Mathematics Education Research Journal
https://doi.org/10.1007/s13394-022-00423-z

ORIGINAL ARTICLE

"You see things that you wouldn't have seen otherwise": enabling elementary preservice teachers to share different ways of seeing mathematics

Julie Alderton[1,6] · Gwen Ineson[2] · Gina Donaldson[3] · Tim Rowland[1] · Chronoula Voutsina[4] · Kirsty Wilson[5]

Received: 3 August 2021 / Revised: 9 May 2022 / Accepted: 29 May 2022

Abstract

Finding opportunities for elementary preservice teachers to engage in mathematics for themselves and to collaborate in their enquiries was the focus of this study, carried out at two English Universities. Preservice teachers on 1-year postgraduate programmes engaged in shared mathematics enquiry, with a focus on growing patterns. We conducted interviews with 15 preservice teachers and analysed the interview data alongside their subsequent lesson plans and lesson evaluations. We explored the awarenesses that emerged through deliberate retrospective analysis of sharing what others were seeing and how this influenced their prospective thinking about their own teaching. Our findings indicate that even when preservice teachers struggle to make sense of what others are seeing, they recognise that some approaches may be more efficient or insightful than others, and that listening to others' ideas is a powerful learning opportunity for teachers and children. This has implications for initial teacher education programmes internationally. There is value in providing preservice teachers with opportunities to engage with mathematics as a shared experience, and in enabling and supporting deliberate retrospective and prospective reflection of this activity.

Keywords Mathematics · Awareness · Elementary preservice teachers · Shared activity · Seeing · Growing patterns

✉ Julie Alderton
jha32@cam.ac.uk

1 University of Cambridge, Cambridge, UK

2 Brunel University London, London, UK

3 Canterbury Christ Church University, Canterbury, UK

4 University of Southampton, Southampton, UK

5 University of Birmingham, Birmingham, UK

6 Faculty of Education, University of Cambridge, 184 Hills Road, Cambridge CB2 8PQ, UK

Published online: 14 June 2022

Introduction

As part of their initial teacher education (ITE), preservice elementary teachers may be asked to work on some mathematics at their own level with their peers. This offers an opportunity to become aware of how others experience mathematical activity in different ways, as well as possibilities for their own teaching of mathematics. But how do preservice teachers make sense of what others experience and how might these opportunities to share different ways of seeing mathematics influence their own thinking about teaching?

In this paper, we report on the emerging awarenesses of preservice elementary teachers who collaborated on visual growing pattern tasks and shared what they saw. Mason (2008) describes awarenesses that can be articulated as explicit and argues for the development of explicit awarenesses in teacher education. Teachers who are aware of what their learners may be attending to, and who have considered possibilities for action, can begin to make conscious choices in their practice. In our teaching sessions, we offered preservice teachers opportunities to develop explicit awarenesses through "working on mathematics for themselves" and "collaborating in their enquiries" (Mason, 2010 pp.42–43). We have previously reported on our analysis of preservice teachers' awarenesses gained through working on mathematics for themselves (Voutsina et al., 2022). Here, we analyse how collaborating in their enquiries sensitised preservice teachers to the value of engaging with others' ways of seeing a growing pattern. We operationalise Mason's (2010) notion of collaborating with others in the context of our customary ways of working with our preservice teachers. That is, we provide opportunities and encouragement for small group peer interaction, rather than a requirement; however, time for whole group sharing and discussion of the mathematics engaged with is always included in taught sessions.

Growing patterns are often constructed from components such as squares, matchsticks or dots and offer visual and concrete contexts for making mathematical generalisations about how a pattern grows. These patterns, including those we used (Fig. 1a, b), can be "seen" in different ways, and hence collaborating with others affords opportunities to become aware of how others "see" the images. Teachers who are aware that learners can see growing patterns in multiple ways can plan appropriate pedagogical opportunities, for example, to share ways of seeing. This visual approach also has potential to support preservice teachers in making meaningful links between different ways of seeing and the algebraic generalisation. We recognise the value of this, given many preservice teachers' anxiety about their

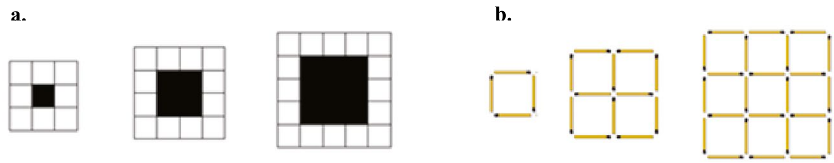


Fig. 1 **a** Flowerbed growing pattern. **b** Matchstick squares growing pattern

algebraic understanding (Wilkie, 2014), which may reflect limited experience of meaningful visual approaches in their own schooling. Simple growing pattern tasks also offer an appropriate context for elementary pupils to reason algebraically, which teachers of young children may not appreciate the significance of (Goulding et al., 2002). This study examines the awarenesses that preservice teachers display when looking back on their experiences of exploring growing patterns with their peers during university-based seminars. We track how these explicit awarenesses may shape their prospective thinking about their teaching.

We finish this section by detailing our research questions before outlining our theoretical framework. We draw on elements of enactivism (Varela, 1999), together with Mason's (2010) notions of retrospection and prospection, and we explain our use of the term deliberate retrospective analysis (Voutsina et al., 2022). We then review literature on shared activity and visual growing pattern tasks in initial teacher education. We explain our methodology, and our approach to analysing our interviews with preservice teachers, together with lesson plans and evaluations from those preservice teachers who subsequently taught visual growing patterns in school. We present extracts from our qualitative data that indicate the value of offering opportunities for novice teachers to share their ways of seeing as part of their preservice mathematics teacher education.

Our research questions are as follows:

1. What kinds of awareness are reported by preservice teachers when they reflect on the diverse ways in which they see growing patterns as part of shared mathematical enquiries?
2. How does preservice teachers' deliberate retrospective analysis of their own experience of seeing and sharing mathematics shape their prospective accounts of their teaching?

There has been limited research, as yet, which explores the role of seeing what others see in mathematics education for practising teachers and for preservice teachers during their initial teacher education. The significance of our contribution lies in the focus on seeing what others see, when engaging with visual growing patterns, to further understand mathematics pedagogy in ITE.

Theoretical framework

We draw on enactivism as our theoretical perspective that informs our work. Enactivism is a theory of cognition which was first articulated by Varela et al. (1991). It is a wide-ranging discourse on learning which theorises knowledge and meaning making from a biological and evolutionary perspective. Maturana and Varela's (1992, p. 26) enactivist perspective that, "All doing is knowing and all knowing is doing", aligns with our interest in the importance of elementary preservice teachers working on mathematics for themselves. The key concepts and complexities of an extensive theory such as enactivism are not possible to address in the space we have here. Instead, we focus on some key aspects that relate directly to our research focus on shared activity and seeing differently. We suggest Reid and Mgombelo (2015) for a detailed introduction to the broader principles of enactivism, which they illustrate with examples of their use in mathematics education.

Enactivism does not consider knowledge a possession but an action, emphasising sense-making as an embodied activity developed within interactions with others and the environment. Coles and Brown (2016) argue that through interaction with others we change ourselves. "'Who we are' is related to 'who other people are to us' through the recurrent patterns of interaction between us and those around us" (Brown & Coles, 2012, p. 223). Mathematical understanding is thereby a collective phenomenon. Davis (1995) notes that even when an individual is working independently on a mathematical task, action is social as it is framed in language and procedures that have arisen in social activity. This process is often termed co-emergent as the individual learner is viewed as inseparable from their social and cultural context. Brown (2015, p. 188) explains, "we are, literally, what we do, our environment having created us as we have created our world". For Brown (2015), her enactivist research in mathematics education has sustained the notion that learning is seeing more, seeing differently, making new connections and becoming aware of new things. Davis (2004) likewise describes learning as a recursive and complex process linked to actions and perceptions in which explorations of current spaces can trigger seeing differently, leading to opening up of new "spaces of possibility" (p. 184). Learning in enactivism is associated, therefore, with change and new actions and perceptions that arise in a specific context.

