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Stationary anisotropic Stokes, Oseen and Navier–Stokes systems: Periodic solutions in $\mathbb{R}^n$

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First, the solution uniqueness, existence and regularity for stationary anisotropic (linear) Stokes and generalised Oseen systems with constant viscosity coefficients in a compressible framework are analysed in a range of periodic Sobolev (Bessel-potential) spaces in $\mathbb{R}^n$. By the Galerkin algorithm and the Brower fixed point theorem, the existence of solution to the stationary anisotropic (nonlinear) Navier–Stokes incompressible system is shown in a periodic Sobolev space for any $n \geq 2$. Then the solution uniqueness and regularity results for stationary anisotropic periodic Navier–Stokes system are established for $n \in \{2, 3, 4\}$.

KEYWORDS

anisotropic Stokes, Oseen and Navier-Stokes equations, existence, higher dimensions, periodic Sobolev spaces, uniqueness and regularity, relaxed ellipticity

MSC CLASSIFICATION

35J57, 35Q30, 46E35, 76D, 76M

1 | INTRODUCTION

Analysis of Stokes and Navier–Stokes equations is an established and active field of research in the applied mathematical analysis; see, for example [1–7], and references therein. In [8–12], this field has been extended to the transmission and boundary-value problems for stationary Stokes and Navier–Stokes equations of anisotropic fluids, particularly, with relaxed ellipticity condition on the viscosity tensor.

In this paper, we present some further results in this direction considering space-periodic solutions in $\mathbb{R}^n$, $n \geq 2$, to the stationary Stokes, generalised Oseen and Navier–Stokes equations of anisotropic fluids, with an emphasis on solution regularity. The periodic setting is interesting on its own, modelling fluid flow in periodic composite structures, and is also a common element of homogenisation theories for inhomogeneous fluids. First, the solution uniqueness, existence and regularity for stationary anisotropic (linear) Stokes and generalised Oseen systems with constant viscosity coefficients in a compressible framework are analysed in a range of periodic Sobolev (Bessel-potential) spaces on n-dimensional flat torus. By the Galerkin algorithm, the linear results are employed to show existence of solution to the stationary anisotropic (nonlinear) Navier–Stokes incompressible system on torus in a periodic Sobolev space for any $n \geq 2$. Then the solution uniqueness and regularity results for stationary anisotropic Navier–Stokes system on torus are established for $n \in \{2, 3, 4\}$. This paper, particularly, extends to the Oseen system and to the Navier–Stokes system for $n > 3$ the results obtained in our paper [13].

2 | ANISOTROPIC STOKES, NAVIER–STOKES AND OSEEN SYSTEMS

Let $\mathfrak{L}$ denote the second-order differential operator represented in the component-wise divergence form as

$$(\mathfrak{L}\mathbf{u})_k := \partial_\alpha \left(a_{kj}^{\alpha\beta} E_{j\beta}(\mathbf{u}) \right),\ k = 1, \dots, n, \tag{2.1}$$

where $\mathbf{u} = (u_1, \dots, u_n)^\top$, $E_{j\beta}(\mathbf{u}) := \frac{1}{2}(\partial_j u_\beta + \partial_\beta u_j)$ are the entries of the symmetric part $\mathbb{E}(\mathbf{u})$ of $\nabla\mathbf{u}$ (the gradient of $\mathbf{u}$) and $a_{kj}^{\alpha\beta}$ are constant components of the viscosity coefficient tensor $\mathbb{A} := \left(a_{kj}^{\alpha\beta} \right)_{1\le i,j,\alpha,\beta\le n}$ (cf. [14]). We also denoted $\partial_j := \frac{\partial}{\partial x_j}$. Here and further on, the Einstein summation convention in repeated indices from 1 to n is used unless stated otherwise.

The following symmetry conditions are assumed (see [15, (3.1),(3.3)]):

$$a_{kj}^{\alpha\beta} = a_{\alpha j}^{k\beta} = a_{k\beta}^{\alpha j}. \tag{2.2}$$

In addition, we require that tensor $\mathbb{A}$ satisfies the (relaxed) ellipticity condition in terms of all *symmetric* matrices in $\mathbb{R}^{n\times n}$ with *zero matrix trace*; see [9, 10]. That is, we assume that there exists a constant $C_{\mathbb{A}} > 0$ such that

$$C_{\mathbb{A}} a_{kj}^{\alpha\beta} \zeta_{k\alpha}\zeta_{j\beta} \ge |\boldsymbol{\zeta}|^2, \forall \boldsymbol{\zeta} = (\zeta_{k\alpha})_{k,\alpha=1,\dots,n} \in \mathbb{R}^{n\times n} \text{ such that } \boldsymbol{\zeta} = \boldsymbol{\zeta}^\top \text{ and } \sum_{k=1}^{n} \zeta_{kk} = 0, \tag{2.3}$$

where $|\boldsymbol{\zeta}|^2 = \zeta_{k\alpha}\zeta_{k\alpha}$, while the superscript $\top$ denotes the transpose of a matrix.

The tensor $\mathbb{A}$ is endowed with the norm

$$\|\mathbb{A}\| := \max\left\{ |a_{kj}^{\alpha\beta}| : k, j, \alpha, \beta = 1 \dots, n \right\}. \tag{2.4}$$

Symmetry conditions (2.2) lead to the following equivalent form of the operator $\mathfrak{L}$

$$(\mathfrak{L}\mathbf{u})_k = \partial_\alpha \left(a_{kj}^{\alpha\beta} \partial_\beta u_j \right),\ k = 1, \dots, n. \tag{2.5}$$

Let $\mathbf{u}$ be an unknown vector field, p be an unknown scalar field, $\mathbf{f}$ be a given vector field and g be a given scalar field. Then the linear equations

$$-\mathfrak{L}\mathbf{u} + \nabla p = \mathbf{f},\ \operatorname{div}\ \mathbf{u} = g \tag{2.6}$$

determine the anisotropic stationary Stokes system with viscosity tensor coefficient $\mathbb{A} = \left(A^{\alpha\beta}\right)_{1\le\alpha,\beta\le n}$ in a compressible framework.

The nonlinear system

$$-\mathfrak{L}\mathbf{u} + \nabla p + (\mathbf{u}\cdot\nabla)\mathbf{u} = \mathbf{f},\ \operatorname{div}\ \mathbf{u} = g \tag{2.7}$$

is called the anisotropic stationary Navier–Stokes system with viscosity tensor coefficient $\mathbb{A} = \left(A^{\alpha\beta}\right)_{1\le\alpha,\beta\le n}$ in a compressible framework.

In addition, we will also consider a linearised version of the anisotropic stationary Navier–Stokes system (2.7),

$$-\mathfrak{L}\mathbf{u} + \nabla p + (\mathbf{U}\cdot\nabla)\mathbf{u} = \mathbf{f},\ \operatorname{div}\ \mathbf{u} = g, \tag{2.8}$$

where $\mathbf{U}$ is a given function. We will call (2.8) the anisotropic stationary Oseen system. If $\mathbf{U} \equiv 0$, then (2.8) reduces to the Stokes system (2.6).

If $g = 0$ in (2.6), (2.7) and (2.8), then these equations are reduced, respectively, to the incompressible anisotropic stationary Stokes, Navier–Stokes and Oseen systems.

In the *isotropic case*, the tensor $\mathbb{A}$ reduces to

$$a_{kj}^{\alpha\beta} = \lambda\delta_{k\alpha}\delta_{j\beta} + \mu\left(\delta_{\alpha j}\delta_{\beta k} + \delta_{\alpha\beta}\delta_{kj}\right),\ 1 \le i, j, \alpha, \beta \le n, \tag{2.9}$$

where λ and μ are real constant parameters with $\mu > 0$ (cf., e.g., Appendix III, Part I, Section 1 in [7]) and $\delta_{k\alpha}$ is the Kronecker symbol. Then (2.5) becomes

$$\mathfrak{L}\mathbf{u} = (\lambda + \mu)\nabla \text{div } \mathbf{u} + \mu\Delta\mathbf{u}. \tag{2.10}$$

Then it is immediate that condition (2.3) is fulfilled (cf. [10]), and thus, our results apply also to the Stokes, Navier–Stokes and Oseen systems in the *isotropic case*. Assuming $\lambda = 0$, $\mu = 1$, we arrive at the classical mathematical formulations of isotropic Stokes, Navier–Stokes and Oseen systems.

3 | SOME PERIODIC FUNCTION SPACES

Let us introduce some function spaces on torus and periodic function spaces (see, e.g., [16, p.26], [17, 18], [19, Chapter 3], [3, Section 1.7.1], [6, Chapter 2], for more details).

Let $n \geq 1$ be an integer and $\mathbb{T}$ be the n-dimensional flat torus that can be parameterised as the semiopen cube $\mathbb{T} = [0, 1)^n \subset \mathbb{R}^n$ (cf. [20, p. 312]). In what follows, $\mathcal{D}(\mathbb{T}) = \mathcal{C}^\infty(\mathbb{T})$ denotes the space of infinitely smooth real or complex functions on the torus. As usual, $\mathbb{N}$ denotes the set of natural numbers, $\mathbb{N}_0$ the set of natural numbers augmented by 0, and $\mathbb{Z}$ the set of integers.

Let $\boldsymbol{\xi} \in \mathbb{Z}^n$ denote the n-dimensional vector with integer components. We will further need also the set

$$\dot{\mathbb{Z}}^n := \mathbb{Z}^n \setminus \{\mathbf{0}\}.$$

Extending the torus parameterisation to $\mathbb{R}^n$, it is often useful to identify $\mathbb{T}$ with the quotient space $\mathbb{R}^n \setminus \mathbb{Z}^n$. Then the space of functions $\mathcal{C}^\infty(\mathbb{T})$ on the torus can be identified with the space of $\mathbb{T}$-periodic (1-periodic) functions $\mathcal{C}^\infty_\# = \mathcal{C}^\infty_\#(\mathbb{R}^n)$ that consists of functions $\phi \in \mathcal{C}^\infty(\mathbb{R}^n)$ such that

$$\phi(\mathbf{x} + \xi) = \phi(\mathbf{x}) \quad \forall \xi \in \mathbb{Z}^n, \quad \forall \mathbf{x} \in \mathbb{R}^n. \tag{3.1}$$

Similarly, the Lebesgue space on the torus $L_p(\mathbb{T})$, $1 \leq p \leq \infty$, can be identified with the periodic Lebesgue space $L_{p\#} = L_{p\#}(\mathbb{R}^n)$ that consists of functions $\phi \in L_{p,\text{loc}}(\mathbb{R}^n)$, which satisfy the periodicity condition (3.1) (for a.e. $\mathbf{x} \in \mathbb{R}^n$).

The space dual to $\mathcal{D}(\mathbb{T})$, that is, the space of linear bounded functionals on $\mathcal{D}(\mathbb{T})$, called the space of torus distributions, is denoted by $\mathcal{D}'(\mathbb{T})$ and can be identified with the space of periodic distributions $\mathcal{D}'_\#$ acting on $\mathcal{C}^\infty_\#$.

The toroidal/periodic Fourier transform mapping a function $g \in \mathcal{C}^\infty_\#$ to a set of its Fourier coefficients $\hat{g}$ is defined as (see, e.g., [19, Definition 3.1.8])

$$\hat{g}(\xi) = [\mathcal{F}_\mathbb{T} g](\xi) := \int_\mathbb{T} e^{-2\pi i \mathbf{x}\cdot\xi} g(\mathbf{x}) d\mathbf{x}, \ \xi \in \mathbb{Z}^n.$$

and can be generalised to the Fourier transform acting on a distribution $g \in \mathcal{D}'_\#$.

For any $\xi \in \mathbb{Z}^n$, let $|\xi| := \left(\sum_{j=1}^n \xi_j^2\right)^{1/2}$ be the Euclidean norm in $\mathbb{Z}^n$ and let us denote

$$\rho(\xi) := (1 + |\xi|^2)^{1/2}.$$

Evidently,

$$\frac{1}{2}\rho(\xi)^2 \leq |\xi|^2 \leq \rho(\xi)^2 \quad \forall \xi \in \dot{\mathbb{Z}}^n. \tag{3.2}$$

Similar to [19, Definition 3.2.2], for $s \in \mathbb{R}$ we define the *periodic/toroidal Sobolev (Bessel-potential) spaces* $H^s_\# := H^s_\#(\mathbb{R}^n) := H^s(\mathbb{T})$, which consist of the torus distributions $g \in \mathcal{D}'(\mathbb{T})$, for which the norm

$$\|g\|_{H^s_\#} := \|\rho^s \hat{g}\|_{\ell_2} := \left(\sum_{\xi \in \mathbb{Z}^n} \rho(\xi)^{2s} |\hat{g}(\xi)|^2\right)^{1/2} \tag{3.3}$$

is finite, that is, the series in (3.3) converges. Here, $\|\cdot\|_{\ell_2}$ is the standard norm in the space of square summable sequences. By [19, Proposition 3.2.6], $H^s_\#$ are Hilbert spaces.

The dual product between $g \in H^s_\#$ and $f \in H^{-s}_\#$, $s \in \mathbb{R}$, is defined (cf. [19, Definition 3.2.8]) as

$$\langle g, f\rangle_{\mathbb{T}} := \langle \hat{g}, \hat{f}\rangle_{\mathbb{Z}^n} := \sum_{\xi\in\mathbb{Z}^n} \hat{g}(\xi)\hat{f}(-\xi). \tag{3.4}$$

If $s = 0$, that is, $g, f \in L_{2\#}$, then (3.4) reduces to

$$\langle g, f\rangle_{\mathbb{T}} = \int_{\mathbb{T}} g(\mathbf{x})f(\mathbf{x})d\mathbf{x}.$$

Hence for any $s \in \mathbb{R}$, the space $H^{-s}_\#$ is adjoint (dual) to $H^s_\#$, that is, $H^{-s}_\# = (H^s_\#)^*$. Similar to, for example, [21, p.76], one can show that

$$\|g\|_{H^s_\#} = \sup_{f\in H^{-s}_\#, f\neq 0} \frac{|\langle g, f\rangle_{\mathbb{T}}|}{\|f\|_{H^{-s}_\#}}. \tag{3.5}$$

For $g \in H^s_\#$, $s \in \mathbb{R}$ and $m \in \mathbb{N}_0$, let us consider the partial sums

$$g_m(\mathbf{x}) = \sum_{\xi\in\mathbb{Z}^n, |\xi|\leq m} \hat{g}(\xi)e^{2\pi i\mathbf{x}\cdot\xi}.$$

Evidently, $g_m \in C^\infty_\#$, $\hat{g}_m(\xi) = \hat{g}(\xi)$ if $|\xi| \leq m$ and $\hat{g}_m(\xi) = 0$ if $|\xi| > m$. This implies that $\|g - g_m\|_{H^s_\#} \to 0$ as $m \to \infty$ and hence we can write

$$g(\mathbf{x}) = \sum_{\xi\in\mathbb{Z}^n} \hat{g}(\xi)e^{2\pi i\mathbf{x}\cdot\xi}, \tag{3.6}$$

where the Fourier series converges in the sense of norm (3.3). Moreover, since g is an arbitrary distribution from $H^s_\#$, this also implies that the space $C^\infty_\#$ is dense in $H^s_\#$ for any $s \in \mathbb{R}$ (cf. [19, Exercise 3.2.9]).

There holds the compact embedding $H^t_\# \hookrightarrow H^s_\#$ if $t > s$, embeddings $H^s_\# \subset C^m_\#$ if $m \in \mathbb{N}_0$, $s > m + n/2$, and moreover, $\bigcap_{s\in\mathbb{R}} H^s_\# = C^\infty_\#$ (cf. [19, Exercises 3.2.10 and 3.2.11 and Corollary 3.2.12]). Note also that the periodic norms on $H^s_\#$ are equivalent to the corresponding standard (nonperiodic) Bessel potential norms on $\mathbb{T}$ as a cubic domain (see, e.g., [17, Section 13.8.1]).

By (3.3), $\|g\|^2_{H^s_\#} = |\hat{g}(\mathbf{0})|^2 + |g|^2_{H^s_\#}$, where

$$|g|_{H^s_\#} := \|\rho^s\hat{g}\|_{\dot{\ell}_2} := \left(\sum_{\xi\in\dot{\mathbb{Z}}^n} \rho(\xi)^{2s}|\hat{g}(\xi)|^2\right)^{1/2}$$

is the seminorm in $H^s_\#$.

For any $s \in \mathbb{R}$, let us also introduce the space

$$\dot{H}^s_\# := \{g \in H^s_\# : \langle g, 1\rangle_{\mathbb{T}} = 0\}. \tag{3.7}$$

The definition implies that if $g \in \dot{H}^s_\#$, then $\hat{g}(\mathbf{0}) = 0$ and

$$\|g\|_{\dot{H}^s_\#} = \|g\|_{H^s_\#} = |g|_{H^s_\#} = \|\rho^s\hat{g}\|_{\dot{\ell}_2}. \tag{3.8}$$

Denoting $\dot{C}^\infty_\# := \{g \in C^\infty_\# : \langle g, 1\rangle_{\mathbb{T}} = 0\}$, then $\bigcap_{s\in\mathbb{R}} \dot{H}^s_\# = \dot{C}^\infty_\#$.

Definition (3.7) also implies that the space adjoint to $\dot{H}^s_\#$ can be expressed as the quotient space,

$$(\dot{H}^s_\#)^* = (H^s_\#)^*/\mathbb{R} = H^{-s}_\#/\mathbb{R}.$$

Identifying the quotient space $H_{\#}^{-s}/\mathbb{R}$ with the space $\dot{H}_{\#}^{-s}$, we then identify the space $\dot{H}_{\#}^{-s}$ with the space dual to $\dot{H}_{\#}^{s}$.

The corresponding spaces of n-component vector functions/distributions are denoted as $\mathbf{L}_{q\#} := (L_{q\#})^n$, $\mathbf{H}_{\#}^{s} := (H_{\#}^{s})^n$, and so forth.

