. Assume $z(m) \geq$ $n_{0}$. Suppose that $E_{n}(K, u)$ occurs and let $U$ be a $(k(n)-1)$-separating set of $G\left(\mathscr{X}_{n}, r_{n}\right)$ with $\operatorname{diam}(U) \leq K r_{n}$. We define its 'pixel version' $\sigma(U):=\mathscr{L}_{z(m(n))} \cap U_{\varepsilon r_{z(m(n))}}$.

Since $\sigma(U) \subset A$, there exists $j \in[\ell]$ such that $\sigma(U) \cap T\left(j, K, r_{z(m(n))}\right) \neq \varnothing$. By our choice of $\varepsilon$, provided $n$ is large enough we have $\operatorname{diam}(\sigma(U)) \leq 2 K r_{z(m(n))}$. Therefore $\sigma(U) \in \cup_{j=1}^{[\ell]} \mathscr{T}_{z(m(n))}^{j}$.

Since $U$ is $(k(n)-1)$-separating for $G\left(\mathscr{X}_{n}, r_{n}\right)$, we have $\mathscr{X}_{n}\left(U_{r_{n}} \backslash U\right)<k(n)$. We claim that $\mathscr{X}_{n}\left(D_{\sigma(U), z(m(n))}\right)<k(n)$ provided $n$ is large enough. Indeed, by the triangle inequality $\sigma(U)_{(1-2 \varepsilon) r_{z(m(n))}} \subset U_{(1-\varepsilon) r_{z(m(n))}} \subset U_{r_{n}}$ (for $n$ large), while $U \subset \sigma(U)_{\varepsilon r_{z(n(m))}}$. Thus $D_{\sigma(U), z(m(n))} \subset U_{r_{n}} \backslash U$, and the claim follows. Also, provided $n$ is large enough, we have $k(n) \leq k^{\prime}(z(m(n)))$. Thus we have the event inclusions

$$
\begin{aligned}
E_{n}(K, u) & \subset \cup_{j=1}^{\ell} \cup_{\sigma \in \mathscr{T}_{z(m(n))}^{j}}\left\{\mathscr{X}_{n}\left(D_{\sigma, z(m(n))}\right)<k(n)\right\} \\
& \subset \cup_{j=1}^{\ell} \cup_{\sigma \in \mathscr{T}_{z(m(n))}^{j}} F_{m(n)}(\sigma)
\end{aligned}
$$

By (3.15), for any $n \in \mathbb{N}$ and $\sigma \subset \mathscr{L}_{n}$ we have $D_{\sigma, n} \supset\left(\sigma_{\varepsilon r_{n}}\right)_{(1-3 \varepsilon) r_{n}} \backslash \sigma_{\varepsilon r_{n}}$. Hence by (3.12), for all large enough $n$ and all $\sigma \in \cup_{j \in[\ell]} \mathscr{T}_{n}^{j}$ we have $\mu\left(D_{\sigma, n}\right) \geq a_{j}(1-3 \varepsilon)^{d} r_{n}^{d}$. A simple coupling shows that, provided $m$ is large, we have

$$
\mathbb{P}\left[\cup_{j \in[\ell]} \cup_{\sigma \in \mathscr{T}_{z(m)}^{j}} F_{m}(\sigma)\right]=\sum_{j=1}^{\ell} O\left(r_{z(m)}^{-b_{j}}\right) \mathbb{P}\left[\operatorname{Bin}\left(z(m),(1-3 \varepsilon)^{d} a_{j} r_{z(m)}^{d}\right)<k^{\prime}(z(m))\right]
$$

By Lemma3.1(b) and our choice of $r_{n}$ and $\varepsilon$, provided $m$ is large, we have

$$
\begin{array}{r}
\mathbb{P}\left[\cup_{j \in[\ell]} \cup_{\sigma \in \mathscr{T}_{z(m)}^{j}} F_{m}(\sigma)\right] \\
=O(1) \sum_{j=1}^{\ell} \exp \left(\left(b_{j} / d\right) \log z(m)-(1-3 \varepsilon)^{d} u a_{j} H\left(\frac{\beta^{\prime}}{(1-3 \varepsilon)^{d} u a_{j}}\right) \log z(m)\right)=O\left(m^{-J \varepsilon}\right),
\end{array}
$$

which is summable in $m$.
It follows from the Borel-Cantelli lemma that almost surely $\cup_{j \in[\ell]} \cup_{\sigma \in \mathscr{T}_{z(m)}^{j}} F_{m}(\sigma)$ occurs only for finitely many $m$ which implies that $E_{n}(K, u)$ occurs for only finitely many $n$. This completes the proof of the case $\beta<\infty$.