One of the core principles of enactivism, according to Davis and Francis (2021), is structural coupling. Maturana (cited in Coles, 2013) contends that the structure of the nervous system and body determines its response to something. Reid and Mgombelo (2015) use the analogy of billiard balls to illustrate how change is triggered but not determined by interactions:

> The motion of a billiard ball struck by another billiard ball is sometimes seen as determined by the force and direction of the ball striking it, but it is actually determined by the structure of the ball being struck. If the ball had the structure of a tennis ball it would move very differently. The ball striking the other ball provides energy, but the structure of the ball being struck determines what happens to that energy. (p. 173)

The structure of human beings is more complex than billiard balls and evolves over our lifetime, altered by every interaction which determines how we perceive the world (Coles, 2013). Structural coupling is the engagement of joint activity; the "intimate entangling of one's attentions and activities with another's" (Davis, 2004, p. 166). Davis (2004) comments that to structurally couple, and there must be sufficient common ground to be able to interact. It is, therefore, part of teachers' pedagogical decision making to provide common experiences necessary for students to engage in productive discussion. Towers and Davis (2002) demonstrate how structural coupling between two mathematics students does not mean identical participation. They illustrate how two students can engage with mathematics differently while working together:

> sometimes to be truly co-operating, at other times to be sitting together while they work but rarely engaging in discussion, and at yet other times seem to be in agreement about the mathematical task but producing entirely different mathematics on paper. (p. 331/2)

Learning occurs as students change their structures, conditioned by particular circumstances, but due to their own complex structure (Davis, 2004).

Alongside our use of enactivism, we also draw on Masons' claims about teaching as a process of directing learners' attention. He states that learning involves shifts in form and focus of attention (Mason, 2010). He recommends that teachers engage with mathematics themselves in order to identify the core awarenesses which lie at the heart of mathematics and to support the way they choose to direct the attention of learners. Unfamiliar mathematical situations can increase teachers' awareness of mathematics and how it is learned. He claims this sensitises teachers' attention (Mason, 2010).

Mason (2002) uses the word "spection", with its link to looking, as a form of analytical reflection. It is possible to reflect in the moment, "spection is being awake in the moment, noticing and responding freshly and creatively in the instant, catching oneself before embarking on habitual behaviour" (p. 86). Spection can take place before or after an action. Mason (1994) elaborates:

> While reflection continues to be an ill-defined and overly used term, I use it to refer to retrospective re-entering of salient moments from the recent past, and attempting to give accounts of these in descriptions which do not embellish, judge or justify. Their purpose is to resonate similar experience in the listener through which they can enter the experience of the describer. To prepare for future actions, I am prospective by mentally imagining myself in a typical situation in which I wish to work differently, and projectively imagining myself responding in the way I wish. (p. 11)

We introduced the term "deliberate retro-spective analysis" in our previous work (Voutsina et al., 2022) to combine the enactivist notion of *deliberate analysis* (Varela, 1999), defined as the way that expert teachers are able to act spontaneously and analyse their actions retrospectively, with Mason's (1994) *spection*, "the retrospective thought on the stairs after an incident when you think of what you could have said or done" (Mason, 1994, p.10). Our application of deliberate retrospective analysis is specific in this study. Our focus is on preservice teachers' actions of doing, seeing and sharing mathematics with their peers at their own level and the emergence of awarenesses in relation to prospective planning of their teaching.

Literature review

The research reported here focusses on shared activity where preservice teachers are asked to reflect on working together on a visual growing pattern task. In this review, we consider a selection of existing empirical research on the use of shared mathematical activities in mathematics education generally and then in initial teacher education. The studies we include here do not necessarily draw on the specific frameworks of awarenesses and spection from Mason (2008) or structural coupling stemming from enactivism (Varela, 1999). In mathematics education, research has considered the impact of learners working together, and some of which is summarised by Bakker et al. (2015) who claim that teaching mathematics should be for dialogue and through dialogue. Hodgen et al. (2018) reported a body of evidence showing that shared learning has a positive effect on mathematical attainment and attitude for all students. Ellis (2011) found that generalisation, in particular, is a collective activity and that publicly sharing generalisations is one of seven generalisation-promoting actions teachers are recommended to use.

Some studies address preservice teachers' shared mathematical activity in ITE. For example, Crespo (2003) reported on a study with elementary preservice teachers on task design. Working together on problem posing provided a shared experience for preservice teachers which helped them to promote and support problem-solving in their own teaching. Wilcox et al. (1991) discussed work with elementary preservice teachers, who worked on mathematics together frequently. The results included a shift in authority in their lessons and increased value for different approaches. Brown (2015) recalled her own teacher education as involving shared reflection on video-taped clips of teaching, and the benefit she gained from hearing multiple perspectives.

The research reported in this study focuses on what preservice teachers see as they engage with growing geometric patterns, and what awarenesses arise from engaging with what others see. The use of visual images in teaching and learning mathematics has been common, often stemming from Bruner's (1966) three modes of representation: the enactive, iconic and symbolic. Many mathematics education texts have discussed the importance of learners engaging with visual representations (for example Liebeck, 1984; Boaler, 2016). The term "seeing" is often used in articulating mathematical learning and this is exemplified by Helliwell and Brown (2020) in terms of their learning as mathematics teacher educators and researchers. Studies such as that of Brayer Ebby (2000) report that preservice teachers, as they work together, can learn to see other people's methods of calculating.

Other research has explored the role of seeing in learning from visual patterns in the mathematics classroom. Radford (2010) identifies the process of the "domestication of the eye", where mathematicians have been culturally inducted to see in particular ways. This is a lengthy process, and Radford exemplifies a teacher's use of gesture and rhythm to direct pupil's attention to what she wants them to see when working on visual patterns. Wilkie and Clarke (2016) categorised the different ways elementary students visualised growing patterns and noted that a few were able to find more than one way to "see" the same geometric structure. Drury (2007) analysed a lesson where learners shared their ways of seeing a visual pattern leading to a generalisation, and the tensions where they felt that their "seeing" did not match that which was intended by the teacher. Chua and Hoyles (2014) reported a study which involved secondary school learners comparing strategies for pattern generalisations. They found that the impact was that teachers included multiple ways of seeing structures of pattern in their teaching, and aligned their choice of generalising strategies with preferences within the class.

In the specific area of preservice teachers' engagement with visual patterns, Hershkowitz et al. (2001) worked with teachers across several countries using the same matchstick pattern that we report using here. They argue for the place of visualisation in mathematical reasoning. Arcavi (2003) goes on to further analyse teachers' responses to the same activity to reflect on the nature of visualisation as both a product and process of mathematics. Vale et al. (2012) report research claiming that preservice teachers can be taught the act of seeing as a problem-solving strategy, which helps them to formulate and identify elegant solutions to problems. Wilkie (2016), who carried out a year-long study with elementary teachers, teaching sequences of geometric pattern generalisation lessons, reported that teachers developed their ability to anticipate a variety of visualisations in future teaching after seeing their students learning to visualise the same pattern in different ways.

Our study contributes to the existing research by focussing on the place of reflecting on how others see visual growing patterns as part of the pedagogy of ITE.

Methodology

Our enactivist approach to the study included recognising that our mathematics teaching and learning was influenced by our beliefs and attitudes, and that this also influenced the way we approached our research and all methodological decisions (Brown, 2015). We provide in this section a detailed outline of our methodological and analytical approach that aims to communicate in a transparent way our processes of analysis (Reid, 1996).

This study was conducted in two of the five universities where the authors of this paper, work on initial teacher education programmes. The 1-year postgraduate ITE programme at one university (U1) included a specialist option focusing on elementary mathematics. A requirement for this programme was that preservice teachers had all continued to study mathematics at an advanced level, and most had completed an undergraduate degree in mathematics, or a closely related subject. All eight preservice teachers on this programme were invited to participate in the study and all accepted. Another university (U2) ran a 1-year elementary generalist postgraduate programme, and seven (out of forty-four invited) preservice teachers on this programme volunteered to be involved in the study. There was no requirement on this programme for preservice teachers to have studied mathematics at an advanced level, although one participant in this study had completed a mathematics undergraduate degree, and one had studied mathematics at advanced level.