Note that the norm $\|\nabla(\cdot)\|_{\mathbf{H}_{\#}^{s-1}}$ is an equivalent norm in $\dot{H}_{\#}^{s}$. Indeed, by (3.6),

$$\nabla g(\mathbf{x}) = 2\pi i \sum_{\xi\in\dot{\mathbb{Z}}^n} \xi e^{2\pi i\mathbf{x}\cdot\xi}\hat{g}(\xi),\ \ \widehat{\nabla g}(\xi) = 2\pi i\xi\hat{g}(\xi)\quad \forall g \in \dot{H}_{\#}^{s},$$

and then (3.2) and (3.8) imply that

$$2\pi^2\|g\|^2_{H_{\#}^{s}} = 2\pi^2\|g\|^2_{\dot{H}_{\#}^{s}} = 2\pi^2|g|^2_{H_{\#}^{s}} \leq \|\nabla g\|^2_{\mathbf{H}_{\#}^{s-1}} \leq 4\pi^2|g|^2_{H_{\#}^{s}} = 4\pi^2\|g\|^2_{\dot{H}_{\#}^{s}} = 4\pi^2\|g\|^2_{H_{\#}^{s}}\quad \forall g \in \dot{H}_{\#}^{s}. \tag{3.9}$$

The vector counterpart of (3.9) takes the form

$$2\pi^2\|\mathbf{v}\|^2_{\mathbf{H}_{\#}^{s}} = 2\pi^2\|\mathbf{v}\|^2_{\dot{\mathbf{H}}_{\#}^{s}} \leq \|\nabla \mathbf{v}\|^2_{(H_{\#}^{s-1})^{n\times n}} \leq 4\pi^2\|\mathbf{v}\|^2_{\dot{\mathbf{H}}_{\#}^{s}} = 4\pi^2\|\mathbf{v}\|^2_{\mathbf{H}_{\#}^{s}}\quad \forall \mathbf{v} \in \dot{\mathbf{H}}_{\#}^{s}. \tag{3.10}$$

We will further need the first Korn inequality

$$\|\nabla \mathbf{v}\|^2_{(L_{2\#})^{n\times n}} \leq 2\|\mathbb{E}(\mathbf{v})\|^2_{(L_{2\#})^{n\times n}}\quad \forall \mathbf{v} \in \mathbf{H}_{\#}^{1} \tag{3.11}$$

that can be easily proved by adapting, for example, the proof in [21, Theorem 10.1] to the periodic Sobolev space; compare also [15, Theorem 2.8].

Let us also define the Sobolev spaces of divergence-free functions and distributions,

$$\dot{\mathbf{H}}_{\#\sigma}^{s} := \left\{\mathbf{w} \in \dot{\mathbf{H}}_{\#}^{s} : \operatorname{div}\ \mathbf{w} = 0\right\},\ s \in \mathbb{R}, \tag{3.12}$$

endowed with the same norm (3.3). Similarly, $\mathbf{C}_{\#\sigma}^{\infty}$ and $\mathbf{L}_{q\#\sigma}$ denote the subspaces of divergence-free vector-functions from $\mathbf{C}_{\#}^{\infty}$ and $\mathbf{L}_{q\#}$, respectively, and so forth.

4 | STATIONARY ANISOTROPIC PERIODIC STOKES SYSTEM

Let $n \geq 2$. In this section, we generalise to the isotropic and anisotropic (linear) Stokes systems in compressible framework and to a range of Sobolev spaces the analysis available in [6, Section 2.2].

For the unknowns $(\mathbf{u}, p) \in \dot{\mathbf{H}}_{\#}^{s} \times \dot{H}_{\#}^{s-1}$ and the given data $(\mathbf{f}, g) \in \dot{\mathbf{H}}_{\#}^{s-2} \times \dot{H}_{\#}^{s-1}$, $s \in \mathbb{R}$, let us consider the Stokes system

$$-\mathfrak{L}\mathbf{u} + \nabla p = \mathbf{f}, \tag{4.1}$$

$$\operatorname{div}\ \mathbf{u} = g, \tag{4.2}$$

that should be understood in the sense of distributions, that is,

$$\langle -\mathfrak{L}\mathbf{u} + \nabla p, \boldsymbol{\phi}\rangle_{\mathbb{T}} = \langle \mathbf{f}, \boldsymbol{\phi}\rangle_{\mathbb{T}}\quad \forall \boldsymbol{\phi} \in \mathbf{C}_{\#}^{\infty}, \tag{4.3}$$

$$\langle \operatorname{div}\ \mathbf{u}, \phi\rangle_{\mathbb{T}} = \langle g, \phi\rangle_{\mathbb{T}}\quad \forall \phi \in C_{\#}^{\infty}. \tag{4.4}$$

For $\boldsymbol{\xi} \in \dot{\mathbb{Z}}^n$, let us employ $\bar{e}_{\xi}(\mathbf{x}) = e^{-2\pi i x\cdot\xi}$ as ϕ in (4.4) and $\bar{e}_{\xi}(\mathbf{x})$, multiplied by the unit coordinate vector, as $\boldsymbol{\phi}$ in (4.3). Then recalling (2.5), we arrive for each $\boldsymbol{\xi} \in \dot{\mathbb{Z}}^n$ at the following algebraic system for the Fourier coefficients, $\hat{u}_j(\boldsymbol{\xi})$, $k = 1, 2, \dots, n$, and $\hat{p}(\boldsymbol{\xi})$.

$$4\pi^2\xi_\alpha a_{kj}^{\alpha\beta}\xi_\beta\hat{u}_j(\boldsymbol{\xi}) + 2\pi i\xi_k\hat{p}(\boldsymbol{\xi}) = \hat{f}_k(\boldsymbol{\xi})\quad \forall \boldsymbol{\xi} \in \dot{\mathbb{Z}}^n,\ k = 1, 2, \dots, n \tag{4.5}$$

$$2\pi i\xi_j\hat{u}_j(\boldsymbol{\xi}) = \hat{g}(\boldsymbol{\xi})\quad \forall \boldsymbol{\xi} \in \dot{\mathbb{Z}}^n. \tag{4.6}$$

System (4.5)–(4.6) can be written in the form

$$\mathfrak{S}(\xi)\begin{pmatrix}\hat{\mathbf{u}}(\xi)\\ \hat{p}(\xi)\end{pmatrix}=\begin{pmatrix}\hat{\mathbf{f}}(\xi)\\ \hat{g}(\xi)\end{pmatrix}\quad\forall\xi\in\dot{\mathbb{Z}}^n, \tag{4.7}$$

where $\mathfrak{S}(\xi)$ is the $(n+1)\times(n+1)$ matrix with entries

$$\mathfrak{S}_{\ell j}(\xi)=\begin{cases}4\pi^2\xi_\alpha a_{\ell j}^{\alpha\beta}\xi_\beta, & \ell,j=1,\dots,n;\\ -2\pi i\xi_\ell, & \ell=1,\dots,n,\ j=n+1;\\ -2\pi i\xi_j, & \ell=n+1,\ j=1,\dots,n;\\ 0, & \ell=j=n+1.\end{cases} \tag{4.8}$$

Here, $i^2=-1$, and $\xi=(\xi_1,\dots,\xi_n)$. The matrix $\mathfrak{S}(\xi)$ is in fact the toroidal/periodic symbol (cf. [19, Section 4.1.1]) of the anisotropic Stokes system (4.1)–(4.2).

Lemma 4.1. *Let $n\geq 2$ and condition (2.3) hold. Then the matrix $\mathfrak{S}(\xi)$ is nonsingular for any $\xi\in\dot{\mathbb{Z}}^n$ and hence the solution of the algebraic system (4.5)–(4.6) can be represented in terms of the inverse matrix $\mathfrak{S}^{-1}(\xi)$ as*

$$\begin{pmatrix}\hat{\mathbf{u}}(\xi)\\ \hat{p}(\xi)\end{pmatrix}=\mathfrak{S}^{-1}(\xi)\begin{pmatrix}\hat{\mathbf{f}}(\xi)\\ \hat{g}(\xi)\end{pmatrix}\quad\forall\xi\in\dot{\mathbb{Z}}^n. \tag{4.9}$$

Moreover, the following estimates hold:

$$|\hat{\mathbf{u}}(\xi)|\leq\hat{C}_{uf}\frac{|\hat{\mathbf{f}}(\xi)|}{|2\pi\xi|^2}+\hat{C}_{ug}\frac{|\hat{g}(\xi)|}{2\pi|\xi|}, \tag{4.10}$$

$$|\hat{p}(\xi)|\leq\hat{C}_{pf}\frac{|\hat{\mathbf{f}}(\xi)|}{2\pi|\xi|}+\hat{C}_{pg}|\hat{g}(\xi)|\quad\forall\xi\in\dot{\mathbb{Z}}^n, \tag{4.11}$$

where

$$\hat{C}_{uf}=2C_{\mathbb{A}},\ \hat{C}_{ug}=\hat{C}_{pf}=1+2C_{\mathbb{A}}\|\mathbb{A}\|,\ \hat{C}_{pg}=\|\mathbb{A}\|(1+2C_{\mathbb{A}}\|\mathbb{A}\|).$$

Proof. Introducing the new variables $\hat{\mathbf{w}}=2\pi\hat{\mathbf{u}}$, $\hat{q}=i\hat{p}$, the algebraic system (4.7) can be represented in the equivalent form

$$\tilde{\mathfrak{S}}(\xi)\begin{pmatrix}\hat{\mathbf{w}}(\xi)\\ \hat{q}(\xi)\end{pmatrix}=\begin{pmatrix}\hat{\mathbf{f}}(\xi)/(2\pi)\\ -i\hat{g}(\xi)\end{pmatrix}\quad\forall\xi\in\dot{\mathbb{Z}}^n, \tag{4.12}$$

where $\tilde{\mathfrak{S}}(\xi)$ is a real matrix with the entries

$$\tilde{\mathfrak{S}}_{\ell j}(\xi)=\begin{cases}\xi_\alpha a_{\ell j}^{\alpha\beta}\xi_\beta, & \ell,j=1,\dots,n;\\ \xi_\ell, & \ell=1,\dots,n,\ j=n+1;\\ \xi_j, & \ell=n+1,\ j=1,\dots,n;\\ 0, & \ell=j=n+1.\end{cases} \tag{4.13}$$

In order to show that $\tilde{\mathfrak{S}}_{\ell j}(\xi)$ is nonsingular for any $\xi\in\dot{\mathbb{Z}}^n$, we use Theorem 7.1. To this end, for a fixed $\xi\in\dot{\mathbb{Z}}^n$, we consider the bilinear forms $a_0:\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}$ and $b_0:\mathbb{R}^n\times\mathbb{R}\to\mathbb{R}$,

$$a_0(\hat{\mathbf{w}},\hat{\mathbf{v}}):=\hat{w}_\ell\xi_\alpha a_{\ell j}^{\alpha\beta}\xi_\beta\hat{v}_j\quad\forall\hat{\mathbf{w}},\hat{\mathbf{v}}\in\mathbb{R}^n, \tag{4.14}$$

$$b_0(\hat{\mathbf{v}},\hat{q}):=\xi_j\hat{v}_j\hat{q}\quad\forall\hat{\mathbf{v}}\in\mathbb{R}^n,\ \hat{q}\in\mathbb{R}, \tag{4.15}$$

as well as the closed subspace V_ξ of $\mathbb{R}^n$ given by

$$V_\xi := \left\{ \hat{\mathbf{v}} \in \mathbb{R}^n : b_0(\hat{\mathbf{v}}, \hat{q}) = 0, \ \ \forall \hat{q} \in \mathbb{R} \right\} = \left\{ \hat{\mathbf{v}} \in \mathbb{R}^n : \xi_j \hat{v}_j = 0 \right\}. \tag{4.16}$$

It is immediate that these bilinear forms are bounded, as they satisfy the estimates:

$$|a_0(\hat{\mathbf{w}}, \hat{\mathbf{v}})| \le \|\mathbb{A}\| |\xi|^2 |\hat{\mathbf{w}}|\,|\hat{\mathbf{v}}|, \quad |b_0(\hat{\mathbf{v}}, \hat{q})| \le |\xi|\,|\hat{\mathbf{v}}|\,|\hat{q}| \quad \forall \hat{\mathbf{w}}, \hat{\mathbf{v}} \in \mathbb{R}^n, \quad \forall \hat{q} \in \mathbb{R}.$$

The symmetry conditions (2.2) allow us to write the bilinear form a_0 as

$$a_0(\hat{\mathbf{w}}, \hat{\mathbf{v}}) = a_{\ell j}^{\alpha\beta} (\hat{\mathbf{w}} \otimes \xi)^s_{\ell\alpha} (\hat{\mathbf{v}} \otimes \xi)^s_{\beta j}, \tag{4.17}$$

where $(\hat{\mathbf{w}} \otimes \xi)^s$ is the symmetric part of the matrix $\hat{\mathbf{w}} \otimes \xi$, that is,

$$(\hat{\mathbf{w}} \otimes \xi)^s_{\ell\alpha} := \frac{1}{2} \left(\hat{w}_\ell \xi_\alpha + \hat{w}_\alpha \xi_\ell \right), \ \ell, \alpha = 1, \dots, n. \tag{4.18}$$

Due to (4.17) and the ellipticity condition (2.3), we obtain that a_0 satisfies the estimate

$$a_0(\hat{\mathbf{v}}, \hat{\mathbf{v}}) \ge c_{\mathbb{A}}^{-1} |(\hat{\mathbf{v}} \otimes \xi)^s|^2 = \frac{1}{2} c_{\mathbb{A}}^{-1} |\hat{\mathbf{v}}|^2 |\xi|^2 \quad \forall \hat{\mathbf{v}} \in \mathbb{R}^n \ \text{ such that } \ \hat{\mathbf{v}} \cdot \xi = 0, \tag{4.19}$$

where $\hat{\mathbf{v}} \cdot \xi = \sum_{\ell=1}^n (\hat{\mathbf{v}} \otimes \xi)^s_{\ell\ell}$ is the trace of the symmetric matrix $(\hat{\mathbf{v}} \otimes \xi)^s$. Therefore, the bounded bilinear form $a_0 : V_\xi \times V_\xi \to \mathbb{R}$ is coercive when $\xi \ne \mathbf{0}$.

In addition, an elementary computation shows that

$$\inf_{\hat{q} \in \mathbb{R} \setminus \{0\}} \sup_{\hat{\mathbf{v}} \in \mathbb{R}^n \setminus \{\mathbf{0}\}} \frac{b_0(\hat{\mathbf{v}}, \hat{q})}{|\hat{\mathbf{v}}|\,|\hat{q}|} \ge |\xi|, \tag{4.20}$$

and accordingly that the bilinear form b_0 satisfies the inf-sup condition with the inf-sup constant $|\xi|$.

By applying Theorem 7.1, we conclude that the modified symbol matrix $\widetilde{\mathfrak{S}}(\xi)$ given by (4.13) is invertible for any $\xi \in \dot{\mathbb{Z}}^n$, and hence that the symbol matrix $\mathfrak{S}(\xi)$ given by (4.8) has the same property and the formula (4.9) is well defined. Moreover, estimates (7.5) and (7.6) are applicable to the real and imaginary parts of system (4.12) and after combining them, we get

$$|\hat{\mathbf{w}}(\xi)| \le \hat{C}_{uf} \frac{|\hat{\mathbf{f}}(\xi)|}{2\pi |\xi|^2} + \hat{C}_{ug} \frac{|\hat{g}(\xi)|}{|\xi|},$$
$$|\hat{q}(\xi)| \le \hat{C}_{pf} \frac{|\hat{\mathbf{f}}(\xi)|}{2\pi |\xi|} + \hat{C}_{pg} |\hat{g}(\xi)| \quad \forall \xi \in \dot{\mathbb{Z}}^n.$$

Recalling that $\hat{\mathbf{w}} = 2\pi \hat{\mathbf{u}}$ and $\hat{q} = i\hat{p}$, we arrive at estimates (4.10) and (4.11). □

Note that the proof of Lemma 4.1 is essentially similar to the proof of [10, Lemma 15] given for the nonperiodic case, where the ellipticity of the anisotropic Stokes system in the sense of Agmon–Douglis–Nirenberg was given but augments it with estimates (4.10) and (4.11).

Remark 4.2. For the isotropic case (2.9), due to (2.10), system (4.5)–(4.6) reduces to

$$4\pi^2 \left[(\lambda + \mu)\xi(\xi \cdot \hat{\mathbf{u}}(\xi)) + \mu |\xi|^2 \hat{\mathbf{u}}(\xi) \right] + 2\pi i \xi \hat{p}(\xi) = \hat{\mathbf{f}}(\xi) \quad \forall \xi \in \dot{\mathbb{Z}}^n, \tag{4.21}$$

$$2\pi i \xi \cdot \hat{\mathbf{u}}(\xi) = \hat{g}(\xi) \quad \forall \xi \in \dot{\mathbb{Z}}^n. \tag{4.22}$$

Taking scalar product of Equation (4.21) with ξ and employing (4.22), we obtain

$$\hat{p}(\xi) = \frac{\xi \cdot \hat{\mathbf{f}}(\xi)}{2\pi i|\xi|^2} + (\lambda + 2\mu)\hat{g}(\xi), \quad \forall \xi \in \dot{\mathbb{Z}}^n, \tag{4.23}$$

and substituting this back to (4.21), we get

$$\hat{\mathbf{u}}(\xi) = \frac{1}{4\pi^2\mu|\xi|^2}\left[\hat{\mathbf{f}}(\xi) - \xi\frac{\xi \cdot \hat{\mathbf{f}}(\xi)}{|\xi|^2}\right] + \xi\frac{\hat{g}(\xi)}{2\pi i|\xi|^2}, \quad \forall \xi \in \dot{\mathbb{Z}}^n \tag{4.24}$$

(cf. [6, Section 2.2] for the case $s = 1$, $g = 0$, $\lambda = 0$ and $\mu = 1$). Expressions (4.24) and (4.23) evidently satisfy estimates (4.10) and (4.11).

The anisotropic Stokes system (4.1)–(4.2) can be rewritten as

$$S\begin{pmatrix}\mathbf{u}\\ p\end{pmatrix} = \begin{pmatrix}\mathbf{f}\\ g\end{pmatrix},$$

where

$$S\begin{pmatrix}\mathbf{u}\\ p\end{pmatrix} := \begin{pmatrix}-\mathfrak{L}\mathbf{u} + \nabla p\\ \operatorname{div}\,\mathbf{u}\end{pmatrix},$$

and for any $s \in \mathbb{R}$,

$$S : \dot{\mathbf{H}}_\#^s \times \dot{H}_\#^{s-1} \to \dot{\mathbf{H}}_\#^{s-2} \times \dot{H}_\#^{s-1} \tag{4.25}$$

is a linear continuous operator.