Now assume $\beta=\infty$. For the rest of the proof assume also that $\varepsilon \in(0,1)$ is such that $u a_{j}(1-\varepsilon)^{d}>1$ for all $j \in[\ell]$. We do not have to go through the subsequence argument as before because the growth of $k(n)$ is super-logarithmic. Now redefine $F_{n}(\sigma):=$ $\left\{\mathscr{X}_{n}\left(D_{\sigma, n}\right)<k(n)\right\}$. If $E_{n}(K, u)$ happens then we now redefine the pixel version of the separating set $U$ as

$$
\sigma(U):=\mathscr{L}_{n} \cap U_{\varepsilon r_{n}}
$$

and enumerate the possible shapes $\sigma$ of the pixel version. Thus we have

$$
E_{n}(K, u) \subset \cup_{j=1}^{\ell} \cup_{\sigma \in \mathscr{T}_{n}^{j}} F_{n}(\sigma)
$$

Using estimates of $\left|\mathscr{T}_{n}^{j}\right|$, we have

$$
\mathbb{P}\left[E_{n}(K, u)\right]=\sum_{j=1}^{\ell} O\left(r_{n}^{-b_{j}}\right) \mathbb{P}\left[\operatorname{Bin}\left(n,(1-3 \varepsilon)^{d} a_{j} r_{n}^{d}\right)<k(n)\right] .
$$

Noticing $r_{n}^{-1}=O\left(n^{1 / d}\right)$, and applying Lemma3.1-(b) leads to

$$
\mathbb{P}\left[E_{n}(K, u)\right]=O\left(n^{b_{j} / d}\right) \sum_{j=1}^{\ell} \exp \left(-(1-3 \varepsilon)^{d} a_{j} u k(n) H\left(\frac{k(n)}{(1-3 \varepsilon)^{d} a_{j} u k(n)}\right)\right)
$$

which is summable in $n$, and the claim follows by the Borel-Cantelli lemma.
The following lemma eliminates the existence of a $(k(n)-1)$-separating pair with both diameters larger than $K r_{n}$.

Lemma 3.11. Let the assumptions of Proposition 3.6 hold. If $\beta=\infty$, let $u>\max _{j} a_{j}^{-1}$ and for $n \in \mathbb{N}$, set $r_{n}=(u k(n) / n)^{1 / d}$. If $\beta<\infty$, let $u>\max _{j \in[\ell]} a_{j}^{-1} \hat{H}_{\beta}\left(b_{j} / d\right)$, and for $n \in \mathbb{N}$ set $r_{n}=(u(\log n) / n)^{1 / d}$. For $K \in \mathbb{N}$ let $H_{n}(K, u)$ denote the event that there exists $a$ $(k(n)-1)$-separating pair $(U, W)$ in $G\left(\mathscr{X}_{n}, r_{n}\right)$ such that $\min (\operatorname{diam}(U), \operatorname{diam}(W)) \geq K r_{n}$. Then there exists $K_{1} \in \mathbb{N}$ such that almost surely $H_{n}\left(K_{1}, u\right)$ occurs for only finitely many $n$.

Proof. Suppose $H_{n}(K, u)$ holds. Then $U_{r_{n} / 2}$ and $W_{r_{n} / 2}$ are disjoint and connected in $A$. One of the components of $A \backslash U_{r_{n} / 2}$ contains $W$, denoted by $W^{\prime}$. Set $U^{\prime}=A \backslash W^{\prime}$. Then $U \subset U^{\prime}, W \subset W^{\prime}$ and $A=W^{\prime} \cup U^{\prime}$. Let $\partial_{W} U:=\overline{W^{\prime}} \cap \overline{U^{\prime}}$. Then $\partial_{W} U$ is connected by the unicoherence of $A$. Moreover, any continuous path in $A$ connecting $U$ and $W$ must pass through $\partial_{W} U$.

Recall $\delta_{1}$ and $K_{0}$ in the assumption (G). We claim (and show in the next few paragraphs) that

$$
\begin{equation*}
\operatorname{diam}\left(\partial_{W} U\right) \geq \frac{1}{2 K_{0}+2} \min \left(\delta_{1} / 3, \operatorname{diam}(W) / 3, \operatorname{diam}(U) / 3\right) \tag{3.16}
\end{equation*}
$$

Suppose the opposite. Setting $b=\operatorname{diam}\left(\partial_{W} U\right)$, we can find $x \in A$ such that $\partial_{W} U \subset B(x, b)$, and we can find $X \in U \backslash B(x, b), Y \in W \backslash B(x, b)$. Since $b<\delta_{1} / 3$, the number of components of $A \backslash B(x, b)$ is at most two. There have to be two components because otherwise $X$ and $Y$ can be connected by a path in $A$ disjoint from $\partial U$, which is a contradiction.

Suppose that $X$ lies in the component of $A \backslash B(x, b)$ having diameter at most $K_{0} b$, denoted by $Q_{X}$, and $Y$ lies in the other component, denoted by $Q_{Y}$ (if it is the other way round we reverse the roles of $X$ and $Y$ in the rest of this argument). We claim that there exists $X^{\prime} \in U$ such that $\operatorname{dist}\left(X, X^{\prime}\right)>\left(2 K_{0}+2\right) b$. If not, then for any $X_{1}, X_{2} \in U$,
we have by triangle inequality that $\operatorname{dist}\left(X_{1}, X_{2}\right) \leq 2\left(2 K_{0}+2\right) b$, yielding that $\operatorname{diam}(U) \leq$ $2\left(2 K_{0}+2\right) b$, contradicting $\operatorname{diam}(U)>3\left(2 K_{0}+2\right) b$ by the negation of (3.16).

We claim that $\operatorname{dist}(X, B(x, b)) \leq K_{0} b$. To see this, using the assumed connectivity of $A$, take a continuous path in $A$ from $X$ to $Y$. The first exit point of this path from $Q_{X}$ lies in $B(x, b)$ (else it would not be an exit point from $Q_{X}$ ) but also in the closure of $Q_{X}$, and hence in $B\left(X, K_{0} b\right)$. This yields the latest claim.