As part of the mainstream mathematics input on the postgraduate programmes, the authors of this paper who taught at U1 and U2 ran a session with their groups of preservice teachers on algebraic reasoning, which included work on the two visual growing patterns below (Fig. 1a, b). The preservice teachers were invited to extend the patterns and to note any generalisations they could make, about the number of slabs (flowerbed pattern, Fig. 1a) or matchsticks (matchstick squares pattern, Fig. 1b), rather than focusing explicitly on finding an expression for the nth term. While some preservice teachers selected to work individually initially, they were invited to collaborate in pairs or small groups to extend their ideas. Following some time to work on each of the problems, they were encouraged to share the various ways in which they could "see" the pattern growing with the wider group, and how this might influence the pedagogical approaches they would use in their teaching. The way that preservice teachers shared their thinking varied. Most showed the diagrams and sketches that they had drawn to support their thinking, others used the available manipulatives to highlight the different ways that they had seen the pattern, and some used language such as "rows and columns" (see Elsa below) or "Ls" (see Terry below). Following this taught session, using a semi-structured approach, the participants in the study were interviewed by another member of the research team about how they had approached the problems, how they had worked with others in their group and how they might draw on these activities in their own teaching. The interviews also offered an opportunity to discuss their previous mathematics experience, including how long they had studied the subject. Data gathered in this phase consisted of copies of the jottings they had made during the taught session and the transcript of the interview during which these jottings were discussed.

The preservice teachers in U1 on the specialist mathematics programme had significantly more programme time for mathematics. There were regular opportunities for these preservice teachers to informally discuss lessons they had taught in school and this work was not formally assessed, and no specific pedagogical approach was prescribed. This approach afforded the opportunity to collect additional data in a follow-up activity with this cohort, which, given time constraints on the programme, was not possible at U2. Additional data collected from participants at U1 involved a class discussion about lessons on growing patterns that they had planned and taught in schools. An audio recording of this discussion was transcribed, and copies of the participants' lesson plans and evaluations constituted an additional data resource. Ethical approval was obtained from both universities involved in the study.

Data analysis

Using an enactivist lens, we value the different perspectives that we, as researchers, bring to the study, and that through collaborative analysis, it can be possible to "see more" (Lozano, 2015, p. 231). The process of our collaborative data analysis

Table 1 Data analysis process

Phase 1	Phase 2
Review of interview transcripts and associated jottings: • Working on mathematics for themselves • Collaborating in their enquiries (Mason, 2010)	Cycle 1 – Analysis to identify: • Retrospection – generalisation activity • Prospection – future teaching approaches Cycle 2 Identification of threads between retrospection of learning and prospection of teaching through data Cycle 3 Identification of themes across threads

is described here and summarised in Table 1 below. The first phase of data analysis involved the project team working in pairs to review the transcripts and associated jottings, to identify the approaches that each participant used while working on the growing pattern problems. The focus during this phase was on the shifts in attention to recursive and functional relationships (Ferrara & Sinclair, 2016) and was reported at two conferences in the UK (Alderton et al., 2017; Rowland et al., 2018) as a way of involving the wider mathematics education community in our work.

Phase one had prompted our interest in Mason's ideas of "spection" (2010), which, he suggests, can be developed through teachers "working on mathematics for themselves" and "collaborating in their enquiries" (pp. 42–43) and phase two of our analysis, which is the subject of this paper, followed up on this, using the notions of retrospection and prospection to interrogate our data further, which we did in three cycles.

In the first cycle of this second phase of analysis, we began by working individually to review transcripts and jottings for all participants, and lesson plans, evaluations and comments in the group discussion (U1), to identify all examples of deliberate retrospective analysis of their work on the growing patterns, and their prospective comments about their future teaching approaches (Mason, 2010). We then refined and agreed these examples, working in pairs, before sharing and agreeing our analysis with the whole team. Figure 2 below exemplifies this process for Andrew (pseudonyms are used throughout), one of our participants from U1, where he makes retrospective comments about his approach to the growing patterns activity, and a prospective comment showing his response to a question about the relevance of these kinds of activities for teaching.

In the second cycle of this phase of analysis, we worked in pairs to identify particular *threads* through the entire data set that we had for each participant, where a specific approach had been identified retrospectively as being helpful or challenging for them when they had "collaborated in their enquiries" during the taught session, and which had then been drawn upon, prospectively, during the interview, their lesson plan, or the follow-up group discussion (U1 only), about their ideas for their

Retro-spection	*I found it quite difficult to actually visualise it another way and to separate it out, and I've, I think it was about the fourth person to speak on different ones, and I'd seen the others, and I've jotted some down, trying to rationalise it, and I was fiddling around with multi-blocks, trying to think about the colourings and the difference. And I actually quite struggled with it because I think, when I saw it in one way, it was a lot harder to see than in others.* **Andrew-U1, Interview**
Pro-spection	*I think it's really important for children to understand that not everyone solves and understands maths the same way.* **Andrew-U1, Interview**

Fig. 2 Framework of analysis informed by Mason (2010)

Jacob (U1): Seeing how other people are 'seeing' the pattern – this is sometimes difficult
John (U1): The usefulness and value of sharing ideas and discussing them with others
Fiona (U2): The value and challenges of working with others
Annie (U1): The value of working with others

Fig. 3 Examples of threads between retrospection and prospection

future teaching. These threads were agreed across the study team and Fig. 3 shows examples of the some of the threads we identified for a sample of four of our participants relating to "collaborating in their enquiries".

In the third and final cycle of this phase of analysis, we made use of inductive thematic analysis (Boyatzis, 1998). It was important to us that this analysis was only undertaken when the threads across retrospective and prospective deliberation were fully identified and agreed for each of our interviews. Threads across all transcripts were searched systematically to identify similarities and differences. This enabled us to work together across the project team, to identify themes across the threads, which came from the interview data itself. For example, in Jacob's thread above, he indicated that he could not always *see* the pattern in the ways that others in the group were describing, and in Annie's thread above, she described valuing the explanations of others about how they saw the patterns growing, which was reinforced later, through her lesson plan, which focussed on allowing time for the pupils to share their different ways of viewing the patterns. These examples, and similar threads for other preservice teachers, were categorised under the theme: awareness of how others *see* the pattern which is the focus of our data presented below.

The trustworthiness of the study has been ascertained through the following aspects of our research process that we have outlined in this section: The interviews were carried in two universities by tutors who were experienced researchers and ITE tutors themselves and who represented other universities. This allowed us to apply aspects of researcher and data collection triangulation to support the credibility of our process (Lincoln & Guba, 1985).

We adopted a lens of analysis based on the ideas of reflective retrospection and prospection by Mason (2010), and we applied this consistently, to filter through the interview transcripts statements and ideas that we tagged as instances of 'retrospection of learning' and instances that we tagged as "prospection of teaching". To identify instances of retrospection, we triangulated data from interview transcripts as well as students' jottings during the session, while instances of ideas related to prospection were triangulated between the interview data, lesson plans and lesson evaluations. This process allowed us to support the confirmability of our interpretations (Nowell et al., 2017).

The process of analysing sets of transcripts individually first, then in pairs and then across the research group, and addressing any inconsistencies through dialogue in research meetings, enabled us to increase the credibility and dependability of our conclusions (Stahl & King, 2020). The findings were consistent across the two universities that we focused on and resonate with our experiences across our own universities. In this paper, we have outlined the different phases of our analysis, to ensure a transparent communication of our processes and we are reporting examples of our findings in detail to allow others to judge their transferability (Nowell, et al., 2017).

Findings

This section presents our findings in relation to an overarching theme that we named "Awareness of how others see the pattern".