Now we are in the position to prove the following assertion.

Theorem 4.3. *Let $n \geq 2$ and condition (2.3) hold.*

(i) For any $(\mathbf{f}, g) \in \dot{\mathbf{H}}_\#^{s-2} \times \dot{H}_\#^{s-1}$, $s \in \mathbb{R}$, the anisotropic Stokes system (4.1)–(4.2) in torus $\mathbb{T}$ has a unique solution $(\mathbf{u}, p) \in \dot{\mathbf{H}}_\#^s \times \dot{H}_\#^{s-1}$, where

$$\mathbf{u}(\mathbf{x}) = \sum_{\xi \in \dot{\mathbb{Z}}^n} e^{2\pi i x\cdot\xi}\hat{\mathbf{u}}(\xi), \;\; p(\mathbf{x}) = \sum_{\xi \in \dot{\mathbb{Z}}^n} e^{2\pi i x\cdot\xi}\hat{p}(\xi) \tag{4.26}$$

with $\hat{\mathbf{u}}(\xi)$ and $\hat{p}(\xi)$ given by (4.9). In addition,

$$\|\mathbf{u}\|_{\dot{\mathbf{H}}_\#^s} \leq C_{uf}\|\mathbf{f}\|_{\dot{\mathbf{H}}_\#^{s-2}} + C_{ug}\|g\|_{\dot{H}_\#^{s-1}}, \tag{4.27}$$

$$\|p\|_{\dot{H}_\#^{s-1}} \leq C_{pf}\|\mathbf{f}\|_{\dot{\mathbf{H}}_\#^{s-2}} + C_{pg}\|g\|_{\dot{H}_\#^{s-1}}, \tag{4.28}$$

where

$$C_{uf} = \frac{1}{\pi^2}C_{\mathbb{A}}, C_{ug} = C_{pf} = \frac{1}{\sqrt{2}\pi}(1 + 2C_{\mathbb{A}}\|\mathbb{A}\|), C_{pg} = \|\mathbb{A}\|(1 + 2C_{\mathbb{A}}\|\mathbb{A}\|),$$

and operator (4.25) is an isomorphism.

(ii) Moreover, if $(\mathbf{f}, g) \in \dot{\mathbf{C}}_\#^\infty \times \dot{C}_\#^\infty$, then $(\mathbf{u}, p) \in \dot{\mathbf{C}}_\#^\infty \times \dot{C}_\#^\infty$.

Proof.

(i) Expressions (4.9) supplemented by the relations $\hat{\mathbf{u}}(\mathbf{0}) = \mathbf{0}$, $\hat{p}(\mathbf{0}) = 0$, following from the inclusions $(\mathbf{u}, p) \in \dot{\mathbf{H}}_{\#}^{s} \times \dot{H}_{\#}^{s-1}$, imply the uniqueness. From estimates (4.10) and (4.11), we obtain the estimates

$$
\begin{aligned}
\|\mathbf{u}\|_{\dot{\mathbf{H}}_{\#}^{s}} &= \left(\sum_{\xi\in\dot{\mathbb{Z}}^n} \rho(\xi)^{2s}|\hat{\mathbf{u}}(\xi)|^2\right)^{1/2} \\
&\le \frac{\hat{C}_{uf}}{4\pi^2}\left(\sum_{\xi\in\dot{\mathbb{Z}}^n} \rho(\xi)^{2s}\frac{|\hat{\mathbf{f}}(\xi)|^2}{|\xi|^4}\right)^{1/2} + \frac{\hat{C}_{ug}}{2\pi}\left(\sum_{\xi\in\dot{\mathbb{Z}}^n} \rho(\xi)^{2s}\frac{|\hat{g}(\xi)|^2}{|\xi|^2}\right)^{1/2} \\
&= \frac{\hat{C}_{uf}}{4\pi^2}\left(\sum_{\xi\in\dot{\mathbb{Z}}^n} \rho(\xi)^{2(s-2)}|\hat{\mathbf{f}}(\xi)|^2\frac{\rho(\xi)^4}{|\xi|^4}\right)^{1/2} + \frac{\hat{C}_{ug}}{2\pi}\left(\sum_{\xi\in\dot{\mathbb{Z}}^n} \rho(\xi)^{2(s-1)}|\hat{g}(\xi)|^2\frac{\rho(\xi)^2}{|\xi|^2}\right)^{1/2} \\
&\le \frac{\hat{C}_{uf}}{2\pi^2}\|\mathbf{f}\|_{\dot{\mathbf{H}}_{\#}^{s-2}} + \frac{\hat{C}_{ug}}{\sqrt{2}\pi}\|g\|_{\dot{H}_{\#}^{s-1}}, \\
\|p\|_{\dot{H}_{\#}^{s-1}} &= \left(\sum_{\xi\in\dot{\mathbb{Z}}^n} \rho(\xi)^{2s-2}|\hat{p}(\xi)|^2\right)^{1/2} \\
&\le \frac{\hat{C}_{pf}}{2\pi}\left(\sum_{\xi\in\dot{\mathbb{Z}}^n} \rho(\xi)^{2s-2}\frac{|\hat{\mathbf{f}}(\xi)|^2}{|\xi|^2}\right)^{1/2} + \hat{C}_{pg}\left(\sum_{\xi\in\dot{\mathbb{Z}}^n} \rho(\xi)^{2s-2}|\hat{g}(\xi)|^2\right)^{1/2} \\
&= \frac{\hat{C}_{pf}}{2\pi}\left(\sum_{\xi\in\dot{\mathbb{Z}}^n} \rho(\xi)^{2(s-2)}|\hat{\mathbf{f}}(\xi)|^2\frac{\rho(\xi)^2}{|\xi|^2}\right)^{1/2} + \hat{C}_{pg}\left(\sum_{\xi\in\dot{\mathbb{Z}}^n} \rho(\xi)^{2(s-1)}|\hat{g}(\xi)|^2\right)^{1/2} \\
&\le \frac{\hat{C}_{pf}}{\sqrt{2}\pi}\|\mathbf{f}\|_{\dot{\mathbf{H}}_{\#}^{s-2}} + \hat{C}_{pg}\|g\|_{\dot{H}_{\#}^{s-1}}.
\end{aligned}
$$

These estimates imply (4.27)–(4.28) and hence inclusions in the corresponding spaces and that operator (4.25) is an isomorphism.

(ii) The inclusion $(\mathbf{f}, g) \in \dot{\mathbf{C}}_{\#}^{\infty} \times \dot{C}_{\#}^{\infty}$ implies that $(\mathbf{f}, g) \in \dot{\mathbf{H}}_{\#}^{s-2} \times \dot{H}_{\#}^{s-1}$ for any $s \in \mathbb{R}$. Then by item (i) $(\mathbf{u}, p) \in \dot{\mathbf{H}}_{\#}^{s} \times \dot{H}_{\#}^{s-1}$ for any $s \in \mathbb{R}$ and hence $(\mathbf{u}, p) \in \dot{\mathbf{C}}_{\#}^{\infty} \times \dot{C}_{\#}^{\infty}$.

□

If $g = 0$ in (4.2), we can reformulate the Stokes system (4.1)–(4.2) as the vector equation

$$
-\mathfrak{L}\mathbf{u} + \nabla p = \mathbf{f} \tag{4.29}
$$

for the unknowns $(\mathbf{u}, p) \in \dot{\mathbf{H}}_{\#\sigma}^{s} \times \dot{H}_{\#}^{s-1}$ and the given data $\mathbf{f} \in \dot{\mathbf{H}}_{\#}^{s-2}$, $s \in \mathbb{R}$. Then Theorem 4.3 implies the following assertion.

Corollary 4.4. *Let $n \ge 2$ and condition (2.3) hold.*

(i) For any $\mathbf{f} \in \dot{\mathbf{H}}_{\#}^{s-2}$, $s \in \mathbb{R}$, the anisotropic Stokes equation (4.29) in torus $\mathbb{T}$ has a unique incompressible solution $(\mathbf{u}, p) \in \dot{\mathbf{H}}_{\#\sigma}^{s} \times \dot{H}_{\#}^{s-1}$, given by (4.9) and (4.26) (and particularly by (4.24), (4.23), (4.26) for the isotropic case (2.9)) with $g = 0$. In addition,

$$
\|\mathbf{u}\|_{\dot{\mathbf{H}}_{\#}^{s}} \le C_{uf}\|\mathbf{f}\|_{\dot{\mathbf{H}}_{\#}^{s-2}}, \tag{4.30}
$$

$$
\|p\|_{\dot{H}_{\#}^{s-1}} \le C_{pf}\|\mathbf{f}\|_{\dot{\mathbf{H}}_{\#}^{s-2}}, \tag{4.31}
$$

where

$$C_{uf} = \frac{1}{\pi^2} C_{\mathbb{A}},\ C_{pf} = \frac{1}{\sqrt{2}\pi}(1 + 2C_{\mathbb{A}}\|\mathbb{A}\|),$$

and the operator

$$\mathcal{L} : \dot{\mathbf{H}}^s_{\#\sigma} \times \dot{H}^{s-1}_{\#} \to \dot{\mathbf{H}}^{s-2}_{\#},$$

where

$$\mathcal{L}(\mathbf{u}, p) := \mathfrak{L}\mathbf{u} - \nabla p, \tag{4.32}$$

is an isomorphism.

(ii) Moreover, if $\mathbf{f} \in \dot{\mathbf{C}}^\infty_{\#}$*, then* $(\mathbf{u}, p) \in \dot{\mathbf{C}}^\infty_{\#\sigma} \times \dot{C}^\infty_{\#}$.

5 | STATIONARY ANISOTROPIC PERIODIC OSEEN SYSTEM

For the unknowns $(\mathbf{u}, p) \in \dot{\mathbf{H}}^s_{\#} \times \dot{H}^{s-1}_{\#}$, let us consider the Oseen system

$$-\mathfrak{L}\mathbf{u} + \nabla p + (\mathbf{U} \cdot \nabla)\mathbf{u} = \mathbf{f}, \tag{5.1}$$

$$\text{div }\mathbf{u} = g \tag{5.2}$$

with given data $(\mathbf{f}, g) \in \dot{\mathbf{H}}^{s-2}_{\#} \times \dot{H}^{s-1}_{\#}$, $s \geq 1$ and a given function $\mathbf{U}$.

The anisotropic Oseen system (5.1)–(5.2) can be rewritten as

$$S_U \begin{pmatrix} \mathbf{u} \\ p \end{pmatrix} = \begin{pmatrix} \mathbf{f} \\ g \end{pmatrix},$$

where

$$S_U \begin{pmatrix} \mathbf{u} \\ p \end{pmatrix} := \begin{pmatrix} -\mathfrak{L}\mathbf{u} + \nabla p + (\mathbf{U} \cdot \nabla)\mathbf{u} \\ \text{div }\mathbf{u} \end{pmatrix} = S \begin{pmatrix} \mathbf{u} \\ p \end{pmatrix} + \begin{pmatrix} (\mathbf{U} \cdot \nabla)\mathbf{u} \\ 0 \end{pmatrix}. \tag{5.3}$$

5.1 | Weak solution to the stationary periodic anisotropic Oseen system

For a fixed $\mathbf{U}$, and the bilinear forms

$$a_{\mathbb{T};\mathbf{U}}(\mathbf{u}, \mathbf{v}) := \left\langle a^{\alpha\beta}_{ij} E_{j\beta}(\mathbf{u}), E_{i\alpha}(\mathbf{v}) \right\rangle_{\mathbb{T}} + \langle (\mathbf{U} \cdot \nabla)\mathbf{u}, \mathbf{v} \rangle_{\mathbb{T}}, \quad \forall \mathbf{u}, \mathbf{v} \in \dot{\mathbf{H}}^1_{\#}, \tag{5.4}$$

$$b_{\mathbb{T}}(\mathbf{v}, q) := -\langle \text{div }\mathbf{v}, q \rangle_{\mathbb{T}}, \quad \forall \mathbf{v} \in \dot{\mathbf{H}}^1_{\#}, \quad \forall q \in \dot{L}_{2\#}, \tag{5.5}$$

let us consider the following mixed variational problem:

Find $(\mathbf{u}, p) \in \dot{\mathbf{H}}^1_{\#} \times \dot{L}_{2\#}$ *such that for given* $\mathbf{f} \in \dot{\mathbf{H}}^{-1}_{\#}$ *and* $g \in \dot{L}_{2\#}$,

$$\begin{cases} a_{\mathbb{T};\mathbf{U}}(\mathbf{u}, \mathbf{v}) + b_{\mathbb{T}}(\mathbf{v}, p) = -\langle \mathbf{f}, \mathbf{v} \rangle_{\mathbb{T}} & \forall \mathbf{v} \in \dot{\mathbf{H}}^1_{\#}, \\ b_{\mathbb{T}}(\mathbf{u}, q) = -\langle g, q \rangle_{\mathbb{T}} & \forall q \in \dot{L}_{2\#}. \end{cases} \tag{5.6}$$

Let us note that the subspace $\dot{\mathbf{H}}^1_{\#\sigma}$ of $\dot{\mathbf{H}}^1_{\#}$ (see 3.12) has also the characterisation

$$\dot{\mathbf{H}}^1_{\#\sigma} = \left\{ \mathbf{w} \in \dot{\mathbf{H}}^1_{\#} : b_{\mathbb{T}}(\mathbf{w}, q) = 0 \quad \forall q \in \dot{L}_{2\#} \right\}. \tag{5.7}$$

Let us now prove the well-posedness result for problem (5.6) (cf. Lemma 3.1 in [8], and Lemma 5 in [10], where similar results were proved for the Stokes system in nonperiodic anisotropic settings).

Theorem 5.1. *Let $n \geq 2$, condition (2.3) hold, and $\mathbf{U} \in \mathbf{L}_{\theta\#\sigma}$, where $\theta \in (2, \infty)$ if $n = 2$, while $\theta \in [n, \infty)$ if $n \geq 3$.*

(i) Then for all given data $\boldsymbol{f} \in \dot{\mathbf{H}}^{-1}_{\#}$ and $g \in \dot{L}_{2\#}$, the variational problem (5.6) has a unique solution $(\boldsymbol{u}, p) \in \dot{\mathbf{H}}^1_{\#} \times \dot{L}_{2\#}$ and

$$\|\boldsymbol{u}\|_{\dot{\mathbf{H}}^1_{\#}} \leq C_{uf} \|\boldsymbol{f}\|_{\dot{\mathbf{H}}^{-1}_{\#}} + C_{ug;U} \|g\|_{\dot{L}_{2\#}}, \tag{5.8}$$

$$\|p\|_{\dot{L}_{2\#}} \leq C_{pf;U} \|\boldsymbol{f}\|_{\dot{\mathbf{H}}^{-1}_{\#}} + C_{pg;U} \|g\|_{\dot{L}_{2\#}}, \tag{5.9}$$

where

$$C_{uf} = \pi^{-2} C_{\mathbb{A}}, \tag{5.10}$$

while the constants $C_{ug;U} = C_{pf;U}$ and $C_{pg;U}$ depend only on $C_{\mathbb{A}}$, $\|\mathbb{A}\|$, n, and $\mathbf{U}$.

(ii) Moreover, the anisotropic Oseen system (5.1)–(5.2) is well-posed in $\dot{\mathbf{H}}^1_{\#} \times \dot{L}_{2\#}$ and its unique solution $(\boldsymbol{u}, p) \in \dot{\mathbf{H}}^1_{\#} \times \dot{L}_{2\#}$ is provided by the solution of the variational problem from item (i).

(iii) The operator

$$S_U : \dot{\mathbf{H}}^1_{\#} \times \dot{L}_{2\#} \to \dot{\mathbf{H}}^{-1}_{\#} \times \dot{L}_{2\#}, \tag{5.11}$$

where S_U is defined by (5.3), is an isomorphism.

Proof.