We show that $X^{\prime}$ and $Y$ have to be in the same component of $A \backslash B(x, b)$. To this end, notice first that $X^{\prime}$ cannot be in $Q_{X}$, because for any $z \in Q_{X}$,

$$
\operatorname{dist}(X, z) \leq K_{0} b<\left(2 K_{0}+2\right) b
$$

Secondly, $X^{\prime}$ cannot be in $B(x, b)$ either because for any $z \in B(x, b)$, we have

$$
\operatorname{dist}(z, X) \leq \operatorname{dist}(X, B(x, b))+2 b \leq\left(K_{0}+2\right) b<\left(2 K_{0}+2\right) b .
$$

Therefore, $X^{\prime}$ has to be in $Q_{Y}$, and we reach again to a contradiction that $X^{\prime}$ and $Y$ can be connected by a path in $A$ disjoint from $\partial U$. We have thus proved (3.16).

Let $\varepsilon \in(0,1 / 9)$ and let $\mathscr{L}_{n}$ be as defined at (3.13) (the $\varepsilon$ does not have to be the same as it was there). Recall that $\mathscr{L}_{n}$ has the covering property that for every $x \in A$ we have $\mathscr{L}_{n} \cap B\left(x, r_{n} \varepsilon\right) \neq \varnothing$ and the spacing property that $\left|\mathscr{L}_{n} \cap B\left(x, r_{n} \varepsilon / 3\right)\right| \leq \ell$ for all such $x$.

Define $D_{W} U=\left\{x \in \mathscr{L}_{n}: B\left(x, \varepsilon r_{n}\right) \cap \partial_{W} U \neq \varnothing\right\}$. Then by the covering property of $\mathscr{L}_{n},\left(D_{W} U\right)_{\varepsilon r_{n}}$ is connected and covers $\partial_{W} U$. That is, $D_{W} U$, as a subset of the metric space $A$, is $\left(2 \varepsilon r_{n}, 1\right)$-connected.

By (3.16) and the occurrence of $H_{n}(K, u)$, we have

$$
2 \varepsilon r_{n}\left|D_{W} U\right| \geq \operatorname{diam}\left(\partial_{W} U\right) \geq \min \left(\delta_{1} / 3, K r_{n} / 3\right) /\left(2 K_{0}+2\right)
$$

Therefore, provided $n$ is large, we have $\left|D_{W} U\right| \geq K /\left(6 \varepsilon\left(2 K_{0}+2\right)\right)$.
We claim that there is a constant $c \in(0, \infty)$, independent of $n$, such that for all $q \in \mathbb{N}$, if $\left|D_{W} U\right|=q$ then $D_{W} U$ can take at most $O\left(r_{n}^{-\max \left(b_{j}\right)} c^{q}\right)$ possible 'shapes'. Indeed, given $x_{0} \in \mathscr{L}_{n}$, set

$$
\mathscr{U}_{n, q}\left(x_{0}\right):=\left\{\sigma \subset \mathscr{L}_{n}:|\sigma|=q, \sigma \text { is }\left(2 \varepsilon r_{n}, 1\right) \text {-connected, } x_{0} \in \sigma\right\} .
$$

Then $D_{W} U \in \cup_{j \in[\ell]} \cup_{x_{0} \in T\left(j, K, r_{n}\right) \cap \mathscr{L}_{n}} \mathscr{U}_{n, q}\left(x_{0}\right)$. By Lemma 3.8, we have $\left|\mathscr{U}_{n, q}\left(x_{0}\right)\right| \leq c^{q}$ for some finite constant $c$. Recall from the proof of Lemma3.10 that $\left|T\left(j, K, r_{n}\right) \cap \mathscr{L}_{n}\right|=$ $O\left(r_{n}^{-b_{j}}\right)$. The claim follows.

For all $n \in \mathbb{N}$, if $x \in \partial_{W} U$ then $\operatorname{dist}(x, U)=r_{n} / 2$. Therefore by the triangle inequality, $\left(D_{W} U\right)_{\varepsilon r_{n} / 5} \subset U_{r}$, while $U \cap\left(D_{W} U\right)_{\varepsilon r_{n} / 5}=\varnothing$; hence $\mathscr{X}_{n} \cap\left(D_{W} U\right)_{\varepsilon r_{n} / 5}=\varnothing$. This, together with the the union bound, yields that

$$
\begin{equation*}
\mathbb{P}\left[H_{n}(K, u)\right] \leq \sum_{q \geq K /\left(6 \varepsilon\left(2 K_{0}+2\right)\right)} \sum_{\sigma} \mathbb{P}\left[\mathscr{X}_{n}\left(\sigma_{\varepsilon r_{n} / 5}\right)<k(n)\right], \tag{3.17}
\end{equation*}
$$

where the second sum is over all possible shapes $\sigma \subset \mathscr{L}_{n}$ of cardinality $q$ that are $\left(2 \varepsilon r_{n}, 1\right)$ connected. Since every point in $A$ is covered at most $\ell$ times, by (3.12) (with $G=\{z\}$ ), there exists $\varepsilon_{1} \in(0,1)$ such that

$$
\mu\left(\sigma_{\varepsilon r_{n} / 5}\right) \geq(1 / \ell) \sum_{z \in \sigma} \mu\left(B\left(z, \varepsilon r_{n} / 5\right)\right) \geq(q / \ell) \varepsilon_{1}\left(\varepsilon r_{n} / 5\right)^{d}
$$