The identified threads of preservice teachers' retrospection of their shared learning experience and prospection of teaching reflected a duality of expressed ideas: When working together, it can be fascinating when others help you see in a way that you would not have seen the pattern yourself, but it can also be challenging when you cannot see what others see and therefore cannot share in their way of thinking about the pattern. We present examples of threads that illustrate this expressed duality in preservice teachers' articulated awarenesses under two sub-themes: "Sharing with others different ways of seeing a pattern: a constructive experience" and "Sharing with others different ways of seeing a pattern: a constructive but challenging experience".

Sharing with others different ways of seeing a pattern: a constructive experience

The extracts below exemplify threads where working with others and seeing "how others see" a pattern was viewed as a constructive, positive experience in preservice teachers' retrospection of their own learning experience during the university session. This theme was subsequently reflected in their comments about how they would prospectively plan for their teaching.

Sophie (U2) commented positively on peer discussion and comparing different approaches as key elements that helped her work out a way of generalising the matchstick pattern.

> But I don't think I, unless my partner had worked out that you could do it by rows … I don't think I would have been able to work it out either… It's only through discussion on our table who, like comparing ideas, that we worked out how to do it.
> (Sophie, U2, Interview, Matchsticks)

Her acknowledgement of the value of sharing ideas with others is subsequently reflected in her position about the emphasis and space that she would give in her own lessons for pupils to discover their own, different ways and share with others, supporting each other's learning.

> I think when you've got a question of lots of ways of doing it, I think, and just making it quite open, not telling them directly how to do a question, but allowing them to test ways out and talk to their friends and … just make, giving them the chance to use their own strategies I think. (Sophie, U2, Interview)

Similarly, John, Annie, Hayley, Terry and Steve from U1 and Elsa from U2 expressed consistently positive views about their experience of sharing different approaches which allowed them to "see" the pattern in different ways. This was then reflected in their comments about how they would organise their teaching of similar activities and in their lesson plan. During the interview, John noted that it might be difficult to move away from one's own view of a pattern if there is no opportunity to share other perspectives. He subsequently planned a lesson on visual patterns (Fig. 4) that encouraged pupils to share and "support each other" and "break down different ways of seeing it".

> I think because when you see it in one way you perhaps can only see it in that way, so when you get a different perspective, then it's, it might make you think about the problem in a way that you wouldn't have, even if you just stared at it yourself, trying to look for as many… ways as you can, because you have a certain perspective … that you can't see past.
> I think they [children] enjoy maths in general a lot more when it's something they can discuss and they're solving and they're trying to … and there's something almost like puzzle-y about it, I guess…
>
> (John, U1, Interview)

Elsa (U2) Terry (U1) and Steve (U1) also expressed the view that sharing ways of seeing the pattern with others helped them recognise certain elements of the pattern that they would not have seen otherwise.

> … because I was visualising as rows, then I would probably not have seen the difference, that actually the columns was the same, if we hadn't discussed it, I could have continued on with rows and tried to find a pattern just in the rows, which there wasn't one, just if you looked at just rows.
>
> (Elsa, U2, Interview, Matchsticks)
> Yeah, I think that's because of the, you're initially drawn to that one and that's the one that you identify with the most. And then other people started talking about seeing Ls and … and then I could see, I could see where they were coming from.
>
> (Terry, U1, Interview, Matchsticks)

Children work on their tables in groups with multilink cubes. They work within mixed attainment talk partners to support each other.

Conclusion

Gather class back on the carpet to discuss what they've done and what they noticed about the pattern. Break down different ways of seeing it.

Fig. 4 Extract from John's lesson plan

> … the reason that I now really enjoy mathematics is the idea of that investigating, of spotting these patterns and actually being able to look at things and be aware that different people have seen it in different ways. And that's, that's an idea that's really been a strength, and, this year, and is an idea that I don't think I would have, I think following this year I've completely re-thought what I thought was maths.
>
> (Steve, U1, Interview)

The extracts from Elsa's and Terry's interviews illustrate two examples of individual, differing ways of seeing the matchsticks pattern. They also communicate the value that these preservice teachers perceive in having the opportunity to be exposed to and discuss alternative ways of seeing the pattern. The third extract above further extends and broadens this reflection. Steve, who had studied mathematics at advanced level at school (i.e. a 2-year subject-based qualification used for entrance to UK Higher Education institutions) and described himself as "reasonably successful" with mathematics, expressed a new, for him, realisation about the value of sharing his work on patterns with others. He noted that "being aware that different people have seen it in different ways" is the essence of mathematics; a realisation that enabled him to "re-think" what mathematics is about. It is interesting to note here a student who had studied "advanced" mathematics coming to a better understanding of his own relationship to the subject as part of mathematical activity in his pre-service teacher education.

In a retrospective account of her work with patterns, Annie (U1) explained four different ways of viewing the flowerbed pattern, which she came to recognise while talking with others and sharing their ways of seeing.

> …. as soon as I saw it, I felt I had it and I visualised, well I drew a little picture here, which was that I took the four, you know the constants of the four corners, they're always there, so you sort of take them off and then you're left with like a cross which is n by n, you know. So you've got 4n and then you've

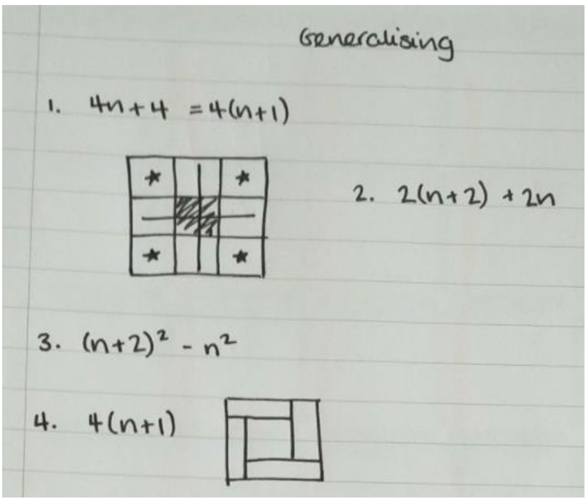


Fig. 5 Annie's jottings when working with the flowerbed pattern

got your four corners, so 4n plus 4. And that was very … straight away (referring to solution 1 in Fig. 5).

(Annie, U1, Interview, Flowerbeds)

… so it was me and Hayley and then Andrew and Jacob here. And Hayley had seen it as … like a strip across the top that was N plus 2 and a strip across the bottom that was N plus 2, and then two more Ns, and I thought that was nice as well (referring to solution 2 in Fig. 5). And after we'd talked about our two approaches to it, we thought, oh I wonder if anyone's thought of another way, and we came up with the, this is our third way. So that was my way at the top, Hayley's way and then, yeah, that it would be one big square with the little square taken out, so find the area of the big square, find the area of the little square and take it out (referring to solution 3 in Fig. 5). And then the one that we didn't think of, which [Tutor] showed us, was this (referring to solution 4 in Fig. 5).
Which is really nice, because that really looks like 4 times n plus 1, which is what it all reduces down to once you simplify it. So I think that's, I never would have seen that though, I always would have been thinking of four corners, whichever way.

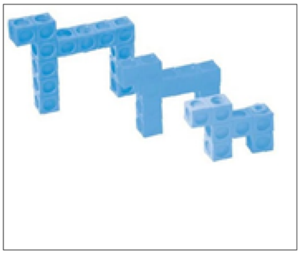
(Annie, U1, Interview, Flowerbeds)

In her subsequent planning of a session with a small group of year 3 (7–8 years old) pupils working on a "dog family" pattern made of multi-link cubes (Fig. 6), Annie adopted an approach to teaching that promotes and encourages plurality of approaches and aspects through which children can see the pattern, drawing attention to the constant elements in a variety of ways.

In her lesson evaluation (Fig. 7), Annie wrote:

In this way, she expressed her satisfaction with a lesson in which pupils were able to visualise the patterns in several ways that were different from the way she had modelled them herself.

Fig. 6 Extracts from Annie's lesson plan



Ask children what stays the same as they make each dog, and what is changing. How many cubes does the head have? Are the legs always the same size? What about the body?

Try to guide children to see the front and back part of the body (shoulders and bottom) as separate from the body, so that they can see that these remain constant. Ask if they are part of the body or not.