(i) We intend to use Theorem 7.1, which requires boundedness of the bilinear form $a_{\mathbb{T};\mathbf{U}}(\cdot,\cdot) : \dot{\mathbf{H}}^1_{\#} \times \dot{\mathbf{H}}^1_{\#} \to \mathbb{R}$ and coercivity of the bilinear form $a_{\mathbb{T};\mathbf{U}}(\cdot,\cdot) : \dot{\mathbf{H}}^1_{\#\sigma} \times \dot{\mathbf{H}}^1_{\#\sigma} \to \mathbb{R}$.
Indeed, by (2.4) and (3.10),

$$\begin{aligned} \left| \left\langle a^{\alpha\beta}_{ij} E_{j\beta}(\mathbf{u}), E_{i\alpha}(\mathbf{v}) \right\rangle_{\mathbb{T}} \right| &\leq n^4 \|\mathbb{A}\| \|\mathbb{E}(\mathbf{u})\|_{L^{n\times n}_{2\#}} \|\mathbb{E}(\mathbf{v})\|_{L^{n\times n}_{2\#}} \\ &\leq n^4 \|\mathbb{A}\| \|\nabla \mathbf{u}\|_{L^{n\times n}_{2\#}} \|\nabla \mathbf{v}\|_{L^{n\times n}_{2\#}} \leq 4\pi^2 n^4 \|\mathbb{A}\| \|\mathbf{u}\|_{\dot{\mathbf{H}}^1_{\#}} \|\mathbf{v}\|_{\dot{\mathbf{H}}^1_{\#}}. \end{aligned} \tag{5.12}$$

On the other hand, by (7.22) with $\mathbf{U}$ for $\mathbf{v}_1$ and $\mathbf{u}$ for $\mathbf{v}_2$,

$$\begin{aligned} \left| \langle (\mathbf{U} \cdot \nabla)\mathbf{u}, \mathbf{v} \rangle_{\mathbb{T}} \right| &\leq \|(\mathbf{U} \cdot \nabla)\mathbf{u}\|_{\mathbf{H}^{-1}_{\#}} \|\mathbf{v}\|_{\mathbf{H}^1_{\#}} \\ &\leq 2\pi C_{2\theta/(\theta-2)\#} \|\mathbf{U}\|_{\mathbf{L}_{\theta\#}} \|\mathbf{u}\|_{\mathbf{H}^1_{\#}} \|\mathbf{v}\|_{\mathbf{H}^1_{\#}}. \end{aligned} \tag{5.13}$$

Hence,

$$|a_{\mathbb{T};\mathbf{U}}(\mathbf{u}, \mathbf{v})| \leq (4\pi^2 n^4 \|\mathbb{A}\| + 2\pi C_{2\theta/(\theta-2)\#} \|\mathbf{U}\|_{\mathbf{L}_{\theta\#}}) \|\mathbf{u}\|_{\dot{\mathbf{H}}^1_{\#}} \|\mathbf{v}\|_{\dot{\mathbf{H}}^1_{\#}}, \tag{5.14}$$

which proves boundedness of the bilinear form $a_{\mathbb{T};\mathbf{U}}(\cdot,\cdot) : \dot{\mathbf{H}}^1_{\#} \times \dot{\mathbf{H}}^1_{\#} \to \mathbb{R}$.
The first Korn inequality (3.11), the relation $\sum_{i=1}^n E_{ii}(\mathbf{w}) = \operatorname{div} \mathbf{w} = 0$ for $\mathbf{w} \in \dot{\mathbf{H}}^1_{\#\sigma}$, the ellipticity condition (2.3), and equivalence of the norm $\|\nabla(\cdot)\|_{L^{n\times n}_{2\#}}$ to the norm $\|\cdot\|_{\dot{\mathbf{H}}^1_{\#}}$ in $\dot{\mathbf{H}}^1_{\#}$ (see (3.10)), imply that

$$\begin{aligned} \left\langle a^{\alpha\beta}_{ij} E_{j\beta}(\mathbf{w}), E_{i\alpha}(\mathbf{w}) \right\rangle_{\mathbb{T}} &\geq C^{-1}_{\mathbb{A}} \|\mathbb{E}(\mathbf{w})\|^2_{L^{n\times n}_{2\#}} \geq \frac{1}{2} C^{-1}_{\mathbb{A}} \|\nabla \mathbf{w}\|^2_{L^{n\times n}_{2\#}} \\ &\geq \pi^2 C^{-1}_{\mathbb{A}} \|\mathbf{w}\|^2_{\dot{\mathbf{H}}^1_{\#}} \quad \forall \mathbf{w} \in \dot{\mathbf{H}}^1_{\#\sigma}. \end{aligned} \tag{5.15}$$

On the other hand, since $\mathbf{U} \in \mathbf{L}_{\theta\#\sigma}$, Section 7.3.4(iii) implies that

$$\langle (\mathbf{U} \cdot \nabla)\mathbf{w}, \mathbf{w} \rangle_{\mathbb{T}} = 0 \quad \forall \mathbf{w} \in \mathbf{H}^1_{\#}.$$

Hence,

$$a_{\mathbb{T};\mathbf{U}}(\mathbf{w},\mathbf{w}) \geq \pi^2 C_{\mathbb{A}}^{-1} \|\mathbf{w}\|^2_{\dot{\mathbf{H}}^1_{\#}} \quad \forall \mathbf{w} \in \dot{\mathbf{H}}^1_{\#\sigma}. \tag{5.16}$$

Inequality (5.16) shows that the bilinear form $a_{\mathbb{T};\mathbf{U}}(\cdot,\cdot) : \dot{\mathbf{H}}^1_{\#\sigma} \times \dot{\mathbf{H}}^1_{\#\sigma} \to \mathbb{R}$ is coercive.

The boundedness of the divergence operator div : $\dot{\mathbf{H}}^1_{\#} \to \dot{L}_{2\#}$ implies that the bilinear form $b_{\mathbb{T}} : \dot{\mathbf{H}}^1_{\#} \times \dot{L}_{2\#} \to \mathbb{R}$ is bounded as well. One can also check that for any $g \in \dot{L}_{2\#}$, the divergence PDE (5.2) has a unique solution in $\dot{\mathbf{H}}^1_{\#}/\dot{\mathbf{H}}^1_{\#\sigma}$. The PDE is equivalent to the algebraic Equation (4.6) for the Fourier coefficients, and the solution is presented as

$$\hat{\mathbf{u}}(\xi) = \frac{\xi}{2\pi i|\xi|^2}\hat{g}(\xi) \quad \forall \xi \in \dot{\mathbb{Z}}^n.$$

Moreover, then the divergence operator

$$-\text{div} : \dot{\mathbf{H}}^1_{\#}/\dot{\mathbf{H}}^1_{\#\sigma} \to \dot{L}_{2\#}$$

is an isomorphism (cf. also, e.g., [22, Lemmas 7-9 in p. 30], [23, Corollary 2.4 and Theorem 2.3 in Chapter 1], [7, Proposition 1.2(i) and Remark 1.4 in Chapter 1], [24, Theorem 3.1] and [9, Theorem 3.1] for some nonperiodic settings). The isomorphism implies that there exists a constant $c_0 > 0$ such that for any $q \in \dot{L}_{2\#}$ there exists $\mathbf{v}_q \in \dot{\mathbf{H}}^1_{\#}$ satisfying the equation $-\text{div}\ \mathbf{v}_q = q$ and the inequality $\|\mathbf{v}\|_{\dot{\mathbf{H}}^1_{\#}} \leq c_0\|q\|_{L_2\#}$. Therefore, the following inequality holds for such $\mathbf{v}$:

$$b_{\mathbb{T}}(\mathbf{v}_q, q) = -\left\langle \text{div}\ \mathbf{v}_q, q\right\rangle_{\mathbb{T}} = \langle q, q\rangle_{\mathbb{T}} = \|q\|^2_{\dot{L}_{2\#}} \geq c_0^{-1}\|\mathbf{v}_q\|_{\dot{\mathbf{H}}^1_{\#}}\|q\|_{\dot{L}_{2\#}}.$$

This, in turn, implies that the bounded bilinear form $b_{\mathbb{T}} : \dot{\mathbf{H}}^1_{\#} \times \dot{L}_{2\#} \to \mathbb{R}$ satisfies the inf-sup condition

$$\inf_{q\in \dot{L}_{2\#}\setminus\{0\}} \sup_{\mathbf{w}\in\dot{\mathbf{H}}^1_{\#}\setminus\mathbf{0}} \frac{b_{\mathbb{T}}(\mathbf{w},q)}{\|\mathbf{w}\|_{\dot{\mathbf{H}}^1_{\#}}\|q\|_{\dot{L}_{2\#}}} \geq \inf_{q\in \dot{L}_{2\#}\setminus\{0\}} \frac{b_{\mathbb{T}}(\mathbf{v}_q,q)}{\|\mathbf{v}_q\|_{\dot{\mathbf{H}}^1_{\#}}\|q\|_{\dot{L}_{2\#}}} \geq c_0^{-1}.$$

Then Theorem 7.1 with $X = \dot{\mathbf{H}}^1_{\#}$, $\mathcal{M} = \dot{L}_{2\#}$ and $V = \dot{\mathbf{H}}^1_{\#\sigma}$ implies that problem (5.6) is well-posed, as asserted.

(ii) Due to (5.3) and (7.23), operator (5.11) is linear and continuous.
The dense embedding of the space $\dot{\mathbf{C}}^{\infty}_{\#}$ in $\dot{\mathbf{H}}^1_{\#}$ shows that system (5.1)–(5.2) has the equivalent mixed variational formulation (5.6) thus proving item (ii).

(iii) Estimates (5.8)–(5.9) imply the existence of a continuous inverse to operator (5.11)

□

5.2 | Solution regularity for the stationary anisotropic Oseen system

For simplicity, we will further limit ourself with the case $\mathbf{U} \in \mathbf{C}^{\infty}_{\#\sigma}$. Regularity of the nonperiodic isotropic Oseen problems with less smooth $\mathbf{U}$ were considered e.g., in [25] and [26], and references therein.

Theorem 5.2. *Let $n \geq 2$ and condition (2.3) hold. Let $\mathbf{U} \in \mathbf{C}^{\infty}_{\#\sigma}$.*

(i) For any $(\mathbf{f}, g) \in \dot{\mathbf{H}}^{s-2}_{\#} \times \dot{H}^{s-1}_{\#}$, $s \geq 1$, the anisotropic Oseen system (5.1)–(5.2) has a unique solution

$$(\mathbf{u}, p) \in \dot{\mathbf{H}}^s_{\#} \times \dot{H}^{s-1}_{\#}. \tag{5.17}$$

In addition, there exists a constant $C_s = C_s(C_{\mathbb{A}}, \|\mathbb{A}\|, n, \mathbf{U}, s) > 0$ such that

$$\|\mathbf{u}\|_{\dot{\mathbf{H}}^s_{\#}} + \|p\|_{\dot{H}^{s-1}_{\#}} \leq C_s\left(\|\mathbf{f}\|_{\dot{\mathbf{H}}^{s-2}_{\#}} + \|g\|_{\dot{H}^{s-1}_{\#}}\right) \tag{5.18}$$

and the operator

$$S_U : \dot{\mathbf{H}}^s_{\#} \times \dot{H}^{s-1}_{\#} \to \dot{\mathbf{H}}^{s-2}_{\#} \times \dot{H}^{s-1}_{\#} \tag{5.19}$$

is an isomorphism; S_U is defined by (5.3).
(ii) Moreover, if $(\mathbf{f}, g) \in \dot{\mathbf{C}}^\infty_\# \times \dot{\mathcal{C}}^\infty_\#$, then $(\mathbf{u}, p) \in \dot{\mathbf{C}}^\infty_\# \times \dot{\mathcal{C}}^\infty_\#$.

Proof. Since $\mathbf{U} \in \mathbf{C}^\infty_{\#\sigma} \subset \mathbf{L}_{\theta\#\sigma}$ for any $\theta > 1$, Theorem 5.1 implies that system (5.1)–(5.2) has a unique solution $(\mathbf{u}, p) \in \dot{\mathbf{H}}^{s_1}_\# \times \dot{H}^{s_1-1}_\#$ with $s_1 = 1$. Then $(\mathbf{U} \cdot \nabla)\mathbf{u} \in \dot{\mathbf{H}}^{s_1-1}_\#$ and

$$\|(\mathbf{U} \cdot \nabla)\mathbf{u}\|_{\dot{\mathbf{H}}^{s_1-1}_\#} \le C_0(\mathbf{U})\|\mathbf{u}\|_{\dot{\mathbf{H}}^{s_1}_\#} \tag{5.20}$$

implying, due to estimate (5.8) in Theorem 5.1,

$$\begin{aligned}\|(\mathbf{U} \cdot \nabla)\mathbf{u}\|_{\dot{\mathbf{H}}^{s_1-1}_\#} &\le C_0(\mathbf{U}) \left(C_{uf}\|\mathbf{f}\|_{\dot{\mathbf{H}}^{s_1-2}_\#} + C_{ug;U}\|g\|_{\dot{H}^{s_1-1}_\#} \right) \\ &\le C_{1s}(\mathbf{U}) \left(\|\mathbf{f}\|_{\dot{\mathbf{H}}^{s-2}_\#} + \|g\|_{\dot{H}^{s-1}_\#} \right).\end{aligned} \tag{5.21}$$

Hence, the couple $(\mathbf{u}, p)$ satisfies the system

$$-\mathfrak{L}\mathbf{u} + \nabla p = \mathbf{f}^{(1)}, \tag{5.22}$$

$$\operatorname{div} \mathbf{u} = g, \tag{5.23}$$

with $\mathbf{f}^{(1)} := \mathbf{f} - (\mathbf{U} \cdot \nabla)\mathbf{u} \in \dot{\mathbf{H}}^{s^{(1)}-2}_\#$, where $s^{(1)} = \min\{s, s_1 + 1\}$. By Theorem 4.3(i), the Stokes system (5.22)–(5.23) has a unique solution in $\dot{\mathbf{H}}^{\tilde{s}}_{\#\sigma} \times \dot{H}^{\tilde{s}-1}_\#$ for any $\tilde{s} \le s^{(1)}$ and thus $(\mathbf{u}, p) \in \dot{\mathbf{H}}^{s^{(1)}}_{\#\sigma} \times \dot{H}^{s^{(1)}-1}_\#$ with the estimate

$$\|\mathbf{u}\|_{\dot{\mathbf{H}}^{s^{(1)}}_\#} + \|p\|_{\dot{H}^{s^{(1)}-1}_\#} \le C^{(1)} \left(\|\mathbf{f}\|_{\dot{\mathbf{H}}^{s-2}_\#} + \|g\|_{\dot{H}^{s-1}_\#} \right) \tag{5.24}$$

implied by estimates (4.27)–(4.28) and (5.21); $C^{(1)} > 0$ is a constant depending only on $C_{\mathbb{A}}$, $\|\mathbb{A}\|$, n, and $\mathbf{U}$. If $s^{(1)} = s$, this proves inclusion (5.17) and estimate (5.18).

Otherwise, $s^{(1)} = s_1 + 1 < s$ and we arrange an iterative process by replacing in the previous paragraph s_1 with $s^{(1)}$ on each iteration until we arrive at the case $s^{(1)} = \min\{s, s_1 + 1\} = s$. Note that in each iteration, s_1 increases by 1, which implies that the iteration process will stop after a finite number of iterations. This proves inclusion (5.17) and estimate (5.18).

The continuity of operator (4.25) and estimate (5.20) together with representation (5.3) imply the continuity of operator (5.19). Along with the existence of a continuous inverse to operator (5.19) implied by estimate (5.18), this means that operator (5.19) is an isomorphism.

Moreover, if $(\mathbf{f}, g) \in \dot{\mathbf{C}}^\infty_\# \times \dot{\mathcal{C}}^\infty_\#$, item (i) implies that $(\mathbf{u}, p) \in \dot{\mathbf{H}}^{s}_{\#\sigma} \times \dot{H}^{s-1}_\#$ for arbitrary s thus giving item (ii) of the theorem. □

6 | STATIONARY ANISOTROPIC PERIODIC NAVIER–STOKES SYSTEM

6.1 | Existence of a weak solution to anisotropic incompressible periodic Navier–Stokes system

In this section, using the Galerkin approximation, we show the existence of a weak solution of the anisotropic Navier–Stokes system in the incompressible case, with general data in L_2-based Sobolev spaces on a flat torus $\mathbb{T}$, for $n \ge 2$.

Let us consider the Navier–Stokes system

$$-\mathfrak{L}\mathbf{u} + \nabla p + (\mathbf{u} \cdot \nabla)\mathbf{u} = \mathbf{f}, \tag{6.1}$$

$$\operatorname{div} \mathbf{u} = 0, \tag{6.2}$$

for the couple of unknowns $(\mathbf{u}, p) \in \dot{\mathbf{H}}^1_\# \times \dot{H}^0_\#$ and the given data $\mathbf{f} \in \dot{\mathbf{H}}^{-1}_\#$. As for the Stokes system, the incompressible Navier–Stokes system (6.1)–(6.2) can be rewritten as one vector equation

$$-\mathfrak{L}\mathbf{u} + \nabla p + (\mathbf{u}\cdot\nabla)\mathbf{u} = \mathbf{f} \tag{6.3}$$

for the unknowns $(\mathbf{u}, p) \in \dot{\mathbf{H}}^1_{\#\sigma} \times \dot{H}^0_{q\#}$, with some $q > 1$, and the given data $\mathbf{f} \in \dot{\mathbf{H}}^{-1}_\#$.

Next, we show the existence of a weak solution of the Navier–Stokes equation, generalising to anisotropic case the Galerkin approximation arguments from [7, Chapter 2] (cf. also [27, Chapter 1, Section 7]).

First of all, let us define the space $\widetilde{\mathbf{V}}_{\#\sigma}$ and its norm as

$$\widetilde{\mathbf{V}}_{\#\sigma} = \dot{\mathbf{H}}^1_{\#\sigma} \cap \mathbf{L}_{n\#}, \qquad \|\mathbf{v}\|_{\widetilde{\mathbf{V}}_{\#\sigma}} = \left(\|\mathbf{v}\|^2_{\dot{\mathbf{H}}^1_\#} + \|\mathbf{v}\|^2_{\mathbf{L}_{n\#}}\right)^{1/2}. \tag{6.4}$$

For the adjoint operator, we have

$$\widetilde{\mathbf{V}}^*_{\#\sigma} = (\dot{\mathbf{H}}^1_{\#\sigma} \cap \mathbf{L}_{n\#})^* = (\dot{\mathbf{H}}^1_{\#\sigma})^* \cup \mathbf{L}_{n/(n-1)\#}. \tag{6.5}$$

If $n \in \{2, 3, 4\}$, then $\widetilde{\mathbf{V}}_{\#\sigma} = \dot{\mathbf{H}}^1_{\#\sigma}$; otherwise $\widetilde{\mathbf{V}}_{\#\sigma}$ is a proper subspace of $\dot{\mathbf{H}}^1_{\#\sigma}$. The space $\widetilde{\mathbf{V}}_{\#\sigma}$ is also the closure of $\dot{\mathbf{C}}^\infty_{\#\sigma}$ in the norm (6.4). Taking into account the mapping properties of operator (7.16), we give, similar to [7, Chapter 2, Equation (1.25)] the following variational formulation of the Navier–Stokes system (6.1)–(6.2), that is, Equation (6.3), for any $n \geq 2$:

For $\mathbf{f} \in \dot{\mathbf{H}}^{-1}_\#$, *find* $\mathbf{u} \in \dot{\mathbf{H}}^1_{\#\sigma}$ *such that*

$$\langle a^{\alpha\beta}_{ij} E_{j\beta}(\mathbf{u}), E_{i\alpha}(\mathbf{v})\rangle_{\mathbb{T}} + \langle (\mathbf{u}\cdot\nabla)\mathbf{u}, \mathbf{v}\rangle_{\mathbb{T}} = \langle \mathbf{f}, \mathbf{v}\rangle_{\mathbb{T}} \quad \forall \mathbf{v} \in \widetilde{\mathbf{V}}_{\#\sigma}. \tag{6.6}$$

Since $\dot{\mathbf{C}}^\infty_{\#\sigma} \subset \widetilde{\mathbf{V}}_{\#\sigma}$, any $\mathbf{u} \in \dot{\mathbf{H}}^1_{\#\sigma}$ satisfying the variational problem (6.6) is also a distributional solution of the Navier–Stokes system (6.1)–(6.2) in the sense of Leray (i.e., for any $\mathbf{v} \in \dot{\mathbf{C}}^\infty_{\#\sigma}$ in (6.6)).

Now we are in a position to prove the following assertion.