Suppose $\beta<\infty$. Set $\varepsilon_{2}:=\left(\varepsilon_{1} / \ell\right)(\varepsilon / 5)^{d}$. By (3.17) and Lemma3.1(b), provided $n$ is large,

$$
\begin{aligned}
& \mathbb{P}\left[H_{n}(K, u)\right] \leq \sum_{q \geq K /\left(6 \varepsilon\left(2 K_{0}+2\right)\right)} O\left(r_{n}^{-\max \left(b_{j}\right)} c^{q}\right) \mathbb{P}\left[\operatorname{Bin}\left(n, \varepsilon_{2} q r_{n}^{d}\right)<(\beta+1) \log n\right] \\
& \quad=O(1) \sum_{q \geq K /\left(6 \varepsilon\left(2 K_{0}+2\right)\right)} c^{q} \exp \left(\left(\max \left(b_{j}\right) / d\right) \log n-\varepsilon_{2} q u H\left(\frac{\beta+1}{\varepsilon_{2} q u}\right) \log n\right)
\end{aligned}
$$

By the continuity of $H(\cdot)$ and the fact that $H(0)=1$, there exists $q_{0}>16 /\left(\varepsilon_{2} u\right)$ such that for any $q>q_{0}$, we have $H\left(\frac{\beta+1}{q \varepsilon_{2} u}\right)>1 / 2$ and $q u \varepsilon_{2}>4 \max \left(b_{j}\right) / d$. Choosing $K=$ $6 \varepsilon\left(2 K_{0}+2\right) q_{0}$ so that $q \geq q_{0}$ in the sum, we see that the exponent of the exponential is bounded above by

$$
\left(\max \left(b_{j}\right) / d\right) \log n-q \varepsilon_{2}(u / 2) \log n \leq-\left(q u \varepsilon_{2} / 4\right) \log n
$$

Therefore, we have for $n$ large that

$$
\begin{aligned}
\mathbb{P}\left[H_{n}(K, u)\right] & =O(1) \sum_{q \geq q_{0}} c^{q} \exp \left(-q u\left(\varepsilon_{2} / 4\right) \log n\right) \\
& =O(1) \sum_{q \geq q_{0}} \exp \left(-q u\left(\varepsilon_{2} / 8\right) \log n\right)=O\left(\exp \left(-q_{0} u\left(\varepsilon_{2} / 8\right) \log n\right)\right)=O\left(n^{-2}\right)
\end{aligned}
$$

The result in this case follows by applying the Borel-Cantelli lemma.
If $\beta=\infty$, then by (3.17) and the estimates of $\left|\cup_{j} \cup_{x_{0}} \mathscr{U}_{n, q}\left(x_{0}\right)\right|$ as previously, we have

$$
\mathbb{P}\left[H_{n}(K, u)\right] \leq \sum_{q \geq K /\left(6 \varepsilon\left(2 K_{0}+2\right)\right)} O\left(r_{n}^{-\max \left(b_{j}\right)} c^{q}\right) \mathbb{P}\left[\operatorname{Bin}\left(n,(q / \ell) \varepsilon_{1}\left(\varepsilon r_{n} / 5\right)^{d}\right)<k(n)\right]
$$

We have $r_{n}^{-\max \left(b_{j}\right)}=O\left(n^{\max \left(b_{j}\right) / d}\right)$, and by Lemma3.1-(b),

$$
\mathbb{P}\left[H_{n}(K, u)\right] \leq \sum_{q \geq K /\left(6 \varepsilon\left(2 K_{0}+2\right)\right)} c^{q} \exp \left(\left(\max \left(b_{j}\right) / d\right) \log n-q \varepsilon_{2} u k(n) H\left(\frac{k(n)}{q \varepsilon_{2} u k(n)}\right)\right)
$$

As before, we can choose $K=K_{1}$ (large) so that the $H(\cdot)$ term in every summand is bounded from below by $1 / 2$. By the super-logarithmic growth of $k(n)$, we conclude that $\mathbb{P}\left[H_{n}(K, u)\right] \leq n^{-2}$ provided $n$ is large, so that the Borel-Cantelli lemma gives the result in this case too.

Proof of Proposition 3.6. If $\beta=\infty$ then let $u>\max _{j \in[\ell]}\left(a_{j}^{-1}\right)$ and set $r(n):=u(k(n) / n)^{1 / d}$. If $\beta<\infty$ then let $u>\max _{j \in[\ell]}\left(a_{j}^{-1} \hat{H}_{\beta}\left(b_{j} / d\right)\right)$ and set $r_{n}:=(u(\log n) / n)^{1 / d}$. By Lemmas 3.10 and 3.11, there exists $K \in \mathbb{N}$ such that almost surely, $E_{n}(K, u) \cup H_{n}(K, u)$ occurs for at most finitely many $n$. By Lemma 3.9, if $M_{n, k}>r_{n}$ then $E_{n}(K, u) \cup H_{n}(K, u)$ occurs. Therefore $M_{n, k(n)} \leq r_{n}$ for all large enough $n$, almost surely, and the result follows.