Fig. 7 Extracts from Annie's lesson evaluation

> The children were all able to describe to me how they visualised their patterns, which parts were changing and which were staying the same. This was great as I learnt that the children had three different ways of visualising the pattern, none of which were the same as the one I then showed them. Some children found it easier to describe their pattern than others. Marie was particularly good at spotting and describing patterns in the numbers.

The examples of threads that we have presented here illustrate preservice teachers' expressed awareness of how, working together with their peers, allowed them to see more in the pattern, see it differently and make connections that they had not made before and which they would probably not have made at all if they had worked individually. This experience was then reflected in their prospection of their teaching, in the form of their explicit awareness of the varied ways in which children in their class may see mathematics but also their awareness that they, as teachers, may see mathematics differently from their pupils.

We consider this explicit awareness that our preservice teachers expressed in the context of deliberate retrospection of their learning experience as being directly linked with Brown's (2015) enactivist view of learning as a recursive process of "seeing more" and "seeing differently, becoming aware of new things" (p. 192). Brown (2015) further elaborates: "agreeing the detail of what we do see is a first step. If we stay with our first viewing it is clear that there is a lot that others might see that we do not and it can be surprising, at the detailed level, what we have simply not seen" (p.193). The examples that we have presented here are illustrative of how, when working together in their enquiries, preservice teachers can step back from the detail of what they see themselves and "direct each other's attention to salient features so that finer distinctions can be made" (Mason, 2010, p.43). We propose that the positive reports of shared activity presented here suggest cases of structural coupling where the existence of sufficient common ground has supported the interaction (Davis, 2004). In this case, we consider that the notion of "sufficient" common ground is evidenced when individuals appear to recognise alternative, differing ways of seeing the pattern and move towards a shared way of perceiving the mathematics activity as a whole, which supports their communication and interaction.

Sharing with others different ways of seeing a pattern: a constructive but challenging experience

Some preservice teachers acknowledged the value of sharing different ideas and ways of seeing a pattern, but they also pointed out potential challenges when working with others who see elements that they cannot recognise themselves.

Jacob (U1) found it fascinating to discover, with others, different ways in which the flowerbed pattern could be seen. But in the case of the matchstick pattern, he experienced some difficulty in seeing it in the way that others saw it.

> Oh, well I just saw things that other people saw that I had no idea, and you know … what did other people see … the four corners (referring to Flowerbeds), a lot of people saw four corners and then bits joining those four corners, I didn't see that at all
> This one (referring to Matchsticks), there were some that other people saw on the matchstick one that I don't think I can see still….
>
> (Jacob, U1, Interview)

Nevertheless, Jacob concluded that sharing different ways in which patterns can be seen and conceptualised is of value when learning mathematics.

> I suppose, yeah, getting ideas off each other and … yeah, then you see things that you wouldn't have seen otherwise
>
> (Jacob, U1, Interview)

In his subsequent lesson plan (Fig. 8), Jacob made reference to building in an opportunity for children to discuss conjectures in the lesson. This suggests planning that promotes sharing of ideas and different visual perspectives in the class, and is in line with Jacob's comments on valuing the notion of "getting ideas off each other" that he had shared at the interview.

Alice (U2) on the other hand, explained that the solution that one group shared during the session did not initially make any sense. In the extract below, she suggested that, when teaching, she will ensure that she considers the fact that working in pairs does not necessarily contribute to extending and supporting one's thinking, even though she acknowledged the value that it may have.

> … a solution was offered, which was a very accurate one, but for me it didn't make sense until someone else had explained it in a different fashion. And I think with children, that's certainly taught me that even working in sort of pairs may not necessarily be enough to extend or push you.
> (Alice, U2, Interview)

In his retrospective account of his work with the two patterns, Andrew (U1) also spoke about the difficulties that he had experienced when sharing his work with others, or hearing about their approaches and struggling to see what they see.

> I thought about that for a while, and I heard other people mention the word 'algebra', so I tried to think of it algebraically, but I was really struggling to put it into … how could you express that, I was thinking in terms of n. … this first

Conclusion (10 mins)

Draw the children back together and highlight some of the things that they have said. If anyone has made a conjecture, ask them to explain it to the class. If anyone noticed anything interesting, highlight it. Allow space for 5 minutes of class discussion.

Fig. 8 Extract from Jacob's lesson plan

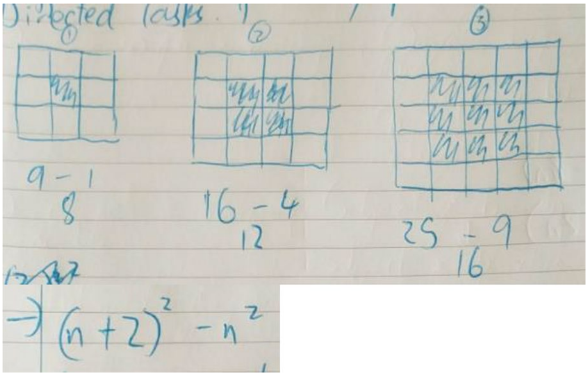


Fig. 9 Andrew's jottings from working on the flowerbed pattern

> term, it is the square (referring to Fig. 9), it's the size of the third term minus the square of 1 by 1. So I thought of it as, OK, so it's the third term, so it's the first term plus 2 to get n plus 2, and to, it was the square of that sides was n plus 2 all squared, take away a square of 1 by 1.
>
> (Andrew, U1, Interview, Flowerbeds)

Andrew perceived the flowerbed itself to be a square-within-a-square, the outer square being 2 units greater, in both dimensions.

Andrew continued:

> I found it quite difficult to actually visualise it another way and to separate it out, and I've, I think it was about the fourth person to speak on different ones, and I'd seen the others … I actually quite struggled with it because I think, when I saw it in one way, it was a lot harder to see than in others.
>
> (Andrew, U1, Interview, Flowerbeds)

Andrew recognised the value of sharing different views and approaches to the pattern with others but also highlighted that, in the classroom, it is important for children to be aware that everyone approaches and understands mathematics in their own way.

> I think it's really important for children to understand that not everyone solves and understands maths the same way.
>
> (Andrew, U1, Interview)

Similarly, Fiona (U2) reflected on her experience of working with others and commented on the challenging aspects of working with others, when there is no shared understanding of the direction that the group is taking towards a solution.

> … I felt like the two people I was working with were, they were on a track, obviously we didn't get to the answer, but they were on a track and I was trying to keep up and I couldn't quite like click in to what they were trying to say, and that was when we were trying to do all this sort of …