Theorem 6.1. *Let $n \geq 2$ and condition (2.3) hold. If $\mathbf{f} \in \dot{\mathbf{H}}^{-1}_\#$, then the anisotropic Navier–Stokes equation (6.3) has a solution $(\mathbf{u}, p) \in \dot{\mathbf{H}}^1_{\#\sigma} \times \dot{H}^0_{q\#}$ (in the sense of distributions), where $q = 2$ for $n \in \{2, 3, 4\}$ and $q = n/(n-2)$ for $n \geq 5$. Moreover, the following estimate holds*

$$\|\mathbf{u}\|_{\dot{\mathbf{H}}^1_\#} \leq \pi^{-2} C_{\mathbb{A}} \|\mathbf{f}\|_{\dot{\mathbf{H}}^{-1}_\#}. \tag{6.7}$$

Proof. As in [7, Chapter 2, Theorem 1.2] and [27, Chapter 1, Theorem 7.1], we will use the Galerkin approximation. First of all, let $\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_l, \dots$ be a system of linearly independent functions from $\dot{\mathbf{C}}^\infty_{\#\sigma}$ that is complete in $\widetilde{\mathbf{V}}_{\#\sigma}$.

For each integer $m \geq 1$, let us look for a solution

$$\mathbf{u}_m = \sum_{l=1}^m \eta_{l,m}\mathbf{w}_l, \ \ \eta_{l,m} \in \mathbb{R} \tag{6.8}$$

of the following discrete analogue of the variational problem (6.6),

$$\langle a^{\alpha\beta}_{ij} E_{j\beta}(\mathbf{u}_m), E_{i\alpha}(\mathbf{w}_k)\rangle_{\mathbb{T}} + \langle (\mathbf{u}_m\cdot\nabla)\mathbf{u}_m, \mathbf{w}_k\rangle_{\mathbb{T}} = \langle \mathbf{f}, \mathbf{w}_k\rangle_{\mathbb{T}} \quad \forall k \in \{1, \dots, m\}. \tag{6.9}$$

For a fixed m, Equations (6.9) give an algebraic system of nonlinear (quadratic) equations for $\eta_{l,m}$, $l \in \{1, \dots, m\}$. Existence of a real solution of this system follows from Lemma 7.2. Indeed, let $\eta := \{\eta_{l,m}\}_{l=1}^m$, $Q(\eta) := \{Q_k(\eta)\}_{k=1}^m$ denote the m-dimensional vectors, where

$$Q_k(\eta) := \langle a^{\alpha\beta}_{ij} E_{j\beta}(\mathbf{u}_m), E_{i\alpha}(\mathbf{w}_k)\rangle_{\mathbb{T}} + \langle (\mathbf{u}_m\cdot\nabla)\mathbf{u}_m, \mathbf{w}_k\rangle_{\mathbb{T}} - \langle \mathbf{f}, \mathbf{w}_k\rangle_{\mathbb{T}},$$

and $\mathbf{u}_m = \mathbf{u}_m(\eta)$ is given by (6.8). Note that $E_{jj}(\mathbf{u}_m) = 0$ since div $\mathbf{u}_m = 0$. Then by representation (6.8), equality (7.25), the ellipticity condition (2.3), the first Korn inequality (3.11) and the norm equivalence inequality (3.10), we obtain

$$\begin{aligned}(Q(\eta), \eta) &= \langle a_{ij}^{\alpha\beta} E_{j\beta}(\mathbf{u}_m), E_{i\alpha}(\mathbf{u}_m)\rangle_{\mathbb{T}} + \langle (\mathbf{u}_m \cdot \nabla)\mathbf{u}_m, \mathbf{u}_m\rangle_{\mathbb{T}} - \langle \mathbf{f}, \mathbf{u}_m\rangle_{\mathbb{T}} \\ &= \langle a_{ij}^{\alpha\beta} E_{j\beta}(\mathbf{u}_m), E_{i\alpha}(\mathbf{u}_m)\rangle_{\mathbb{T}} - \langle \mathbf{f}, \mathbf{u}_m\rangle_{\mathbb{T}} \\ &\geq C_{\mathbb{A}}^{-1} \|\mathbb{E}(\mathbf{u}_m)\|^2_{(L_{2\#})^{n\times n}} - \|\mathbf{f}\|_{\dot{\mathbf{H}}_\#^{-1}} \|\mathbf{u}_m\|_{\dot{\mathbf{H}}_\#^{1}} \\ &\geq \pi^2 C_{\mathbb{A}}^{-1} \|\mathbf{u}_m\|^2_{\dot{\mathbf{H}}_\#^{1}} - \|\mathbf{f}\|_{\dot{\mathbf{H}}_\#^{-1}} \|\mathbf{u}_m\|_{\dot{\mathbf{H}}_\#^{1}} \\ &= (\pi^2 C_{\mathbb{A}}^{-1} \|\mathbf{u}_m\|_{\dot{\mathbf{H}}_\#^{1}} - \|\mathbf{f}\|_{\dot{\mathbf{H}}_\#^{-1}}) \|\mathbf{u}_m\|_{\dot{\mathbf{H}}_\#^{1}}.\end{aligned} \tag{6.10}$$

Thus, $(Q(\eta), \eta) \geq 0 \quad \forall \eta : |\eta| = \rho$, where ρ is sufficiently large (so that $\|\mathbf{u}_m(\eta)\|_{\dot{\mathbf{H}}_\#^{1}} \geq C_{\mathbb{A}} \pi^{-2} \|\mathbf{f}\|_{\dot{\mathbf{H}}_\#^{-1}} \ \forall \eta : |\eta| = \rho$). Hence, by Lemma 7.2, there exists $\eta = \{\eta_{l,m}\}_{l=1}^m$ such that $|\eta| \leq \rho$ and $Q(\eta) = 0$, and then $\mathbf{u}_m(\eta)$ solves (6.9).

Multiplying Equations (6.9) by $\{\eta_{k,m}\}$ and summing them up in $k \in \{1, \dots, m\}$, we obtain

$$\langle a_{ij}^{\alpha\beta} E_{j\beta}(\mathbf{u}_m), E_{i\alpha}(\mathbf{u}_m)\rangle_{\mathbb{T}} + \langle (\mathbf{u}_m \cdot \nabla)\mathbf{u}_m, \mathbf{u}_m\rangle_{\mathbb{T}} = \langle \mathbf{f}, \mathbf{u}_m\rangle_{\mathbb{T}}. \tag{6.11}$$

Similar to (6.10), this implies

$$\begin{aligned}\pi^2 C_{\mathbb{A}}^{-1} \|\mathbf{u}_m\|^2_{\dot{\mathbf{H}}_\#^{1}} &\leq C_{\mathbb{A}}^{-1} \|\mathbb{E}(\mathbf{u}_m)\|^2_{(L_{2\#})^{n\times n}} \\ &\leq \langle a_{ij}^{\alpha\beta} E_{j\beta}(\mathbf{u}_m), E_{i\alpha}(\mathbf{u}_m)\rangle_{\mathbb{T}} \\ &= \langle \mathbf{f}, \mathbf{u}_m\rangle_{\mathbb{T}} \leq \|\mathbf{f}\|_{\dot{\mathbf{H}}_\#^{-1}} \|\mathbf{u}_m\|_{\dot{\mathbf{H}}_\#^{1}}.\end{aligned} \tag{6.12}$$

Thus, for any $m = 1, 2, \dots,$

$$\|\mathbf{u}_m\|_{\dot{\mathbf{H}}_\#^{1}} \leq \pi^{-2} C_{\mathbb{A}} \|\mathbf{f}\|_{\dot{\mathbf{H}}_\#^{-1}}. \tag{6.13}$$

This means that the sequence $\mathbf{u}_m$ is bounded in $\dot{\mathbf{H}}_\#^1$ and in $\dot{\mathbf{H}}_{\#\sigma}^1$ and thus there exists a subsequence $\mathbf{u}_{m'}$ weakly converging in $\dot{\mathbf{H}}_\#^1$ and in $\dot{\mathbf{H}}_{\#\sigma}^1$ to a function $\mathbf{u} \in \dot{\mathbf{H}}_{\#\sigma}^1$. On the other hand, since $\dot{\mathbf{H}}_\#^1$ is compactly embedded in $\dot{\mathbf{L}}_{2\#}$ and $\dot{\mathbf{H}}_{\#\sigma}^1$ is compactly embedded in $\dot{\mathbf{L}}_{2\#\sigma}$, there exists a subsequence of $\mathbf{u}_{m'}$, for which we will use the same notation, that strongly converges in $\dot{\mathbf{L}}_{2\#}$ and $\dot{\mathbf{L}}_{2\#\sigma}$ to $\mathbf{u}$.

Then due to Lemma 7.3, we can take limit in (6.9) as $m \to \infty$ to obtain

$$\langle a_{ij}^{\alpha\beta} E_{j\beta}(\mathbf{u}), E_{i\alpha}(\mathbf{v})\rangle_{\mathbb{T}} + \langle (\mathbf{u} \cdot \nabla)\mathbf{u}, \mathbf{v}\rangle_{\mathbb{T}} = \langle \mathbf{f}, \mathbf{v}\rangle_{\mathbb{T}} \tag{6.14}$$

for any $\mathbf{v} \in \{\mathbf{w}_k\}_{k=1}^\infty$. Since by definition the set $\mathbf{v} \in \{\mathbf{w}_k\}_{k=1}^\infty$ is complete in $\widetilde{\mathbf{V}}_{\#\sigma}$ and operator (7.16) is bounded and continuous, we conclude that Equation (6.14) holds for any $\mathbf{v} \in \widetilde{\mathbf{V}}_{\#\sigma}$; that is, $\mathbf{u}$ solves variational problem (6.6), and moreover, (6.13) implies that $\mathbf{u}$ satisfies estimate (6.7).

After $\mathbf{u} \in \dot{\mathbf{H}}_{\#\sigma}^1$ satisfying (6.6) is obtained, the pressure $p \in \dot{\mathcal{D}}'_\#$ can be found from Equation (6.1) rewritten using notation (2.1) as

$$\nabla p = \mathbf{f} - (\mathbf{u} \cdot \nabla)\mathbf{u} + \mathfrak{L}\mathbf{u} \tag{6.15}$$

and understood in the sense of distributions. In the right-hand side of (6.15), $\mathbf{f} \in \dot{\mathbf{H}}_\#^{-1}$, $\mathfrak{L}\mathbf{u} \in \dot{\mathbf{H}}_\#^{-1}$, while $(\mathbf{u}\cdot\nabla)\mathbf{u} \in \dot{\mathbf{H}}_\#^{-1}$ if $n \in \{2, 3, 4\}$, and $(\mathbf{u} \cdot \nabla)\mathbf{u} \in \dot{\mathbf{L}}_{n/(n-1)\#}$ if $n \geq 5$ (cf. (7.15) and (7.16)). This implies that in fact $p \in \dot{L}_{2\#}$ if $n \in \{2, 3, 4\}$, and $p \in \dot{L}_{2\#} \cup \dot{H}^1_{n/(n-1)\#} \subset \dot{L}_{n/(n-2)\#}$ if $n \geq 5$ (cf. [2, Theorem IX.3.1, Remark IX.3.1], [3, Section 5.1]). □

6.2 | Solution uniqueness for the anisotropic periodic Navier–Stokes system

In this section, we show that under additional constraint on the norm of the given data, the weak solution of the Navier–Stokes equation (6.3) is unique.

For the uniqueness in the nonperiodic setting, for the isotropic case (2.10) with $\lambda = 0$ and $\mu = 1$, compare, for example, [4, Lemma 3.1]; for the anisotropic case, compare [9, Theorem 5.4], [11, Theorem 7.3].

Theorem 6.2. *Let $n \in \{2,3,4\}$ and condition (2.3) hold. Let $\mathbf{f} \in \dot{\mathbf{H}}_{\#}^{-1}$ and*

$$\|\mathbf{f}\|_{\dot{\mathbf{H}}_{\#}^{-1}} < \frac{\pi^3}{\sqrt{2}} C_{\mathbb{A}}^{-2} C_{4\#}^{-2}, \tag{6.16}$$

with the constants $C_{\mathbb{A}}$ and $C_{4\#}$ from the ellipticity condition (2.3) and the Sobolev embedding inequality (7.10), respectively. Then the anisotropic Navier–Stokes equation (6.3) has a unique solution $(\mathbf{u}, p) \in \dot{\mathbf{H}}_{\#\sigma}^{1} \times \dot{H}_{\#}^{0}$.

Proof. Assume that the Navier–Stokes equation (6.3) and thus the variational problem (6.6) have two solutions $(\mathbf{u}^{(1)}, p^{(1)})$ and $(\mathbf{u}^{(2)}, p^{(2)})$ in the space $\dot{\mathbf{H}}_{\#\sigma}^{1} \times \dot{H}_{\#}^{0}$. Note that $\widetilde{\mathbf{V}}_{\#\sigma} = \dot{\mathbf{H}}_{\#\sigma}^{1}$ if $n \in \{2,3,4\}$. Then the first Korn inequality (3.11), the ellipticity condition (2.3), the variational formulation (6.6) and the norm equivalence inequality (3.10) give for $k \in \{1,2\}$,

$$\begin{aligned}
\|\nabla \mathbf{u}^{(k)}\|^2_{(L_{2\#})^{n\times n}} &\le 2\|\mathbb{E}(\mathbf{u}^{(k)})\|^2_{(L_{2\#})^{n\times n}} \\
&\le 2C_{\mathbb{A}} \langle a_{ij}^{\alpha\beta} E_{j\beta}(\mathbf{u}^{(k)}), E_{i\alpha}(\mathbf{u}^{(k)})\rangle_{\mathbb{T}} \\
&= 2C_{\mathbb{A}} \langle \mathbf{f}, \mathbf{u}^{(k)}\rangle_{\mathbb{T}} \le 2C_{\mathbb{A}} \|\mathbf{f}\|_{\dot{\mathbf{H}}_{\#}^{-1}} \|\mathbf{u}^{(k)}\|_{\dot{\mathbf{H}}_{\#}^{1}} \\
&\le \frac{\sqrt{2}}{\pi} C_{\mathbb{A}} \|\mathbf{f}\|_{\dot{\mathbf{H}}_{\#}^{-1}} \|\nabla \mathbf{u}^{(k)}\|_{(L_{2\#})^{n\times n}}.
\end{aligned} \tag{6.17}$$

Hence,

$$\|\nabla \mathbf{u}^{(k)}\|_{(L_{2\#})^{n\times n}} \le \frac{\sqrt{2}}{\pi} C_{\mathbb{A}} \|\mathbf{f}\|_{\dot{\mathbf{H}}_{\#}^{-1}}, \ k \in \{1,2\}. \tag{6.18}$$

The variational formulation (6.6) also implies

$$\left\langle a_{ij}^{\alpha\beta} E_{j\beta}(\mathbf{u}^{(1)} - \mathbf{u}^{(2)}), E_{i\alpha}(\mathbf{v}\ )\right\rangle_{\mathbb{T}} + \left\langle (\mathbf{u}^{(1)} \cdot \nabla)\mathbf{u}^{(1)} - (\mathbf{u}^{(2)} \cdot \nabla)\mathbf{u}^{(2)}, \mathbf{v}\right\rangle_{\mathbb{T}} = 0 \quad \forall \mathbf{v} \in \dot{\mathbf{H}}_{\#\sigma}^{1}. \tag{6.19}$$

Then by choosing $\mathbf{v} = \mathbf{u}^{(1)} - \mathbf{u}^{(2)}$ in (6.19), we obtain

$$\begin{aligned}
&\left\langle a_{ij}^{\alpha\beta} E_{j\beta}(\mathbf{u}^{(1)} - \mathbf{u}^{(2)}), E_{i\alpha}(\mathbf{u}^{(1)} - \mathbf{u}^{(2)})\right\rangle_{\mathbb{T}} \\
&= -\left\langle \left((\mathbf{u}^{(1)} - \mathbf{u}^{(2)}) \cdot \nabla\right) \mathbf{u}^{(1)}, \mathbf{u}^{(1)} - \mathbf{u}^{(2)}\right\rangle_{\mathbb{T}} \\
&\quad - \left\langle (\mathbf{u}^{(2)} \cdot \nabla)(\mathbf{u}^{(1)} - \mathbf{u}^{(2)}), \mathbf{u}^{(1)} - \mathbf{u}^{(2)}\right\rangle_{\mathbb{T}}.
\end{aligned} \tag{6.20}$$

Due to the membership of $\mathbf{u}^{(1)}$ and $\mathbf{u}^{(2)}$ in $\dot{\mathbf{H}}_{\#\sigma}^{1}$, relation (7.25) yields

$$\left\langle (\mathbf{u}^{(2)} \cdot \nabla)(\mathbf{u}^{(1)} - \mathbf{u}^{(2)}), \mathbf{u}^{(1)} - \mathbf{u}^{(2)}\right\rangle_{\mathbb{T}} = 0, \tag{6.21}$$

which shows that Equation (6.20) reduces to

$$\left\langle a_{ij}^{\alpha\beta} E_{j\beta}(\mathbf{u}^{(1)} - \mathbf{u}^{(2)}), E_{i\alpha}(\mathbf{u}^{(1)} - \mathbf{u}^{(2)})\right\rangle_{\mathbb{T}} = -\left\langle \left((\mathbf{u}^{(1)} - \mathbf{u}^{(2)}) \cdot \nabla\right) \mathbf{u}^{(1)}, \mathbf{u}^{(1)} - \mathbf{u}^{(2)}\right\rangle_{\mathbb{T}}. \tag{6.22}$$

On the other hand, in view of condition (2.3) and the first Korn inequality (3.11), we deduce that

$$\|\nabla(\mathbf{u}^{(1)} - \mathbf{u}^{(2)})\|^2_{(L_{2\#})^{n\times n}} \le 2C_{\mathbb{A}} \left\langle a_{ij}^{\alpha\beta} E_{j\beta}(\mathbf{u}^{(1)} - \mathbf{u}^{(2)}), E_{i\alpha}(\mathbf{u}^{(1)} - \mathbf{u}^{(2)})\right\rangle_{\mathbb{T}}. \tag{6.23}$$

By inequalities (7.20), (7.10), (3.9) and (6.18), we obtain

$$\begin{aligned}
&\left|\left\langle \left((\mathbf{u}^{(1)} - \mathbf{u}^{(2)}) \cdot \nabla \right) \mathbf{u}^{(1)}, \mathbf{u}^{(1)} - \mathbf{u}^{(2)} \right\rangle_{\mathbb{T}}\right| \\
&\le \|\mathbf{u}^{(1)} - \mathbf{u}^{(2)}\|^2_{\mathbf{L}_{4\#}} \|\nabla \mathbf{u}^{(1)}\|_{(L_{2\#})^{n\times n}} \\
&\le C^2_{4\#} \|\mathbf{u}^{(1)} - \mathbf{u}^{(2)}\|^2_{\mathbf{H}^1_\#} \|\nabla \mathbf{u}^{(1)}\|_{(L_{2\#})^{n\times n}} \\
&\le \frac{1}{\sqrt{2}\pi^3} C_{\mathbb{A}} C^2_{4\#} \|\nabla(\mathbf{u}^{(1)} - \mathbf{u}^{(2)})\|^2_{(L_{2\#})^{n\times n}} \|\mathbf{f}\|_{\dot{\mathbf{H}}^{-1}_\#}.
\end{aligned} \tag{6.24}$$

Then equalities (6.22)–(6.24) imply that

$$\|\nabla(\mathbf{u}^{(1)} - \mathbf{u}^{(2)})\|^2_{(L_{2\#})^{n\times n}} \le \frac{\sqrt{2}}{\pi^3} C^2_{\mathbb{A}} C^2_{4\#} \|\nabla(\mathbf{u}^{(1)} - \mathbf{u}^{(2)})\|^2_{(L_{2\#})^{n\times n}} \|\mathbf{f}\|_{\dot{\mathbf{H}}^{-1}_\#}. \tag{6.25}$$

Assumption (6.16) shows that estimate (6.25) is possible only if $\mathbf{u}^{(1)} - \mathbf{u}^{(2)} = \mathbf{0}$.