### 3.4 Proof of Theorem 2.5

In this subsection we go back to the mathematical framework in Section 2, that is, we make the assumptions in the statement of Theorem 2.5. In particular we return to assuming $A$ is a convex polytope in $\mathbb{R}^{d}$ with $d \geq 2$, and the probability measure $\mu$ has a density $f$. We shall check the conditions required in order to apply Proposition 3.6 ,

To check these conditions, we shall use the following lemma and notation.
Lemma 3.12. [8, Lemma 6.12] Suppose $\varphi, \varphi^{\prime}$ are faces of $A$ with $D(\varphi)>0$ and $D\left(\varphi^{\prime}\right)=$ $d-1$, and with $\varphi \backslash \varphi^{\prime} \neq \varnothing$. Then $\varphi^{o} \cap \varphi^{\prime}=\varnothing$ and $K\left(\varphi, \varphi^{\prime}\right)<\infty$, where we set

$$
\begin{equation*}
K\left(\varphi, \varphi^{\prime}\right):=\sup _{x \in \varphi^{o}} \frac{\operatorname{dist}(x, \partial \varphi)}{\operatorname{dist}\left(x, \varphi^{\prime}\right)} \tag{3.18}
\end{equation*}
$$

Now define

$$
\begin{equation*}
K(A):=\max \left\{K\left(\varphi, \varphi^{\prime}\right): \varphi, \varphi^{\prime} \in \Phi(A), D(\varphi)>0, D\left(\varphi^{\prime}\right)=d-1, \varphi \backslash \varphi^{\prime} \neq \varnothing\right\} \tag{3.19}
\end{equation*}
$$

Then $K(A)<\infty$ since $A$ is a finite polytope.
For $j \in\{0,1, \ldots, d\}$ let $\Phi_{j}(A)$ denote the collection of $j$-dimensional faces of $A$. For any $D \subset A$ and $r>0$ set $D_{r}=\{x \in A: B(x, r) \cap D \neq \varnothing\}$.
Lemma 3.13. The restriction of Lebesgue measure to $A$ has the doubling property. Moreover the conditions $(O),(B),(T)$ and $(G)$ are met.

Proof. First we verify the doubling property. By the proof of Lemma 3.3, there exists $b>0$ such that $\inf _{x \in A, r \in(0, b]} r^{-d} \operatorname{Vol}(B(x, r) \cap A)>0$. Since $\operatorname{Vol}(B(x, 2 r) \cap A)$ is at most $2^{d} \theta_{d} r^{d}$ for $r \leq b$, and is at $\operatorname{most} \operatorname{Vol}(A)$ for all $r$, the doubling property follows.

Points of $A$ can be ordered by using the lexicographic ordering inherited from $\mathbb{R}^{d}$, thus (O). Since $A$ is convex, for all $x \in A$ and $r>0$ the set $B(x, r) \cap A$ is convex and hence connected, implying (B). All convex polytopes are simply connected, and therefore unicoherent [5] Lemma 9.1], hence (T). Condition (G) follows immediately from Proposition 3.14 , which we prove below.

Proposition 3.14. Let A be a convex finite polytope in $\mathbb{R}^{d}$. Let $N(\cdot)$ denote the number of components of a set. There exists $\delta_{1}>0$ such that for any $x \in A$ any $r \in\left(0, \delta_{1}\right)$, we have $N(A \backslash B(x, r)) \leq 2$. Moreover, in the case that $N(A \backslash B(x, r))=2$, the diameter of the smaller component is at most $c r$, where $c$ is a constant depending only on $A$.

Proof of Proposition 3.14. Write $B$ for $B(x, r)$. Our first observation is that if $y \in A \backslash B$, then there is at least one vertex $v \in \Phi_{0}(A)$ such that the line segment $[y, v]$ is contained in $A \backslash B$. Indeed, if this failed then for each $v \in \Phi_{0}(A)$ there would exist a point $u(v) \in$ $[y, v] \cap B$. But then since $A$ is convex, $y$ would lie in the convex hull of $\left\{v: v \in \Phi_{0}(A)\right\}$, and therefore also in the convex hull of $\left\{u(v): v \in \Phi_{0}(A)\right\}$. Indeed, there exist $\alpha_{v} \geq 0$ with $\sum_{v \in \Phi_{0}(A)} \alpha_{v}=1$ such that $y=\sum_{v \in \Phi_{0}(A)} \alpha_{v} v$, and there exists $\beta_{v} \in[0,1]$ such that $u(v)=$ $\beta_{v} y+\left(1-\beta_{v}\right) v$. Substituting $v$ by $u(v)$ and rearranging terms shows that $y=\sum_{v} \alpha_{v}^{\prime} u(v)$ with some nonnegative $\alpha_{v}^{\prime}$ and $\sum_{v} \alpha_{v}^{\prime}=1$, thus the claim. But then since $B$ is convex we would have $y \in B$, a contradiction.

We refer to the one-dimensional faces $\varphi \in \Phi_{1}(A)$ as edges of $A$. Our second observation is that if the number of edges of $A$ that intersect $B$ is at most 1 , then $A \backslash B$ is connected. Indeed, in this case, for any distinct $v, v^{\prime} \in \Phi_{0}(A)$ there is a path along edges of $A$ from $v$ to $v^{\prime}$ that avoids $B$. For example, if $v, v^{\prime}$ lie in the same two-dimensional face $\varphi$ of $A$ then since $B$ intersects at most one edge of the polygon $\varphi$, there is a path from $v$ to $v^{\prime}$ along the edges of $\varphi$ avoiding $B$. Therefore all $v \in \Phi_{0}(A)$ lie in the same component of $A \backslash B$, so using the first observation we deduce that $A \backslash B$ is connected.