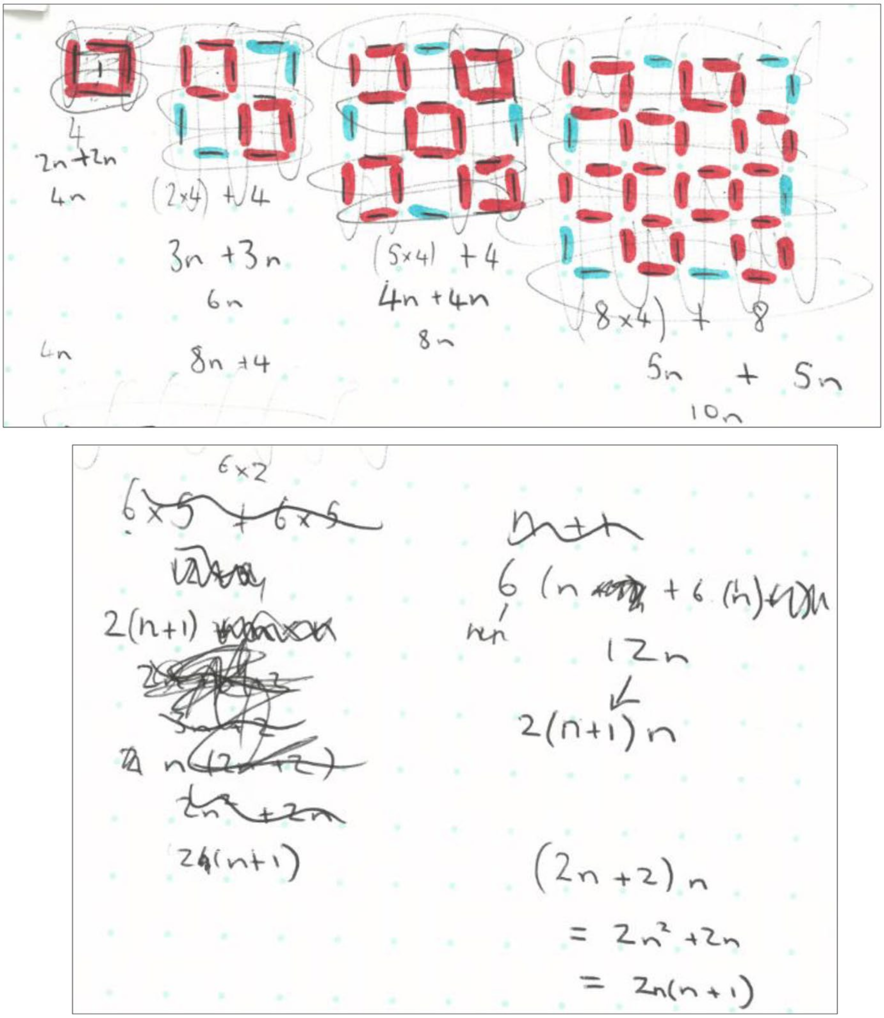


Fig. 10 Emily's jottings and working on the matchstick pattern

(Fiona, U2, Interview)

Fiona's position related to the need, sometimes, to "block out" what others say, in order to experience the feeling of satisfaction that comes with achieving a solution to a generalisation problem yourself, individually, rather than by working with others, were also echoed in Emily's (U1) thoughts.

Starting from one corner of each case (Fig. 10), Emily identified a number of disjoint squares (so that the matchsticks in these squares are only counted once when she multiplies by 4). These include the square diagonally opposite the one that she started from, and the other corners in alternate figures (only). This then leaves some matchsticks on the boundary uncounted, and they have to be added to the matches in those disjoint squares.

> Emily referred to the importance of doing maths for yourself first, before sharing with others.
> … I liked how there was all the different things available, everyone was having a go at it themself and we were kind of encouraged to do, to think about it ourself but also talk about it, but not necessarily talk about it straight away and then … collaborate with other people, so that approach, and having all the different ways of doing it
>
> (Emily, U1, Interview)

The importance of working on a problem for yourself first, before sharing with others, also appeared in her expressed views about how children may act in a classroom learning situation.

> … if their partner's done it a completely different answer and they've found the answer, I think a kid would think, oh I have to stop and I have to do it their way because they've got the answer. So I think that kind of thing, encouraging the kids to follow their own thought process would be good.
>
> (Emily, U1, Interview)

When planning a lesson on generalisation for a year 5 class (9–10-year-old pupils) using a linear matchstick pattern, Emily planned for children to work together in pairs or groups, despite her expressed concerns about children working in groups as part of her interview. Her plan allowed time for children to share different methods and included prompts for children to think about whether the identified rules are the same and whether they need to be the same.

The examples of threads presented here illustrate some preservice teachers' awareness of the challenges that may emerge when, in shared mathematics activity, individuals do not necessarily share the same ways of seeing mathematics or struggle to see what others see in mathematics. Reports of difficulty in sharing ways of seeing may suggest occasions where, in collaborative activity, structural coupling between individuals did not necessarily entail identical participation (Towers & Davis, 2002). In cases where this is not resolved, it may suggest a lack of common ground that would have been necessary (Davis, 2004) for sense-making within structural coupling to occur.

Brown (2015) notes that what people see is an example of past learning, "Hence two people cannot see the same thing nor share the same awareness. However, we can communicate because we can talk about the details of common experiences and, in doing so, the gap between interpretations can be reduced" (p. 189). Preservice teachers who expressed difficulties related to not seeing what others see, still noted and included, in their prospection of teaching, opportunities for pupils to communicate and discuss their different ways of seeing in the class. This suggests an expressed awareness of the power and value of communication, as noted by Brown (2015), to reduce the gap of interpretation and to support shared learning.

Discussion

In this study, we took an enactivist approach, recognising the ways in which we interact with our environments and with those around us, to work with elementary preservice teachers on visual growing patterns. Through subsequent interviews, we prompted their deliberate retrospective analysis (Voutsina et al., 2022) to identify the awarenesses that emerged through their engagement in these collaborative activities. We traced these threads of awarenesses to their prospective ideas about their own teaching, and we explored evidence of these awarenesses in their own teaching.

Drawing on the enactivist literature, we see mathematical understanding as a collective phenomenon (Brown & Coles, 2012), where individuals are inseparable from their context. Using this to frame our work, we planned opportunities for preservice teachers to work on the growing pattern problems collaboratively, and to share their multiple ways of "seeing" the patterns. Preservice teachers have many experiences and recurrent patterns of interaction prior to and during their initial teacher education programmes, both in elementary classrooms and university sessions, which contribute to how they develop awarenesses and make connections. We do not claim any causal relationship between the opportunities we offered and their prospective thinking about teaching mathematics. However, we do point to some general principles.

The approach that each of us takes in our primary mathematics work in initial teacher education is to support our preservice teachers to work together in their mathematical enquiries (Mason, 2010), while also sensitising them to the struggles that the pupils they go on to teach may experience. For example, in the algebraic reasoning session described in this paper, the starting point had been generalising about simple repeating patterns and had progressed to linear, and then to quadratic sequences. Preservice teachers collaborated on these activities, sharing their thinking. Through the act of deliberate retrospective analysis, our data shows evidence of preservice teachers drawing on pedagogic approaches discussed and used in University taught sessions within their own teaching. Annie, for example, retrospectively drew attention to the "constant" four corners in the flowerbed problem, and she later drew on the key pedagogic tool of looking for sameness and difference in her lesson on the dog family, when she planned to ask the children "what stays the same as they make each dog, and what is changing?", so that she would draw attention to the fact that "these remain constant".

Our preservice teachers often join our programmes with a particular view of mathematics, based on their own experiences of learning mathematics in school, which tended to be an individual endeavour, centred around rules and procedures, and which typically promoted a view of mathematics as a set of truths (Ernest, 1991). Research shows that teachers typically teach in the way that they were taught and it is actually very difficult to shift this focus (Powell, 1992). However, our research indicates that through deliberate retrospective analysis, preservice teachers can recognise the benefit of sharing different ways of "seeing" patterns growing, and when prompted to prospectively consider their future teaching, they suggest that they would adopt this pedagogic approach. For example, despite him experiencing some difficulty when alternative ways of seeing the flowerbed problem were shared in the teaching group, Andrew reflected that it was important for "children to understand that not everyone solves and understands maths the same way" (Andrew, U1, Interview).

Most examples in the threads from our data indicated a positive view of the shared mathematical enquiries. For example, our data show that Sophie, John, Annie, Hayley, Terry, Steve and Elsa all found the experience of sharing the different ways of "seeing" the pattern grow helpful in coming to their own understanding because they were enabled to see the pattern in a different way. They also all suggested that this would be an approach they would use in their own teaching, and in some cases, we have evidence that they incorporated it into a planned lesson on growing patterns. However, Jacob, Alice, Andrew and Fiona all expressed some tensions about the extent to which they had found the experience of sharing various approaches helpful. For some, this was about wanting to make sense of it themselves, and they found alternative approaches difficult to follow. For others, it was a concern that they felt they were not making progress with the problem, and they could not follow the explanations of others in the groups. However, despite each of these expressed difficulties with the sharing experiences, they each indicated that they felt it was important to include opportunities for pupils to share different approaches, or different ways of "seeing" the problems in their own teaching. One preservice teacher, Emily, had approached the matchstick problem by working individually until she had found a way of explaining how the pattern was growing. When highlighting her prospective ideas about working with pupils on these types of problems, she indicated that she would encourage them to work individually initially, so that they are not swayed by the approaches of others. However, later in the programme, Emily shared her lesson plan, which indicated that she would now encourage pupils to work in pairs/groups; there was no suggestion that she would encourage an individual approach as the lesson began.