Hence, Equation (6.3) implies $\nabla(p^{(1)} - p^{(2)}) = 0$ in Ω. Then $p^{(1)} - p^{(2)}$ is a constant, that is, $p^{(1)} = p^{(2)}$ in $\dot{H}^0_\#$. □

Note that in the second inequality in (6.24), we used that by the Sobolev embedding theorem, $\|\mathbf{u}^{(1)} - \mathbf{u}^{(2)}\|_{\mathbf{L}_{4\#}} \le C_{4\#}\|\mathbf{u}^{(1)} - \mathbf{u}^{(2)}\|_{\mathbf{H}^1_\#}$, which is however not available for the dimensions $n > 4$. This limited our uniqueness proof to the cases $n \in \{2, 3, 4\}$ only.

6.3 | Solution regularity for the anisotropic periodic Navier–Stokes system

In this section, we show that the regularity of a solution of the anisotropic incompressible Navier–Stokes system on $\mathbb{T}^n$, $n \in \{2, 3, 4\}$, is completely determined by the regularity of its right-hand side, as for the Stokes system. To prove this, we use the inclusions of the nonlinear term $(\mathbf{u}^{(1)} \cdot \nabla)\mathbf{u}$ given by Theorem 7.5 and the unique solvability of corresponding (linear) Stokes system along with the bootstrap argument. This is sufficient to prove the regularity for $n \in \{2, 3\}$. However, to prove the regularity for $n = 4$, we needed to accommodate more subtle norm estimates from [28] first.

Theorem 6.3. *Let $n \ge 2$ and condition (2.3) hold.*

(i) Let $s_1 > -1 + n/2$. If $(\mathbf{u}, p) \in \dot{\mathbf{H}}^{s_1}_{\#\sigma} \times \dot{H}^{s_1-1}_\#$ is a solution of the anisotropic Navier–Stokes equation (6.3) with a right-hand side $\mathbf{f} \in \dot{\mathbf{H}}^{s_2-2}_\#$, where $s_2 > s_1$, then $(\mathbf{u}, p) \in \dot{\mathbf{H}}^{s_2}_{\#\sigma} \times \dot{H}^{s_2-1}_\#$.

(ii) Moreover, if $\mathbf{f} \in \dot{\mathbf{C}}^\infty_\#$, then $(\mathbf{u}, p) \in \dot{\mathbf{C}}^\infty_\# \times \dot{C}^\infty_\#$.

Proof.

(i) Let $(\mathbf{u}, p) \in \dot{\mathbf{H}}^{s_1}_{\#\sigma} \times \dot{H}^{s_1-1}_\#$ be a solution of (6.3) with $\mathbf{f} \in \dot{\mathbf{H}}^{s_2-2}_\#$. Then by Theorem 7.5, for the nonlinear term, we have the inclusion $(\mathbf{u}^{(1)} \cdot \nabla)\mathbf{u} \in \dot{\mathbf{H}}^{t_1}_\#$ with $t_1 = 2s_1 - 1 - n/2$ if $s_1 < n/2$, with $t_1 = s_1 - 1$ if $s_1 > n/2$, and with any $t_1 \in (s_1 - 2, s_1 - 1)$ (and we can further use $t_1 = s_1 - 3/2$ for certainty) if $s_1 = n/2$. Hence, the couple $(\mathbf{u}, p)$ satisfies the equation

$$-\mathfrak{L}\mathbf{u} + \nabla p = \mathbf{f}^{(1)} \tag{6.26}$$

with $\mathbf{f}^{(1)} := \mathbf{f} - (\mathbf{u}^{(1)} \cdot \nabla)\mathbf{u} \in \dot{\mathbf{H}}^{s^{(1)}-2}_\#$, where $s^{(1)} = \min\{s_2, t_1 + 2\}$. By Corollary 4.4(i), the linear Equation (6.26) has a unique solution in $\dot{\mathbf{H}}^{s}_{\#\sigma} \times \dot{H}^{s-1}_\#$ for any $s \le s^{(1)}$ and thus $(\mathbf{u}, p) \in \dot{\mathbf{H}}^{s^{(1)}}_{\#\sigma} \times \dot{H}^{s^{(1)}-1}_\#$. If $s^{(1)} = s_2$, which we call case (a), this proves item (i) of the theorem.

Otherwise, we have case (b), when $s^{(1)} < s_2$, that is, $s^{(1)} = t_1 + 2$, by the definition of $s^{(1)}$. Then we arrange an iterative process by replacing s_1 with $s^{(1)} = t_1 + 2$ on each iteration until we arrive at case (a), thus proving item (i) of the theorem. Note that in case (b),

$$s^{(1)} - s_1 \ge \delta := \min\{s_1 + 1 - n/2, 1, 1/2\} > 0$$

in the first iteration, and δ does not decrease in the next iterations since s_1 increases. This implies that the iteration process will reach the case (a) and stop after a finite number of iterations.

(ii) If $\mathbf{f} \in \dot{\mathbf{C}}^\infty_\#$, then for any $s_2 \in \mathbb{R}$, we have $\mathbf{f} \in \dot{\mathbf{H}}^{s_2-2}_\#$ and item (i) implies that $(\mathbf{u}, p) \in \dot{\mathbf{H}}^{s_2}_{\#\sigma} \times \dot{H}^{s_2-1}_\#$. Hence, $(\mathbf{u}, p) \in \dot{\mathbf{C}}^\infty_\# \times \dot{\mathcal{C}}^\infty_\#$.

□

Combining Theorems 6.1 and 6.3, we obtain the following assertion on existence and regularity of solution to the Navier–Stokes system on torus for $n \in \{2, 3\}$.

Theorem 6.4. *Let $n \in \{2, 3\}$ and condition (2.3) hold.*

(i) If $\mathbf{f} \in \dot{\mathbf{H}}^{s-2}_\#$, $s \geq 1$, then the anisotropic Navier–Stokes equation (6.3) has a solution $(\mathbf{u}, p) \in \dot{\mathbf{H}}^{s}_{\#\sigma} \times \dot{H}^{s-1}_\#$.
(ii) Moreover, if $\mathbf{f} \in \dot{\mathbf{C}}^\infty_\#$, then Equation (6.3) has a solution $(\mathbf{u}, p) \in \dot{\mathbf{C}}^\infty_\# \times \dot{\mathcal{C}}^\infty_\#$.

Note that in the *isotropic case* (2.9) with $\lambda = 0$, similar results for the Navier–Stokes system in flat torus as well as in domains of $\mathbb{R}^n$ are available (e.g., in [2–5, 7]).

One can easily check that the arguments leading to the regularity Theorem 6.4 are at this stage not applicable for $n \geq 4$ since to apply them we would need the existence of $(\mathbf{u}, p)$ in $\dot{\mathbf{H}}^{s_1}_{\#\sigma} \times \dot{H}^{s_1-1}_\#$ for $s_1 > -1 + n/2$, which Theorem 6.1 does not provide. Nevertheless, in the following assertion, it appeared to be possible to accommodate arguments of [28] to our anisotropic periodic setting and to prove the solution existence with $s_1 = 2$ and then the regularity for $n = 4$.

Theorem 6.5. *Let $n \in \{2, 3, 4\}$ and condition (2.3) hold. If $\mathbf{f} \in \dot{\mathbf{H}}^0_\#$, then the anisotropic Navier–Stokes equation (6.3) has a solution $(\mathbf{u}, p) \in \dot{\mathbf{H}}^2_{\#\sigma} \times \dot{H}^1_\#$ and the estimate*

$$\|\mathbf{u}\|_{\dot{\mathbf{H}}^2_\#} + \|p\|_{\dot{H}^1_\#} \leq C_1 \|\mathbf{f}\|_{\dot{\mathbf{H}}^0_\#} + C_{-1} \|\mathbf{f}\|^2_{\dot{\mathbf{H}}^{-1}_\#} \tag{6.27}$$

holds for some constants $C_1, C_{-1} \geq 0$.

Proof. Let $(\mathbf{u}, p) \in \dot{\mathbf{H}}^1_{\#\sigma} \times \dot{H}^0_\#$ be the solution of (6.3) provided by Theorem 6.1 and thus satisfying (6.7). Let $\mathbf{u}_k \in \dot{\mathbf{C}}^\infty_{\#\sigma}$ be a sequence converging to $\mathbf{u}$ in $\dot{\mathbf{H}}^1_{\#\sigma}$ and let us consider the following Oseen equation (a linearised version of Equation 6.3) for $\left(\widetilde{\mathbf{u}}_k, \tilde{p}_k\right)$,

$$-\mathfrak{L}\widetilde{\mathbf{u}}_k + \nabla \tilde{p}_k = \mathbf{f} - (\mathbf{u}_k \cdot \nabla)\widetilde{\mathbf{u}}_k. \tag{6.28}$$

By Theorem 5.2, for every k, there exists a solution $\left(\widetilde{\mathbf{u}}_k, \tilde{p}_k\right) \in \dot{\mathbf{H}}^2_{\#\sigma} \times \dot{H}^1_\#$ of the linear Oseen equation (6.28). Moreover, since $\mathbf{u}_k \in \dot{\mathbf{C}}^\infty_{\#\sigma}$, estimate (5.8) from Theorem 5.1 implies that

$$\|\widetilde{\mathbf{u}}_k\|_{\dot{\mathbf{H}}^1_\#} \leq C_{\mathbb{A}} \pi^{-2} \|\mathbf{f}\|_{\dot{\mathbf{H}}^{-1}_\#}. \tag{6.29}$$

In addition, considering (6.28) as a linear anisotropic Stokes equation with a given right-hand side, by Corollary 4.4, we obtain the estimate

$$\|\widetilde{\mathbf{u}}_k\|_{\dot{\mathbf{H}}^2_{\#\sigma}} + \|\tilde{p}_k\|_{\dot{H}^1_\#} \leq C \left(\|\mathbf{f}\|_{\dot{\mathbf{H}}^0_\#} + \|(\mathbf{u}_k \cdot \nabla)\widetilde{\mathbf{u}}_k\|_{\dot{\mathbf{H}}^0_\#} \right) \tag{6.30}$$

with $C = C_{uf} + C_{pf}$.

Let $\theta \in \mathbb{R}$ be such that $2 < \theta < \infty$ if $n = 2$, while $\theta = n$ if $n \in \{3, 4\}$. Employing Lemma 7.7 with $\mathbf{u}_k$ for u, the sequence $\{\mathbf{u}_k\}$ for $K_{\theta\#}$, and $\nabla\widetilde{\mathbf{u}}_k$ for w, we obtain by (3.10) and (6.29) that for any $\delta > 0$ there exists a constant $\widetilde{C}_\delta = C_\delta(K_{\theta\#}) > 0$ such that

$$\begin{aligned} \|(\mathbf{u}_k \cdot \nabla)\widetilde{\mathbf{u}}_k\|_{\dot{\mathbf{H}}^0_\#} &\leq \delta \|\nabla \widetilde{\mathbf{u}}_k\|_{(H^1_\#)^{n\times n}} + \widetilde{C}_\delta \|\mathbf{u}_k\|_{\mathbf{L}_{\theta\#}} \|\nabla \widetilde{\mathbf{u}}_k\|_{(L_{2\#})^{n\times n}} \\ &\leq 2\pi\delta \|\widetilde{\mathbf{u}}_k\|_{\dot{\mathbf{H}}^2_\#} + 2\pi \widetilde{C}_\delta \|\mathbf{u}_k\|_{\mathbf{L}_{\theta\#}} \|\widetilde{\mathbf{u}}_k\|_{\dot{\mathbf{H}}^1_\#} \\ &\leq 2\pi\delta \|\widetilde{\mathbf{u}}_k\|_{\dot{\mathbf{H}}^2_\#} + 2\pi^{-1} C_{\mathbb{A}} C_{\theta\#} \widetilde{C}_\delta \|\mathbf{u}_k\|_{\dot{\mathbf{H}}^1_\#} \|\mathbf{f}\|_{\dot{\mathbf{H}}^{-1}_\#}. \end{aligned} \tag{6.31}$$

Here, we also took into account that for $2 \leq n \leq 4$, there exists a constant $C_{\theta\#} > 0$ independent of $\mathbf{v}$, such that $\|\mathbf{v}\|_{\mathbf{L}_{\theta\#}} \leq C_{\theta\#} \|\mathbf{v}\|_{\mathbf{H}^1_\#}$ for any $\mathbf{v} \in \mathbf{H}^1_\#$, due to the Sobolev embedding theorem.

Since $\mathbf{u}_k$ converges to $\mathbf{u}$ in $\dot{\mathbf{H}}^1_{\#\sigma}$, estimate (6.7) means that there exists $k_0 \in \mathbb{N}$ such that for $k \geq k_0$ (which we will further assume)

$$\|\mathbf{u}_k\|_{\dot{\mathbf{H}}^1_\#} \leq 2\|\mathbf{u}\|_{\dot{\mathbf{H}}^1_\#} \leq 2\pi^{-2} C_{\mathbb{A}} \|\mathbf{f}\|_{\dot{\mathbf{H}}^{-1}_\#}$$

and hence (6.31) implies

$$\|(\mathbf{u}_k\cdot\nabla)\widetilde{\mathbf{u}}_k\|_{\dot{\mathbf{H}}^0_\#}\le 2\pi\delta\|\widetilde{\mathbf{u}}_k\|_{\dot{\mathbf{H}}^2_\#}+4\pi^{-3}C^2_{\mathbb{A}}C_{\theta\#}\widetilde{C}_\delta\|\mathbf{f}\|^2_{\dot{\mathbf{H}}^{-1}_\#}. \tag{6.32}$$

Substituting (6.32) in (6.30) with $\delta = 1/(4\pi C)$, we obtain

$$\|\widetilde{\mathbf{u}}_k\|_{\dot{\mathbf{H}}^2_{\#\sigma}}+\|\tilde{p}_k\|_{\dot{H}^1_\#}\le 2C\|\mathbf{f}\|_{\dot{\mathbf{H}}^0_\#}+C_{-1}\|\mathbf{f}\|^2_{\dot{\mathbf{H}}^{-1}_\#}, \tag{6.33}$$

where $C_{-1} := 8\pi^{-3}CC^2_{\mathbb{A}}C_{\theta\#}\widetilde{C}_\delta$.

Together with (6.28), this implies that there exist subsequences of $\{\widetilde{\mathbf{u}}_k\}$ and $\{\tilde{p}_k\}$ weakly converging in $\dot{\mathbf{H}}^2_{\#\sigma}$ and $\dot{H}^1_\#$, respectively, to a weak solution $(\widetilde{\mathbf{u}},\tilde{p})\in\dot{\mathbf{H}}^2_{\#\sigma}\times\dot{H}^1_\#\subset\dot{\mathbf{H}}^1_{\#\sigma}\times\dot{H}^0_\#$ of the Oseen equation

$$-\mathfrak{L}\widetilde{\mathbf{u}}+\nabla\tilde{p}=\mathbf{f}-(\mathbf{u}\cdot\nabla)\widetilde{\mathbf{u}} \tag{6.34}$$

satisfying the estimate

$$\|\widetilde{\mathbf{u}}\|_{\dot{\mathbf{H}}^2_{\#\sigma}}+\|\tilde{p}\|_{\dot{H}^1_\#}\le 2C\|\mathbf{f}\|_{\dot{\mathbf{H}}^0_\#}+C_{-1}\|\mathbf{f}\|^2_{\dot{\mathbf{H}}^{-1}_\#}. \tag{6.35}$$

Since $(\mathbf{u},p)\in\dot{\mathbf{H}}^1_{\#\sigma}\times\dot{H}^0_\#$ satisfy (6.3), the solution uniqueness for the Oseen equation (6.34) with a fixed $\mathbf{u}$ in $\dot{\mathbf{H}}^1_{\#\sigma}\times\dot{H}^0_\#$ (see Theorem 5.1(ii)) implies that $(\mathbf{u},p)=(\widetilde{\mathbf{u}},\tilde{p})\in\dot{\mathbf{H}}^2_{\#\sigma}\times\dot{H}^1_\#$ and estimate (6.27) holds with $C_1 = 2C$. □

Note that in the third inequality in (6.31) we used the Sobolev embedding theorem that implied that there exists a constant $C_{\theta\#}>0$ independent of $\mathbf{v}$, such that $\|\mathbf{v}\|_{\mathbf{L}_{\theta\#}}\le C_{\theta\#}\|\mathbf{v}\|_{\mathbf{H}^1_\#}$ for any $\mathbf{v}\in\mathbf{H}^1_\#$. This estimate is however not available for the dimensions $n>4$, which limited our regularity proof in Theorem 6.5 to the cases $n\in\{2,3,4\}$ only.