Recall the definition of $K(A)$ at (3.19). Our third observation is that if $\operatorname{dist}(v, B) \geq$ $3 r K(A)$ for all $v \in \Phi_{0}(A)$ then $A \backslash B$ is connected. Indeed, suppose dist $(v, B) \geq 3 r K(A)$ for all $v \in \Phi_{0}(A)$. Suppose $\varphi, \varphi^{\prime}$ are distinct edges of $A$ with $B \cap \varphi \neq \varnothing$, and pick $y \in B \cap \varphi$. Then $\operatorname{dist}(y, \partial \varphi) \geq 3 r K(A)$ so that by (3.18), $\operatorname{dist}\left(y, \varphi^{\prime}\right) \geq 3 r K(A) / K\left(\varphi, \varphi^{\prime}\right) \geq 3 r$. Hence by the triangle inequality $\operatorname{dist}\left(B, \varphi^{\prime}\right) \geq 3 r-2 r=r$, so that $B \cap \varphi^{\prime}=\varnothing$. Hence $B$ intersects at most one edge of $A$, and by our second observation $A \backslash B$ is connected.

Suppose $\operatorname{dist}(v, B) \leq 3 r K(A)$ for some $v \in \Phi_{0}(A)$. Provided $r$ is small enough, this cannot happen for more than one $v \in \Phi_{0}(A)$. If $u, u^{\prime} \in \Phi_{0}(A) \backslash\{v\}$, then $v \notin\left[u, u^{\prime}\right]$ so $\operatorname{dist}\left(v,\left[u, u^{\prime}\right]\right)>0$. Therefore provided $r$ is small enough, $\left[u, u^{\prime}\right] \subset A \backslash B$. Thus provided $r$ is small enough, all vertices $u \in \Phi_{0}(A) \backslash\{v\}$ lie in the same component of $A \backslash B$. If also $v$ lies in this component, then (by our first observation) $A \backslash B$ is connected.

Thus $A \backslash B$ is disconnected only if $v$ lies in a different component of $A \backslash B$ than all the other vertices. In that case, for $y \in A \backslash B$, if $[y, v] \subset A \backslash B$ then $y$ is in the same component as $v$; otherwise (by our first observation) $y$ lies in the same component as all of the other vertices, and thus $A \backslash B$ has exactly two components.

If $A \backslash B$ has two components, and $y \in A \backslash B$ with $\|y-v\|>(3 K(A)+2) r$, then we claim $[y, v] \cap B \neq \varnothing$. Indeed, for each $u \in \Phi_{0}(A) \backslash\{v\}$ the ray from $v$ in the direction of $u$ passes through $B$. But then by an argument based on the convexity of both $A$ and $B$, the ray from $v$ in the direction of $y$ must also pass through $B$. Since $\operatorname{dist}(v, B) \leq 3 r K(A)$ and $\operatorname{diam}(B)=2 r$, this ray must pass through $B$ at a distance at most $(3 K(A)+2) r$ from $v$, i.e. before it reaches $y$, and the claim follows. Therefore $y$ lies in the component of $A \backslash B$ that does not contain $v$, and thus the component containing $v$ has diameter at most $(3 K(A)+2) r$.

To apply Proposition 3.6, we need to define a partition of $A$ for each small $r>0$, then estimate the corresponding covering numbers and $\mu$-measures in (3.12).

Taking into account a variety of boundary effects near $\partial A$, one should consider separately regions near different faces of $A$. It is however not trivial to construct this partition in such a way that we can obtain tight $\mu$-measure estimates in (3.12). The matter is complicated by the fact that the set $G$ in (3.12) that intersects a region near $\varphi$ is potentially close to a lower dimensional face lying inside $\partial \varphi$. We can avoid the boundary complications by constructing inductively from regions near to the highest dimensional face to the lowest, with increasing 'thickness'. The partition made of $T(\varphi, r)$ 's defined below and the left-over interior region is defined for this purpose.

Let $\left(K_{j}\right)_{j \in \mathbb{N}}$ be an increasing sequence with $K_{1}=1$, and with $K_{j+1}>(2 K(A)+1) K_{j}$ for each $j \in \mathbb{N}$. For instance, we could take $K_{j}=(2 K(A)+2)^{j-1}$.

Now for each $r>0$ and $\varphi \in \Phi(A)$, define the set

$$
T(\varphi, r):=\varphi_{r K_{d-D(\varphi)}} \backslash \cup_{\varphi^{\prime} \in \Phi(A): \varphi^{\prime} \subsetneq \varphi}\left(\varphi^{\prime}\right)_{r K_{d-D\left(\varphi^{\prime}\right)}}
$$

where the $T$ stands for 'territory'. Also define $T(A, r):=A \backslash \cup_{\varphi \in \Phi(A)} \varphi_{r K_{d-D(\varphi)}}$ For each $\varphi \in \Phi^{*}(A)$, we have $T(\varphi, r) \neq \varnothing$ for all $r$ sufficiently small. Hence, there exists $r_{0}>0$ such that for all $\varphi$ and all $r<r_{0}, T(\varphi, r) \neq \varnothing$. Moreover, territories of distinct faces are disjoint, as we show in the following lemma.