There is some research that has explored the difficulties experienced when pupils do not see what others do (for example Drury, 2007) but there is little research which has explored the impact of preservice or in-service teachers struggling to "see" what their pupils see. Our research indicates that even when preservice teachers struggle to make sense of what others are seeing, they recognise that some approaches may be more efficient or insightful than others, and therefore, listening to others' ideas is a powerful learning opportunity for teachers and children. Our preservice teachers valued this approach and subsequently used it in their teaching. We believe experiences during teacher preparation can support dispositions to using a pedagogy which values multiple approaches.

We propose that through the encouragement of deliberate retrospective analysis, preservice teachers can be sensitised to their experiences during the taught sessions, where awareness of different pedagogic approaches were heightened. Preservice teachers were immersed in the experience of sharing different ways of seeing, offering opportunities to sensitise them to the value of this approach, and a model for how they might work with their own pupils. Our enactivist approach helped us to recognise the importance of the environment, as well as the affective element of doing mathematics together.

The approach we have taken in this study drew on the enactivist principle of valuing and recognising the benefit of multiple perspectives to come to a shared understanding. The data we present here highlights the ways that we, the researchers in this study, were structurally coupled (Davis, 2004) with our participants in the study, the preservice teachers. We show that they had different experiences of seeing the patterns within the session, and as ITE providers we need to be aware of these, and to highlight these differences in our teaching, so that we raise their awareness of the range of experiences pupils in their own classes will have.

One limitation of the study was that our research design did not involve observation of this group of preservice teachers teaching with growing pattern activities. Therefore, we have not captured the sensitisation to different ways of seeing that may have occurred in the classroom, for example, if a pupil saw in the mathematics something that the preservice teachers were not expecting and how this was addressed in the moment. Also, we did not have the opportunity to trigger and collect data of preservice teachers' deliberate retrospective analysis of their teaching focusing on growing patterns.

We suggest that it would be fruitful for future research to follow the same group of preservice teachers from their reflective retrospection of their own learning and doing of mathematics, to prospection of teaching, then to in-classroom practices and realisations, and through to retrospection of teaching. This might capture, in a more complete and holistic way, preservice teachers' sensitisation to issues related to the diversity of 'seeing' in mathematics and how this diversity and richness can be used for effective pedagogy, its affordances and any challenges. This could also help shed light to the reasons why ideas expressed as part of one's retrospection of own learning and doing in mathematics, may not necessarily be reflected in their classroom practice during their preservice education placements (for example, the case of Emily).

Conclusion

Our study advances the call for collaboration in mathematics by considering mathematics as a shared experience within ITE. Inspired by Mason's (2010) recommendation that teachers work collaboratively on mathematics, we have promoted collective mathematical activity in our teacher education programmes. Our preservice teacher participants reported that they develop awarenesses about teaching and learning mathematics from sharing what they see, and engaging with how others see a visual growing pattern.

"Seeing" mathematics is problematic (Radford, 2010) and teachers cannot assume that learners see as they do. Not all the preservice teachers we worked with found it easy to learn from sharing what they see. However, engaging in their deliberate retrospective reflections allowed us to trace and to interpret the way in which they appear to make sense of their collective mathematical experiences in university-based sessions, and to draw on them in their prospective anticipation of their teaching. Our preservice teacher participants began to develop a disposition to see mathematics as a learner does.

Our findings have significant implications for the pedagogy of mathematics ITE. We argue for shared mathematical activity to be included in the experiences of preservice teachers. Although university-based time is limited, there is a value in providing elementary preservice teachers with opportunities to engage with mathematics as a shared experience, and to be supported in reflecting on the process together. We recognise the challenge that mathematics tutors face in ensuring that preservice teachers of various levels of experience in mathematics can engage confidently in mathematics together. Furthermore, we argue for allocating time and support for deliberate retrospective and prospective reflection (Voutsina et al., 2022), although we acknowledge the time constraints on ITE programmes. Vale et al. (2019) argue that teachers need to be able to anticipate the range of solutions that their students may produce in response to generalising tasks as well as the reasoning that underlies these solutions. We extend this position and argue that enabling preservice teachers to see mathematics as others see it and to become aware of the challenges associated with understanding the way in which others may see mathematics is essential for the development of educators who, in their own classrooms, will be able to anticipate their students' solutions and underlying reasoning and thus be proactive in their planning for appropriate pedagogical actions.

Considering mathematics ITE from an enactivist perspective has enabled us to identify some important and valuable awarenesses which preservice teachers report that they develop, and the pedagogy which supports this process. We have identified the development of these awarenesses as a form of structural coupling between preservice teachers as learners themselves. Further research might proceed to consider the impact of shared mathematics activity between teacher educators and preservice teachers, and between teacher educators themselves, in order to consider mathematics pedagogy in ITE as an enactment of structural coupling from several perspectives.

Acknowledgements We wish to thank the preservice teachers who participated in this research.

Author contribution All authors made an equally shared contribution to the wider research project and the writing of this paper.

Declarations

Ethics approval The project and data collection obtained ethical approval from the University of Birmingham and University of Cambridge. The research followed the British Educational Research Association Ethical Guidelines (BERA, 2018). British Ethical Research Association [BERA], 2018. *Ethical Guidelines for Educational Research*. 4th ed. [ebook] London: BERA, p.29. Available at: https://www.bera.ac.uk/wp-content/uploads/2018/06/BERA-Ethical-Guidelines-for-Educational-Research_4thEdn_2018.pdf?noredirect=1

Consent to participate All research participants were informed about the purposes of the study and provided their informed consent before their participation in the project.

References

Alderton, J., Donaldson, G., Ineson, G., Rowland, T., Voutsina, C. & Wilson, K. (2017). Pre-service primary teachers' approaches to mathematical generalisation. In F. Curtis (Ed.) *Proceedings of the British Society for Research into Learning Mathematics*, 37(3), 1-6.

Arcavi, A. (2003). The role of visual representations in the learning of mathematics. *Educational Studies in Mathematics, 52*, 215–241. https://doi.org/10.1023/A:1024312321077

Bakker, A., Smit, J., & Wegerif, R. (2015). Scaffolding and dialogic teaching in mathematics education: Introduction and review. *ZDM, 47*, 1047–1065. https://doi.org/10.1007/s11858-015-0738-8

Boaler, J., Chen, L., Williams, C., & Cordero, M. (2016). Seeing as understanding: The importance of visual mathematics for our brain and learning. *Journal of Applied and Computational Mathematics, 5*(5), 1–6. https://doi.org/10.4172/2168-9679.1000325

Boyatzis, R. E. (1998). *Transforming qualitative information: Thematic analysis and code development*. Sage.

Brayer Ebby, C. (2000). Learning to teach mathematics differently: The interaction between coursework and fieldwork for preservice elementary teachers. *Journal for Mathematics Teacher Education, 3*, 69–97. https://doi.org/10.1023/A:1009969527157

Brown, L., & Coles, A. (2012). Developing 'deliberate analysis' for learning mathematics and for mathematics teacher education: How the enactive approach to cognition frames refection. *Educational Studies in Mathematics, 80*(1–2), 217–231. https://doi.org/10.1007/s10649-012-9389-7

Brown, L. (2015). Researching as an enactivist mathematics education researcher. *ZDM, 47*, 185–196. https://doi.org/10.1007/s11858-015-0686-3

Bruner, J. (1966). *Toward a theory of instruction*. Belkapp Press.

Chua, B. L., & Hoyles, C. (2014). Generalisation of linear figural patterns in secondary school mathematics. *The Mathematics Educator, 15*(2), 1–30.