Combining Theorems 6.5 and 6.3, we obtain the following assertion on existence and regularity of solution to the Navier–Stokes system on torus for $n\in\{2,3,4\}$. Note that for $n\in\{2,3\}$, the assertion is already obtained in Theorem 6.4.

Theorem 6.6. *Let $n\in\{2,3,4\}$ and condition (2.3) hold.*

(i) If $\mathbf{f}\in\dot{\mathbf{H}}^{s-2}_\#$, $s\ge 2$, then the anisotropic Navier–Stokes equation (6.3) has a solution $(\mathbf{u},p)\in\dot{\mathbf{H}}^s_{\#\sigma}\times\dot{H}^{s-1}_\#$.
(ii) Moreover, if $\mathbf{f}\in\dot{\mathbf{C}}^\infty_\#$, then Equation (6.3) has a solution $(\mathbf{u},p)\in\dot{\mathbf{C}}^\infty_\#\times\dot{C}^\infty_\#$.

7 | SOME AUXILIARY RESULTS

7.1 | Abstract mixed variational formulations

Let us produce the well-posedness result for the abstract mixed formulation, related to Babuška [29] and Brezzi [30, Theorem 1.1] (see also Theorem 2.34 and Remark 2.35(i) in Ern and Guermond [31] and Brezzi and Fortin [32]).

Theorem 7.1. *Let X and $\mathcal{M}$ be two real Hilbert spaces. Let $a(\cdot,\cdot) : X\times X\to\mathbb{R}$ and $b(\cdot,\cdot) : X\times\mathcal{M}\to\mathbb{R}$ be bounded bilinear forms. Let $f\in X'$ and $g\in\mathcal{M}'$. Let V be the subspace of X defined by*

$$V := \{v\in X : b(v,q)=0\quad\forall q\in\mathcal{M}\}. \tag{7.1}$$

Assume that $a(\cdot,\cdot) : V\times V\to\mathbb{R}$ is coercive, which means that there exists a constant $C_a>0$ such that

$$a(w,w)\ge C_a^{-1}\|w\|^2_X\quad\forall w\in V, \tag{7.2}$$

and that $b(\cdot,\cdot) : X\times\mathcal{M}\to\mathbb{R}$ satisfies the Babuška–Brezzi condition

$$\inf_{q\in\mathcal{M}\setminus\{0\}}\sup_{v\in X\setminus\{0\}}\frac{b(v,q)}{\|v\|_X\|q\|_{\mathcal{M}}}\ge C_b^{-1}, \tag{7.3}$$

with some constant $C_b>0$. Then the mixed variational formulation

$$\begin{cases} a(u,v)+b(v,p) = f(v) & \forall v\in X,\\ b(u,q) = g(q) & \forall q\in\mathcal{M}\end{cases} \tag{7.4}$$

has a unique solution $(u, p) \in X \times \mathcal{M}$ *and*

$$\|u\|_X \le C_a \|f\|_{X'} + C_b(1 + \|a\| C_a)\|g\|_{\mathcal{M}'}, \tag{7.5}$$

$$\|p\|_{\mathcal{M}} \le C_b(1 + \|a\| C_a)\|f\|_{X'} + \|a\| C_b^2 (1 + \|a\| C_a)\|g\|_{\mathcal{M}'}, \tag{7.6}$$

where $\|a\|$ *is the norm of the bilinear form* $a(\cdot, \cdot)$.

7.2 | Brower fixed point theorem application

In the main text, we need the following well-known result that follows from the Brower fixed point theorem (see, e.g., [27, Chapter 1, Lemma 4.3], [2, Lemma IX.3.1]).

Lemma 7.2. *Let* $\eta \to Q(\eta)$ *be a continuous map of* $\mathbb{R}^m$ *to itself, such that for some* $\rho > 0$,

$$(Q(\eta), \eta) \ge 0 \quad \forall \eta : |\eta| = \rho. \tag{7.7}$$

Here, for $\eta = \{\eta_j\}, \zeta = \{\zeta_j\} \in \mathbb{R}^m$, *we denote*

$$(\eta, \zeta) := \sum_{j=1}^{m} \eta_j \zeta_j, \ |\eta| := (\eta, \eta)^{1/2}.$$

Then there exists η *such that* $|\eta| \le \rho$ *and* $Q(\eta) = 0$.

7.3 | Advection term properties

Let the quadratic operator $\boldsymbol{B} : \mathbf{w} \mapsto \boldsymbol{B}\mathbf{w}$ be defined as $\boldsymbol{B}\mathbf{w} := (\mathbf{w} \cdot \nabla)\mathbf{w}$.

Let in this section the dimension $n \ge 2$. To formulate assertions valid both for $n = 2$ and $n > 2$, let us define the set I_n as

$$I_2 := (2, \infty), \ I_n := [n, \infty) \ \text{if} \ n > 2. \tag{7.8}$$

Let $\theta \in I_n$ and let us denote $q_\theta := 2\theta/(\theta - 2)$. Then

$$2 < 2\theta/(\theta - 2) < \infty \quad \text{if} \quad n = 2; \ 2 < 2\theta/(\theta - 2) \le 2n/(n - 2) \ \text{if} \ n > 2. \tag{7.9}$$

By the Sobolev embedding theorem (see, e.g., [33, Section 2.2.4, Corollary 2]), for any $\theta \in I_n$ the space $\mathbf{H}^1_\#$ is continuously embedded in the space $\mathbf{L}_{2\theta/(\theta-2)\#}$ and there exists a constant $C_{2\theta/(\theta-2)\#} > 0$ independent of $\mathbf{v}$, such that

$$\|\mathbf{v}\|_{\mathbf{L}_{2\theta/(\theta-2)\#}} \le C_{2\theta/(\theta-2)\#} \|\mathbf{v}\|_{\mathbf{H}^1_\#} \quad \forall \mathbf{v} \in \mathbf{H}^1_\#, \quad \forall \theta \in I_n. \tag{7.10}$$

By the Hölder inequality, for any $\theta \in I_n$

$$|\langle (\mathbf{v}_1 \cdot \nabla)\mathbf{v}_2, \mathbf{v}_3 \rangle_{\mathbb{T}}| \le \|\mathbf{v}_1\|_{\mathbf{L}_{2\theta/(\theta-2)\#}} \|\nabla \mathbf{v}_2\|_{(L_{2\#})^{n\times n}} \|\mathbf{v}_3\|_{\mathbf{L}_{\theta\#}} \forall \mathbf{v}_1 \in \mathbf{L}_{2\theta/(\theta-2)\#}, \mathbf{v}_2 \in \mathbf{H}^1_\#, \mathbf{v}_3 \in \mathbf{L}_{\theta\#}, \quad \forall \theta \in I_n. \tag{7.11}$$

Due to (7.10) and (3.10), inequality (7.11) gives

$$|\langle (\mathbf{v}_1 \cdot \nabla)\mathbf{v}_2, \mathbf{v}_3 \rangle_{\mathbb{T}}| \le 2\pi C_{2\theta/(\theta-2)\#} \|\mathbf{v}_1\|_{\mathbf{H}^1_\#} \|\mathbf{v}_2\|_{\mathbf{H}^1_\#} \|\mathbf{v}_3\|_{\mathbf{L}_{\theta\#}} \forall \mathbf{v}_1, \mathbf{v}_2 \in \mathbf{H}^1_\#, \mathbf{v}_3 \in \mathbf{L}_{\theta\#}, \quad \forall \theta \in I_n. \tag{7.12}$$

This implies that the trilinear form $\langle(\mathbf{v}_1\cdot\nabla)\mathbf{v}_2,\mathbf{v}_3\rangle_{\mathbb{T}}$ is bounded and continuous on $\mathbf{H}^1_\#\times\mathbf{H}^1_\#\times\mathbf{L}_{\theta\#},\quad\forall\theta\in I_n$. Taking into account that the space $\mathbf{L}_{\theta/(\theta-1)\#}$ is dual to the space $\mathbf{L}_{\theta\#}$, this implies that if $\mathbf{v}_1,\mathbf{v}_2\in\mathbf{H}^1_\#$, then

$$(\mathbf{v}_1\cdot\nabla)\mathbf{v}_2\in\mathbf{L}_{\theta/(\theta-1)\#},\quad\forall\theta\in I_n,$$

and the following estimate holds for the bilinear term

$$\|(\mathbf{v}_1\cdot\nabla)\mathbf{v}_2\|_{\mathbf{L}_{\theta/(\theta-1)\#}}\le 2\pi C_{2\theta/(\theta-2)\#}\|\mathbf{v}_1\|_{\mathbf{H}^1_\#}\|\mathbf{v}_2\|_{\mathbf{H}^1_\#},\quad\forall\theta\in I_n.$$

This means that the quadratic operator

$$\boldsymbol{B}:\mathbf{H}^1_\#\to\mathbf{L}_{\theta/(\theta-1)\#},\quad\forall\theta\in I_n\tag{7.13}$$

is bounded and continuous.

Moreover, if $\mathbf{v}_1\in\mathbf{H}^1_{\#\sigma}$ and $\mathbf{v}_2\in\mathbf{H}^1_\#$, then

$$\int_{\mathbb{T}}(\mathbf{v}_1\cdot\nabla)\mathbf{v}_2dx=\int_{\mathbb{T}}\partial_i(v_{1,i}\mathbf{v}_2)dx-\int_{\mathbb{T}}(\operatorname{div}\ \mathbf{v}_1)\mathbf{v}_2dx=0,\tag{7.14}$$

and hence $(\mathbf{v}_1\cdot\nabla)\mathbf{v}_2\in\dot{\mathbf{L}}_{\theta/(\theta-1)\#},\quad\forall\theta\in I_n$, implying the boundedness and continuity of the quadratic operator

$$\boldsymbol{B}:\dot{\mathbf{H}}^1_{\#\sigma}\to\dot{\mathbf{L}}_{\theta/(\theta-1)\#},\quad\forall\theta\in I_n.\tag{7.15}$$

For $n\ge 3$, taking $\theta=n$, this particularly implies that the quadratic operator

$$\boldsymbol{B}:\dot{\mathbf{H}}^1_{\#\sigma}\to\widetilde{\mathbf{V}}^*_{\#\sigma}=(\dot{\mathbf{H}}^1_{\#\sigma})^*\cup\mathbf{L}_{n/(n-1)\#}\tag{7.16}$$

is also bounded and continuous. In fact, operator (7.16) is bounded and continuous also for $n=2$.

Indeed, for $n=2$, estimate (7.12) with $\theta=4$ leads to the estimate

$$|\langle(\mathbf{v}_1\cdot\nabla)\mathbf{v}_2,\mathbf{v}_3\rangle_{\mathbb{T}}|\le 2\pi C_{4\#}\|\mathbf{v}_1\|_{\mathbf{H}^1_\#}\|\mathbf{v}_2\|_{\mathbf{H}^1_\#}\|\mathbf{v}_3\|_{\mathbf{L}_{4\#}}\forall\mathbf{v}_1,\mathbf{v}_2\in\mathbf{H}^1_\#,\,\mathbf{v}_3\in\mathbf{L}_{4\#}.\tag{7.17}$$

Assume now that $\mathbf{v}_3\in\mathbf{H}^1_\#$ and again take into account that by the Sobolev embedding theorem the space $\mathbf{H}^1_\#$ is continuously embedded in the space $\mathbf{L}_{4\#}$ and

$$\|\mathbf{v}_3\|_{\mathbf{L}_{4\#}}\le C_{4\#}\|\mathbf{v}_3\|_{\mathbf{H}^1_\#}\quad\forall\mathbf{v}_3\in\mathbf{H}^1_\#,\,n=2.\tag{7.18}$$

Due to (7.18), inequality (7.17) gives

$$|\langle(\mathbf{v}_1\cdot\nabla)\mathbf{v}_2,\mathbf{v}_3\rangle_{\mathbb{T}}|\le 2\pi C^2_{4\#}\|\mathbf{v}_1\|_{\mathbf{H}^1_\#}\|\mathbf{v}_2\|_{\mathbf{H}^1_\#}\|\mathbf{v}_3\|_{\mathbf{H}^1_\#}\forall\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3\in\mathbf{H}^1_\#,\,n=2.$$

This implies that the trilinear form $\langle(\mathbf{v}_1\cdot\nabla)\mathbf{v}_2,\mathbf{v}_3\rangle_{\mathbb{T}}$ is bounded and continuous on $\mathbf{H}^1_\#\times\mathbf{H}^1_\#\times\mathbf{H}^1_\#$. Taking into account that the space $\mathbf{H}^{-1}_\#$ is dual to the space $\mathbf{H}^1_\#$, this implies that if $\mathbf{v}_1,\mathbf{v}_2\in\mathbf{H}^1_\#$, then

$$(\mathbf{v}_1\cdot\nabla)\mathbf{v}_2\in\mathbf{H}^{-1}_\#,\,n=2,$$

and the following estimate holds for the bilinear operator

$$\|(\mathbf{v}_1\cdot\nabla)\mathbf{v}_2\|_{\mathbf{H}^{-1}_\#}\le 2\pi C^2_{4\#}\|\mathbf{v}_1\|_{\mathbf{H}^1_\#}\|\mathbf{v}_2\|_{\mathbf{H}^1_\#},\,n=2.$$

This means that the quadratic operator

$$\boldsymbol{B} : \mathbf{H}^1_\# \to \mathbf{H}^{-1}_\#, \, n = 2$$

is bounded and continuous. Moreover, if $\mathbf{v}_1 \in \mathbf{H}^1_{\#\sigma}$ and $\mathbf{v}_2 \in \mathbf{H}^1_\#$, then by (7.14) we also obtain $(\mathbf{v}_1 \cdot \nabla)\mathbf{v}_2 \in \dot{\mathbf{H}}^{-1}_\#$ and the continuity of the quadratic operator

$$\boldsymbol{B} : \dot{\mathbf{H}}^1_{\#\sigma} \to \dot{\mathbf{H}}^{-1}_\#, \, n = 2. \tag{7.19}$$

Since $\mathbf{H}^{-1}_\# = (\mathbf{H}^1_\#)^* \subset (\mathbf{H}^1_{\#\sigma})^* \subset (\dot{\mathbf{H}}^1_{\#\sigma})^* \cup \mathbf{L}_{n/(n-1)\#} = \widetilde{\mathbf{V}}^*_{\#\sigma}$, the continuity of operator (7.19) implies also the boundedness and continuity of operator

$$\boldsymbol{B} : \dot{\mathbf{H}}^1_{\#\sigma} \to \widetilde{\mathbf{V}}^*_{\#\sigma}, \, n = 2,$$

that is, of operator (7.16) also for $n = 2$.

Let us also give another well known result, which proof can be easily accommodated to the periodic case from the one found, for example, in [7, Chapter 2, Lemma 1.5]; see also [27, Chapter 1, Teorem 7.1].

Lemma 7.3. *Let $\mathbf{u}_k$ converge to $\mathbf{u}$ weakly in $\dot{\mathbf{H}}^1_{\#\sigma}$ and strongly in $\dot{\mathbf{L}}_{2\#\sigma}$. Then $\langle \boldsymbol{B}\mathbf{u}_k, \mathbf{v}\rangle_{\mathbb{T}} \to \langle (\mathbf{u} \cdot \nabla)\mathbf{u}, \mathbf{v}\rangle_{\mathbb{T}} \quad \forall \mathbf{v} \in \dot{\mathbf{C}}^\infty_{\#\sigma}$.*

Similar to (7.11), we have

$$|\langle (\mathbf{v}_1 \cdot \nabla)\mathbf{v}_2, \mathbf{v}_3\rangle_{\mathbb{T}}| \le \|\mathbf{v}_1\|_{\mathbf{L}_{\theta\#}} \|\nabla \mathbf{v}_2\|_{(L_{2\#})^{n\times n}} \|\mathbf{v}_3\|_{\mathbf{L}_{2\theta/(\theta-2)\#}} \forall \mathbf{v}_1 \in \mathbf{L}_{\theta\#}, \mathbf{v}_2 \in \mathbf{H}^1_\#, \mathbf{v}_3 \in \mathbf{L}_{2\theta/(\theta-2)\#}. \tag{7.20}$$

Due to (7.10), inequality (7.20) implies

$$|\langle (\mathbf{v}_1 \cdot \nabla)\mathbf{v}_2, \mathbf{v}_3\rangle_{\mathbb{T}}| \le 2\pi C_{2\theta/(\theta-2)\#} \|\mathbf{v}_1\|_{\mathbf{L}_{\theta\#}} \|\mathbf{v}_2\|_{\mathbf{H}^1_\#} \|\mathbf{v}_3\|_{\mathbf{H}^1_\#} \forall \mathbf{v}_1 \in \mathbf{L}_{\theta\#}, \quad \forall \mathbf{v}_2, \mathbf{v}_3 \in \mathbf{H}^1_\#. \tag{7.21}$$

This implies that if $\mathbf{v}_1 \in \mathbf{L}_{\theta\#}$, $\mathbf{v}_2 \in \mathbf{H}^1_\#$, then $(\mathbf{v}_1 \cdot \nabla)\mathbf{v}_2 \in \mathbf{H}^{-1}_\#$ and the following estimate holds for the bilinear operator

$$\|(\mathbf{v}_1 \cdot \nabla)\mathbf{v}_2\|_{\mathbf{H}^{-1}_\#} \le 2\pi C_{2\theta/(\theta-2)\#} \|\mathbf{v}_1\|_{\mathbf{L}_{\theta\#}} \|\mathbf{v}_2\|_{\mathbf{H}^1_\#}. \tag{7.22}$$

Moreover, if $\mathbf{v}_1 \in \mathbf{L}_{\theta\#\sigma}$, $\mathbf{v}_2 \in \mathbf{H}^1_\#$, then again (7.14) holds, implying that by (7.22),

$$\begin{aligned} &(\mathbf{v}_1 \cdot \nabla)\mathbf{v}_2 \in \dot{\mathbf{H}}^{-1}_\#, \\ &\|(\mathbf{v}_1 \cdot \nabla)\mathbf{v}_2\|_{\dot{\mathbf{H}}^{-1}_\#} \le 2\pi C_{2\theta/(\theta-2)\#} \|\mathbf{v}_1\|_{\mathbf{L}_{\theta\#}} \|\mathbf{v}_2\|_{\mathbf{H}^1_\#}. \end{aligned} \tag{7.23}$$

The divergence theorem and periodicity imply the following identity for any $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \in \mathbf{C}^\infty_\#$.

$$\begin{aligned} \langle (\mathbf{v}_1 \cdot \nabla)\mathbf{v}_2, \mathbf{v}_3\rangle_{\mathbb{T}} &= \int_{\mathbb{T}} \nabla \cdot (\mathbf{v}_1(\mathbf{v}_2 \cdot \mathbf{v}_3))\, d\mathbf{x} - \langle (\nabla \cdot \mathbf{v}_1)\mathbf{v}_3 + (\mathbf{v}_1 \cdot \nabla)\mathbf{v}_3, \mathbf{v}_2\rangle_{\mathbb{T}} \\ &= -\langle (\mathbf{v}_1 \cdot \nabla)\mathbf{v}_3, \mathbf{v}_2\rangle_{\mathbb{T}} - \langle (\nabla \cdot \mathbf{v}_1)\mathbf{v}_3, \mathbf{v}_2\rangle_{\mathbb{T}}. \end{aligned} \tag{7.24}$$

In view of (7.24), we obtain the identity

$$\langle (\mathbf{v}_1 \cdot \nabla)\mathbf{v}_2, \mathbf{v}_3\rangle_{\mathbb{T}} = -\langle (\mathbf{v}_1 \cdot \nabla)\mathbf{v}_3, \mathbf{v}_2\rangle_{\mathbb{T}} \quad \forall \mathbf{v}_1 \in \mathbf{C}^\infty_{\#\sigma}, \mathbf{v}_2, \mathbf{v}_3 \in \mathbf{C}^\infty_\#,$$

and hence the following well-known formula for any $\mathbf{v}_1 \in \mathbf{C}^\infty_{\#\sigma}$, $\mathbf{v}_2 \in \mathbf{C}^\infty_{\#}$,

$$\langle (\mathbf{v}_1 \cdot \nabla)\mathbf{v}_2, \mathbf{v}_2 \rangle_{\mathbb{T}} = 0. \tag{7.25}$$

(i) The dense embedding of the space $\mathbf{C}^\infty_{\#}$ into $\mathbf{H}^1_{\#}$ and into $\mathbf{L}_{\theta\#}$, the dense embedding of the space $\mathbf{C}^\infty_{\#\sigma}$ into $\mathbf{H}^1_{\#\sigma}$, and estimate (7.12) ensuring the boundedness of the corresponding dual product in (7.25) imply that relation (7.25) holds also for any $\mathbf{v}_1 \in \mathbf{H}^1_{\#\sigma}$ and $\mathbf{v}_2 \in \mathbf{H}^1_{\#} \cap \mathbf{L}_{\theta\#}$.