Lemma 3.15. There exists $r_{0}>0$ such that for all $r \in\left(0, r_{0}\right)$, and any distinct $\varphi, \varphi^{\prime} \in$ $\Phi^{*}(A)$, it holds that $T(\varphi, r) \cap T\left(\varphi^{\prime}, r\right)=\varnothing$. Moreover, if $\varphi, \varphi^{\prime} \in \Phi(A)$ with $\varphi \backslash \varphi^{\prime} \neq \varnothing$, and $y \in T(\varphi, r)$, then $B(y, r)$ does not intersect $\varphi^{\prime}$.

Proof. We can (and do) assume without loss of generality that $\varphi \backslash \varphi^{\prime} \neq \varnothing$ and $\varphi^{\prime} \backslash \varphi \neq \varnothing$. Indeed, if $\varphi \subset \varphi^{\prime}$, then by construction $T\left(\varphi^{\prime}, r\right) \cap T(\varphi, r)=\varnothing$.

If $\varphi$ is a vertex, then $\operatorname{dist}\left(\varphi, \varphi^{\prime}\right)>0$ so that $T(\varphi, r) \cap T\left(\varphi^{\prime}, r\right)=\varnothing$ for all $r$ small. So it suffices to consider the case where $D(\varphi)>0$ and $D\left(\varphi^{\prime}\right)>0$.

Let $j:=d-D(\varphi)$ and $j^{\prime}:=d-D\left(\varphi^{\prime}\right)$. We can and do assume $j^{\prime} \leq j \leq d-1$.
If there exists $x \in T(\varphi, r) \cap T\left(\varphi^{\prime}, r\right)$, then we can find $z \in \varphi, z^{\prime} \in \varphi^{\prime}$ such that $\|x-z\| \leq$ $r K_{j}$ and $\left\|x-z^{\prime}\right\| \leq r K_{j^{\prime}}$. Therefore $\operatorname{dist}\left(z, \varphi^{\prime}\right) \leq r\left(K_{j}+K_{j^{\prime}}\right) \leq 2 r K_{j}$.

On the other hand, since $x \in T(\varphi, r), \operatorname{dist}(x, \partial \varphi) \geq r K_{j+1}$, and so by the triangle inequality, $r K_{j+1}-r K_{j} \leq \operatorname{dist}(z, \partial \varphi) \leq K(A) \operatorname{dist}\left(z, \varphi^{\prime}\right)$, where the last inequality comes from (3.18). Combining the estimates leads to $K_{j+1} \leq(2 K(A)+1) K_{j}$, which is a contradiction. The first claim follows.

Moving to the second claim, let $\varphi, \varphi^{\prime} \in \Phi(A)$ with $\varphi \backslash \varphi^{\prime} \neq \varnothing$. Suppose $y \in \varphi_{r}^{\prime}$. Set

$$
\tilde{\Phi}:=\left\{\psi \in \Phi(A): \psi \subsetneq \varphi^{\prime}, y \in \psi_{K_{D-d(\psi)}}\right\} .
$$

If $\tilde{\Phi}=\varnothing$ then $y \in T\left(\varphi^{\prime}, r\right)$. Otherwise, choose $\psi \in \tilde{\Phi}$ of minimal dimension. Then $y \in T(\psi, r)$. Either way, $y \notin T(\varphi, r)$ by the first claim. Therefore $T(\varphi, r) \cap \varphi_{r}^{\prime}=\varnothing$.

As a last ingredient for applying Proposition 3.6, for each $J>1$ and $r \in(0,1)$, we construct a partition of $A$ and show (3.12) for all $G$ with diameter at most $J r$. The coefficients $a_{j}$ depend on the location of $G$ in relation to faces of $A$.
Lemma 3.16. Let $J \in \mathbb{N}$ and $\varepsilon>0$. Then the following hold:
(i) For each $\varphi \in \Phi(A)$ we have $\kappa(T(\varphi, 2 J r), r)=O\left(r^{-D(\varphi)}\right)$ as $r \downarrow 0$. Moreover we have $\kappa\left(A \backslash \cup_{\varphi \in \Phi(A)} T(\varphi, 2 J r), r\right)=O\left(r^{-d}\right)$ as $r \downarrow 0$.
(ii) For all small $r>0$ and any $G \subset A$ with $\operatorname{diam}(G) \leq J r$, if it intersects $T(\varphi, 2 J r)$ for some $\varphi \in \Phi^{*}(A)$, then

$$
\begin{equation*}
\mu\left(G_{r} \backslash G\right) \geq(1-\varepsilon) f_{\varphi} \rho_{\varphi} r^{d} . \tag{3.20}
\end{equation*}
$$

Proof. Item (i) follows by the definition of $T(\varphi, r)$. Indeed, $\varphi$ is contained in a bounded region within a $D(\varphi)$-dimensional affine space, and therefore can be covered by $O\left(r^{-D(\varphi)}\right)$ balls of radius $r$. If we then take balls of radius $r\left(1+2 J K_{d-D(\varphi)}\right)$ with the same centres, they will cover $T(\varphi, 2 J r)$, and one can then cover each of the larger balls with a fixed number of balls of radius $r$.