Coles, A. (2013). Being alongside: For the teaching and learning of mathematics. *Sense Publishers*. https://doi.org/10.1007/978-94-6209-212-9

Coles, A., & Brown, L. (2016). Task design for ways of working: Making distinctions in teaching and learning mathematics. *Journal of Mathematics Teacher Education, 19*(2–3), 149–168. https://doi.org/10.1007/s10857-015-9337-4

Crespo, S. (2003). Learning to pose mathematical problems: Exploring changes in preservice teachers' practices. *Educational Studies in Mathematics, 52*, 243–270. https://doi.org/10.1023/A:1024364304664

Davis, B. (1995). Why teach mathematics? Mathematics education and enactivist theory. *For the Learning of Mathematics, 15*(2), 2–9.

Davis, B. (2004*). Inventions of teaching a genealogy*. L. Erlbaum Associates.

Davis, B., & Francis, K. (2021). "Enactivism" in *Discourses on Learning in Education*. https://learningdiscourses.com

Drury, H. (2007). Do you see what I see? Issues arising from a lesson on expressing generality. In D. Pitta-Pantazi & G. Philippou (Eds.), *Proceedings of the Fifth Congress of the European Society for Research in Mathematics Education* (pp. 862–871). Larnaca, Cyprus: University of Cyprus.

Ellis, A. B. (2011). Generalizing-promoting actions: How classroom collaborations can support students' mathematical generalizations. *Journal for Research in Mathematics Education, 42*(4), 308–345. https://doi.org/10.5951/jresematheduc.42.4.0308

Ernest, P. (1991). *The philosophy of mathematics education*.

Ferrara, F., & Sinclair, N. (2016). An early algebra approach to pattern generalisation: Actualising the virtual through words, gestures and toilet paper. *Educational Studies in Mathematics, 92*(1), 1–19. https://doi.org/10.1007/s10649-015-9674-3

Goulding, M., Rowland, T., & Barber, P. (2002). Does it matter? Primary teacher trainees' subject knowledge in mathematics. *British Educational Research Journal, 28*(5), 689–704. https://doi.org/10.1080/0141192022000015543a

Helliwell, T., & Brown, J. (2020). Seeing more, seeing differently as mathematics teacher educator-researchers. *For the Learning of Mathematics, 40*, 9–14.

Hershkowitz, R., Arcavi, A., & Bruckheimer, M. (2001). Reflections on the status and nature of visual reasoning - the case of the matches. *International Journal of Mathematical Education in Science and Technology, 32*(2), 255–265. https://doi.org/10.1080/00207390010010917

Hodgen, J., Foster, C., Marks, R., & Brown, M. (2018). *Evidence for review of mathematics teaching: Improving mathematics in key stages two and three: Evidence review*. London: Education Endowment Foundation. Retrieved July 20, 2021 from: https://educationendowmentfoundation.org.uk/evidence-summaries/evidence-reviews/improving-mathematics-in-key-stages-two-and-three

Liebeck, P. (1984). *How children learn mathematics*. Penguin.

Lincoln, Y. S., & Guba, E. G. (1985). *Naturalistic inquiry*. Sage.

Lozano, M. (2015). Using enactivism as a methodology to characterise algebraic learning. *ZDM, 47*, 223–234. https://doi.org/10.1007/s11858-015-0678-3

Mason, J. (1994). Researching from the inside in mathematics education: Locating an I-You relationship. *Extended version of a Plenary address to PME XVII Lisbon July 1994*. Centre for Mathematics Education, Open University, Milton Keynes, UK. Retrieved July 12, 2021 from: https://www.researchgate.net/publication/288975126_Researching_from_the_Inside_in_Mathematics_Education.

Mason, J. (2002). *The discipline of noticing*. Routledge.

Mason, J. (2008). Being mathematical with and in front of learners. In B. Jaworski & T. Wood (Eds.), *The mathematics teacher educator as a developing professional* (pp. 31–55). Sense Publishers.

Mason, J. (2010). Attention and intention in learning about teaching through teaching. In R. Leikin & R. Zazkis (Eds.), *Learning through teaching mathematics: Development of teachers' knowledge and expertise in practice* (pp. 23–47). Springer.

Maturana, H., & Varela, F. (1992) *The Tree of Knowledge: The biological roots of human understanding, Revised Edition. Shambhala Publications.*

Nowell, L. S., Norris, J. M., White, D. E., & Moules, N. J. (2017). Thematic analysis: Striving to meet the trustworthiness criteria. *International Journal of Qualitative Methods, 16*, 1–13.

Powell, R. R. (1992). The influences of prior experience on pedagogical constructs of traditional and non-traditional preservice teachers. *Teaching and Teacher Education, 8*(3), 225–238. https://doi.org/10.1016/0742-051X(92)90022-U

Radford, L. (2010). Algebraic thinking from a cultural semiotic perspective. *Research in Mathematics Education, 12*(1), 1–19. https://doi.org/10.1080/14794800903569741

Reid, D. (1996). Enactivism as a methodology. In L. Puig & A. Gutiérrez (Eds.), *Proceedings of the 20th annual conference of the International Group for the Psychology of Mathematics Education* (pp. 203–210). Valencia: PME.

Reid, D., & Mgombelo, J. (2015). Survey of key concepts in enactivist theory and methodology. *ZDM, 47*, 171–183. https://doi.org/10.1007/s11858-014-0634-7

Rowland, T., Ineson, G., Alderton, J., Donaldson, G., Voutsina, C., & Wilson, K. (2018). Primary pre-service teachers: reasoning and generalisation. In J. Golding, N. Bretscher, C. Crisan, E. Geraniou, J. Hodgen & C. Morgan (Eds). *Research Proceedings of the 9th British Congress on Mathematics Education* (pp. 159–166). BCME

Stahl, N. A., & King, J. R. (2020). Expanding approaches for research: Understanding and using trustworthiness in qualitative research. *Journal of Developmental Education, 44*(1), 26–28.

Towers, J., & Davis, B. (2002). Structuring occasions. *Educational Studies in Mathematics, 49*, 313–340. https://doi.org/10.1023/A:1020245320040

Vale, C., Widjaja, W., Doig, B., & Groves, S. (2019). Anticipating students' reasoning and planning prompts in structured problem-solving lessons. *Mathematics Education Research Journal, 31*, 1–25. https://doi.org/10.1007/s13394-018-0239-5

Vale, I., Pimentel, T., Cabrita, I., Barbosa, A., & Fonseca, L. (2012). Pattern problem solving tasks as a means to foster creativity in mathematics. In T. Y. Tso (Ed.), *Proceedings of the 36th Conference of the International Group for the Psychology of Mathematics Education* 4 (pp. 171–178). Taipei, Taiwan, July 18–22.

Varela, F., Thompson, E., & Rosch, E. (1991). *The embodied mind: Cognitive science and human experience*. MIT Press.

Varela, F. (1999). *Ethical know-how: Action, wisdom, and cognition*. Stanford University Press.

Voutsina, C., Alderton, J., Wilson, K., Ineson, G., Donaldson, G. & Rowland, T. (2022) Preservice teachers' expressed awarenesses: emerging threads of retro-spection of learning and pro-spection of teaching. *Journal of Mathematics Teacher Education 25*(2), 191–215. https://doi.org/10.1007/s10857-020-09484-y

Wilcox, S., Schram, P., Lappan, G., & Lanier, P. (1991). The role of a learning community in changing preservice teachers' knowledge and beliefs about mathematics education. *For the Learning of Mathematics, 11*(3), 31–39.

Wilkie, K. J. (2014). Upper primary school teachers' mathematical knowledge for teaching functional thinking in algebra. *Journal of Mathematics Teacher Education, 17*(5), 397–428. https://doi.org/10.1007/s10857-013-9251-6

Wilkie, K. J. (2016). Learning to teach upper primary school algebra: Changes to teachers' mathematical knowledge for teaching functional thinking. *Mathematics Education Research Journal, 28*(2), 245–275. https://doi.org/10.1007/s13394-015-0151-1

Wilkie, K. J., & Clarke, D. M. (2016). Developing students' functional thinking in algebra through different visualisations of a growing pattern's structure. *Mathematics Education Research Journal, 28*(2), 223–243. https://doi.org/10.1007/s13394-015-0146-y