(ii) Particularly, for $n = 2$, $\mathbf{H}^1_{\#} \cap \mathbf{L}_{\theta\#} = \mathbf{H}^1_{\#}$ $\forall \theta \in (2, \infty)$; for $n \in \{3, 4\}$, we can choose $\theta = n$ and take into account that $\mathbf{H}^1_{\#} \cap \mathbf{L}_{n\#} = \mathbf{H}^1_{\#}$. Hence, if $n \in \{2, 3, 4\}$, (7.25) holds for any $\mathbf{v}_1 \in \mathbf{H}^1_{\#\sigma}$ and $\mathbf{v}_2 \in \mathbf{H}^1_{\#}$.

(iii) Taking into account also the dense embedding of $\mathbf{C}^\infty_{\#\sigma}$ into $\mathbf{L}_{\theta\#\sigma}$, along with estimate (7.21) ensuring the boundedness of the corresponding dual product in the left hand side of (7.25), we conclude that when $n \geq 2$, (7.25) holds for any $\mathbf{v}_1 \in \mathbf{L}_{\theta\#\sigma}$ and $\mathbf{v}_2 \in \mathbf{H}^1_{\#}$.

Due to Theorem 1 in Section 4.6.1 of [33] and equivalence of the Bessel potential norms on square and norms (3.3) for the Sobolev spaces on torus, we have the following assertion.

Theorem 7.4. *Let $n \geq 1$, $\tilde{s}_1 \leq \tilde{s}_2$ and $\tilde{s}_1 + \tilde{s}_2 > 0$.*

(i) If $\tilde{s}_2 < n/2$, then there exists a constant $C_1 = C_1(\tilde{s}_1, \tilde{s}_2, n)$ such that for any $v_1 \in H^{\tilde{s}_1}_{\#}$ and $v_2 \in H^{\tilde{s}_2}_{\#}$, we have $v_1 v_2 \in H^{\tilde{s}_1+\tilde{s}_2-n/2}_{\#}$ and

$$\|v_1 v_2\|_{H^{\tilde{s}_1+\tilde{s}_2-n/2}_{\#}} \leq C_1 \|v_1\|_{H^{\tilde{s}_1}_{\#}} \|v_2\|_{H^{\tilde{s}_2}_{\#}}. \tag{7.26}$$

(ii) If $\tilde{s}_2 > n/2$, then there exists a constant $C_2 = C_2(\tilde{s}_1, \tilde{s}_2, n)$ such that for any $v_1 \in H^{\tilde{s}_1}_{\#}$ and $v_2 \in H^{\tilde{s}_2}_{\#}$, we have $v_1 v_2 \in H^{\tilde{s}_1}_{\#}$ and

$$\|v_1 v_2\|_{H^{\tilde{s}_1}_{\#}} \leq C_2 \|v_1\|_{H^{\tilde{s}_1}_{\#}} \|v_2\|_{H^{\tilde{s}_2}_{\#}}. \tag{7.27}$$

Theorem 7.4 immediately leads to the following result.

Theorem 7.5. *Let $n \geq 2$.*

(i) If $0 < s < n/2$ then the quadratic operators

$$\boldsymbol{B} : \mathbf{H}^s_{\#} \to \mathbf{H}^{2s-1-n/2}_{\#}, \tag{7.28}$$

$$\boldsymbol{B} : \dot{\mathbf{H}}^s_{\#\sigma} \to \dot{\mathbf{H}}^{2s-1-n/2}_{\#} \tag{7.29}$$

are well defined, continuous and bounded; that is, there exists $C_{n,s} > 0$ such that

$$\|\boldsymbol{B}\mathbf{w}\|_{\mathbf{H}^{2s-1-n/2}_{\#}} \leq C_{n,s} \|\mathbf{w}\|^2_{\mathbf{H}^s_{\#}} \quad \forall \mathbf{w} \in \mathbf{H}^s_{\#}. \tag{7.30}$$

(ii) If $s > n/2$, then the quadratic operators

$$\boldsymbol{B} : \mathbf{H}^s_{\#} \to \mathbf{H}^{s-1}_{\#}, \tag{7.31}$$

$$\boldsymbol{B} : \dot{\mathbf{H}}^s_{\#\sigma} \to \dot{\mathbf{H}}^{s-1}_{\#} \tag{7.32}$$

are well defined, continuous and bounded; that is, there exists $C_{n,s} > 0$ such that

$$\|\boldsymbol{B}\mathbf{w}\|_{\mathbf{H}^{s-1}_{\#}} \leq C_{n,s} \|\mathbf{w}\|^2_{\mathbf{H}^s_{\#}} \quad \forall \mathbf{w} \in \mathbf{H}^s_{\#}. \tag{7.33}$$

Proof. If a function $\mathbf{w}$ is periodic, then evidently the function $\boldsymbol{B}\mathbf{w}$ is periodic as well.

(i) Let $0 < s < n/2$. Theorem 7.4(i) implies estimate (7.30) and then the boundedness of operator (7.28). Further, if $\mathbf{u} \in \dot{\mathbf{H}}^s_{\#\sigma}$, then due to the periodicity,

$$\langle \boldsymbol{B}\mathbf{u}, 1 \rangle_{\mathbb{T}} = \langle \mathbf{u} \cdot \nabla)\mathbf{u}, 1 \rangle_{\mathbb{T}} = -\langle (\mathrm{div}\ \mathbf{u})\mathbf{u}, 1 \rangle_{\mathbb{T}} = \mathbf{0}.$$

Together with estimate (7.30), this implies that quadratic operator (7.29) is well defined and bounded.
Let $\mathbf{w}, \mathbf{w}' \in \mathbf{H}^1_{\#\sigma}$. Then by (7.26), we obtain

$$\begin{aligned}\|\boldsymbol{B}\mathbf{w} - \boldsymbol{B}\mathbf{w}'\|_{\mathbf{H}^{2s-1-n/2}_\#} &\le \|(\mathbf{w}\cdot\nabla)\mathbf{w} - (\mathbf{w}'\cdot\nabla)\mathbf{w}'\|_{\mathbf{H}^{2s-1-n/2}_\#}\\ &\le \|((\mathbf{w}-\mathbf{w}')\cdot\nabla)\mathbf{w} + (\mathbf{w}'\cdot\nabla)(\mathbf{w}-\mathbf{w}')\|_{\mathbf{H}^{2s-1-n/2}_\#}\\ &\le C_{n,s}\|\mathbf{w}-\mathbf{w}'\|_{\mathbf{H}^s_\#}\left(\|\mathbf{w}\|_{\mathbf{H}^s_\#} + \|\mathbf{w}'\|_{\mathbf{H}^s_\#}\right).\end{aligned}$$

This estimate shows that operator (7.28) and (7.29) are continuous.

(ii) Let $s > n/2$. Theorem 7.4(ii) implies estimate (7.33). Then by the same arguments as in item (i), one can prove that operators (7.31) and (7.32) are also well defined, bounded and continuous. □

The following assertion is essentially an adaptation of a result from [28, Lemma and Remark 2.3(ii)] to the periodic setting for the particular case of L_2-based Sobolev spaces.

Lemma 7.6. *Let $n \ge 2$ and $\theta \in I_n$. Then for any $u \in L_{\theta\#}$ and any $\epsilon > 0$, there exists a constant $c_\epsilon(u) > 0$ depending only on u, n and ϵ, such that*

$$\int_{\mathbb{T}} |u|^2|w|^2 dx \le \epsilon\|w\|^2_{H^1_\#} + c_\epsilon(u)\|w\|^2_{L_{2\#}} \quad \forall w \in H^1_\#. \tag{7.34}$$

Proof. Let us recall that $q_\theta := 2\theta/(\theta-2) > 2$; see (7.8) and (7.9). Due to the Sobolev embedding theorem, the space $H^1_\#$ is continuously embedded in $L_{q_\theta\#}$ and hence $w \in L_{q_\theta\#}$. By the Hölder inequality, this implies that $|u|^2|w|^2 \in L_{1\#}$ and integral in the left hand side of (7.34) is bounded.

Let us employ the contradiction argument and assume that there exist $u \in L_{\theta\#}$ and $\epsilon > 0$ such that for any constant $c > 0$ there exists w_c, such that

$$\int_{\mathbb{T}} |u|^2|w_c|^2 dx > \epsilon\|w_c\|^2_{H^1_\#} + c\|w_c\|^2_{L_{2\#}}. \tag{7.35}$$

Inequality (7.35) does not hold if $w_c = 0$ a.e. on $\mathbb{T}$; hence, we can assume that $\|w_c\|_{H^1_\#} \ne 0$. Inequality (7.35) then implies that for any c it will also hold for $\widetilde{w}_c = w_c/\|w_c\|_{H^1_\#}$. We evidently have $\|\widetilde{w}_c\|_{H^1_\#} = 1$ and by the Sobolev embedding theorem $\|\widetilde{w}_c\|_{L_{q_\theta\#}}$ is bounded by a constant that does not depend on c. By the Hölder inequality for $u \in L_{\theta\#}$ and $\widetilde{w}_c \in L_{q_\theta\#}$, we have

$$\|u\|^2_{L_{\theta\#}}\|\widetilde{w}_c\|^2_{L_{q_\theta\#}} \ge \int_{\mathbb{T}} |u|^2|\widetilde{w}_c|^2 dx > \epsilon + c\|\widetilde{w}_c\|^2_{L_{2\#}}. \tag{7.36}$$

Choosing $c \in \mathbb{N}$ and taking limit of (7.36) as $c \to \infty$, the inequality implies that

$$\||\widetilde{w}_c|^2\|_{L_{1\#}} = \|\widetilde{w}_c\|^2_{L_{2\#}} \to 0 \text{ as } c \to \infty. \tag{7.37}$$

Note that $q_\theta/2 > 1$. The boundedness of sequence $\|\widetilde{w}_c\|_{L_{q_\theta\#}}$ is equivalent to the boundedness of the sequence $\||\widetilde{w}_c|^2\|_{L_{q_\theta/2\#}}$ and implies that there exists a subsequence of $\{|\widetilde{w}_c|^2\}$ weakly converging in $L_{q_\theta/2\#}$ (and hence in $L_{1\#}$) to a function $W \in L_{q_\theta/2\#}$, and then by (7.37), $W = 0$. Then the integral in inequality (7.36) converges to zero, and the inequality implies that $\epsilon = 0$, which contradicts the assumption $\epsilon > 0$. This implies (7.34). □

Let us now prove a stronger version of Lemma 7.6 (cf. [28, Remark 2.3(iii)]).

Lemma 7.7. *Let $n \ge 2$ and $\theta \in I_n$. Let $K_{\theta\#}$ be a compact subset of $L_{\theta\#}$. Then for any $\delta > 0$ there exist constants $\widetilde{C}_\delta(K_{\theta\#}) > 0$ and $C_\delta(K_{\theta\#}) > 0$ (depending only on n, θ, $K_{\theta\#}$ and δ) such that for all $w \in H^1_\#$ and $u \in K_{\theta\#}$,*

$$\|uw\|_{L_{2\#}} \le \delta\|w\|_{H^1_\#} + \widetilde{C}_{\delta(K_{\theta\#})}\|u\|_{L_{\theta\#}}\|w\|_{L_{2\#}} \tag{7.38}$$

$$\le \delta\|w\|_{H^1_\#} + C_\delta(K_{\theta\#})\|w\|_{L_{2\#}}. \tag{7.39}$$

Proof. Let us first prove that for any $\epsilon > 0$, there exist a constant $\tilde{c}_\epsilon > 0$, depending only on n, θ and ϵ, such that

$$\int_{\mathbb{T}} |u|^2 |w|^2 dx \leq \epsilon \|w\|^2_{H^1_\#} + \tilde{c}_\epsilon \|u\|^2_{L_{\theta\#}} \|w\|^2_{L_{2\#}}. \tag{7.40}$$

Let us recall that $q_\theta := 2\theta/(\theta - 2) > 2$ (cf. (7.8) and (7.9)). For $u = 0$ or $w = 0$, inequality (7.40) evidently holds. To prove that it holds also for $\|u\|_{L_{\theta\#}} \neq 0$ and $\|w\|_{H^1_\#} \neq 0$ let us assume the contrary, namely, that there exists $\epsilon > 0$ such that for any constant $c > 0$ there exists $u_c \in K_{\theta\#}$ and $w_c \in H^1_\#$ such that

$$\int_{\mathbb{T}} |u_c|^2 |w_c|^2 dx > \epsilon \|w_c\|^2_{H^1_\#} + c\|u_c\|^2_{L_{\theta\#}} \|w_c\|^2_{L_{2\#}}. \tag{7.41}$$

Inequality (7.41) does not hold if $w_c = 0$; hence, we can assume that $\|w_c\|_{H^1_\#} \neq 0$. Inequality (7.41) then implies that for any c it will also hold for $\widetilde{w}_c = w_c/\|w_c\|_{H^1_\#}$. We evidently have $\|\widetilde{w}_c\|_{H^1_\#} = 1$, and by the Sobolev embedding theorem, all $\|\widetilde{w}_c\|_{L_{q_\theta\#}}$ are bounded by a constant that does not depend on c. By the Hölder inequality for $u_c \in L_{\theta\#}$ and $\widetilde{w}_c \in L_{q_\theta\#}$, inequality (7.41) reduces to

$$\|u_c\|^2_{L_{\theta\#}} \|\widetilde{w}_c\|^2_{L_{q_\theta\#}} \geq \int_{\mathbb{T}} |u_c|^2 |\widetilde{w}_c|^2 dx > \epsilon + c\|u_c\|^2_{L_{\theta\#}} \|\widetilde{w}_c\|^2_{L_{2\#}}. \tag{7.42}$$

Let us choose the sequence $\{c\} = \mathbb{N}$. Inequality (7.42) implies that

$$\|\widetilde{w}_c\|^2_{L_{q_\theta\#}} > c\|\widetilde{w}_c\|^2_{L_{2\#}}. \tag{7.43}$$

Since all $\|\widetilde{w}_c\|_{L_{q_\theta\#}}$ are bounded by a constant independent of c, inequality (7.43) implies that

$$\|\widetilde{w}_c\|_{L_{2\#}} \to 0 \text{ as } c \to \infty. \tag{7.44}$$

The boundedness of sequence $\|\widetilde{w}_c\|_{L_{q_\theta\#}}$ is equivalent to the boundedness of sequence $\||\widetilde{w}_c|^2\|_{L_{q_\theta/2\#}}$ and implies that there exists a subsequence of $\{|\widetilde{w}_c|^2\}$ weakly converging in $L_{q_\theta/2\#}$ (and hence in $L_{1\#}$) to a function $W \in L_{q_\theta/2\#}$, and then by (7.44), $W = 0$. Here, we took into account that $q_\theta/2 > 1$. On the other hand, since the set $K_{\theta\#}$ is a compact subset of $L_{\theta\#}$, there exists a subsequence of $\{|u_c|^2\}$ (with the indices c belonging to the subset of indices of the previous subsequence) strongly converging in $L_{\theta/2\#}$.

Then (see, e.g., Problem 6.33 in [34]) the integral in inequality (7.42) converges to zero, and the inequality implies that $\epsilon = 0$, which contradicts the assumption $\epsilon > 0$. This implies (7.40), which after choosing $\epsilon = \delta^2$ and $\widetilde{C}_\delta = \sqrt{\tilde{c}_\epsilon}$ leads to (7.38). The set $K_{\theta\#}$ is compact and hence bounded in $L_{\theta\#}$, which then implies inequality (7.39). □

Note that Lemma 7.7 particularly holds true when $K_{\theta\#}$ is a sequence converging in $L_{\theta\#}$.

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CONFLICT OF INTEREST STATEMENT

This work does not have any conflicts of interest.

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