For (ii), let $G \subset A$ with $\operatorname{diam}(G) \leq J r$. Suppose first that $G \cap T(\varphi, 2 J r) \neq \varnothing$ for some $\varphi \in \Phi(A)$. Let $x_{0} \in G \cap T(\varphi, 2 J r)$. Then $G_{r} \subset B\left(x_{0}, 2 J r\right)$. By Lemma 3.15, we see that $B\left(x_{0}, 2 J r\right)$ does not intersect any $\varphi^{\prime} \in \Phi(A)$ with $\varphi \backslash \varphi^{\prime} \neq \varnothing$. It follows that

$$
\begin{equation*}
B\left(x_{0}, 2 J r\right) \cap A=B\left(x_{0}, 2 J r\right) \cap\left(z_{0}+\mathscr{K}_{\varphi}\right) \tag{3.21}
\end{equation*}
$$

where $\mathscr{K}_{\varphi}$ is the cone determined by $\varphi$ and $z_{0}$ is the point of $\varphi$ closest to $x_{0}$.
Set $D(x, r):=B(x, r) \cap\left(x+\mathscr{K}_{\varphi}\right)$. We claim that for any $x \in G$, we have $D(x, r) \subset$ $A$. Indeed, given $y \in D(x, r)$, we can write $y=z_{0}+\left(x-z_{0}\right)+(y-x)=: z_{0}+\theta_{1}+\theta_{2}$. Here $\theta_{1}, \theta_{2} \in \mathscr{K}_{\varphi}$. By convexity and scale invariance of $\mathscr{K}_{\varphi}$, we have $\theta_{1}+\theta_{2} \in \mathscr{K}_{\varphi}$ so $y \in z_{0}+\mathscr{K}_{\varphi}$. Also $\left\|y-x_{0}\right\| \leq\|y-x\|+\left\|x-x_{0}\right\| \leq 2 J r$, and hence $y \in A$ by (3.21), as claimed.

It follows that (with $\oplus$ denoting Minkowski addition)

$$
\mu\left(G_{r} \backslash G\right) \geq \mu((G \oplus D(o, r)) \backslash G) \geq \operatorname{Vol}((G \oplus D(o, r)) \backslash G) \inf _{x \in G \oplus D(o, r)} f(x)
$$

By the Brunn-Minkowski inequality [5, Section 5.3], we have $\operatorname{Vol}(G \oplus D(o, r)) \geq \operatorname{Vol}(G)+$ $\operatorname{Vol}(D(o, r))=\operatorname{Vol}(G)+\rho_{\varphi} r^{d}$. The claim (3.20) follows by the continuity of $f$ on $\partial A$.

As for the case $\varphi=A$, suppose now that $G \cap T(A, 2 J r) \neq \varnothing$. Taking $x \in G \cap T(A, 2 J r)$ we have $\operatorname{dist}(x, \partial A) \geq 2 J r$, and hence $\operatorname{dist}(G, \partial A) \geq 2 J r-J r=J r$. Therefore $G_{r} \subset A$, so by the Brunn-Minkowski inequality

$$
\mu\left(G_{r} \backslash G\right) \geq f_{0} \operatorname{Vol}((G \oplus B(o, r)) \backslash G) \geq f_{0} \theta_{d} r^{d}
$$

In this case $f_{\varphi}=f_{0}$ and $\rho_{\varphi}=\theta_{d}$, and the claim (3.20) follows in this case too, completing the proof of (ii).

Proof of Theorem 2.5] By (2.9), and Theorem 2.1, it suffices to prove the upper bound. We shall do this by applying Proposition 3.6 in the situation of Theorem 2.5.

By Lemma 3.13, the restriction to $A$ of Lebesgue measure has the doubling property, and conditions (O), (B), (T) and (G) are satisfied

To apply Proposition 3.6, we need to define (for each $K \in \mathbb{N}$ and each $r \in\left(0, r_{0}(K)\right)$ ) a finite partition $\{T(j, K, r)\}$. For this we take the sets $T(\varphi, 2 K r), \varphi \in \Phi^{*}(A)$. By Lemma 3.15, and the definition of $T(A, r)$, for each $K \in \mathbb{N}$ there exists $r_{0}(K)>0$ such that for $r \in\left(0, r_{0}(K)\right)$ the sets $T(\varphi, 2 K r), \varphi \in \Phi^{*}(A)$, do indeed partition $A$.

For each $\varphi \in \Phi^{*}(A)$, using Lemma 3.16-(i) we have the condition (3.11) in Proposition 3.6, where the constant denoted $b_{j}$ there is equal to $D(\varphi)$. Also, using Lemma 3.16-(ii) we have the condition (3.12) in proposition 3.6 where the constant denoted $a_{j}$ there is equal to $(1-\varepsilon) f_{\varphi} \rho_{\varphi}$.

Suppose $\beta<\infty$. By applying Proposition 3.6 in the manner described above we see that for $\varepsilon>0$, we have

$$
\limsup _{n \rightarrow \infty} n\left(M_{n, k(n)}\right)^{d} / \log n \leq \max _{\varphi \in \Phi^{*}(A)}\left(\frac{\hat{H}_{\beta}(D(\varphi) / d)}{(1-\varepsilon) f_{\varphi} \rho_{\varphi}}\right)
$$

and the result follows. If $\beta=\infty$, using corresponding part of Proposition 3.6 gives the result in this case too.

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