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# Locally Minimum-Variance Filtering of 2-D Systems over Sensor Networks with Measurement Degradations: A Distributed Recursive Algorithm

Fan Wang, Zidong Wang, Jinling Liang and Jun Yang

***Abstract*—This paper tackles the recursive filtering problem for an array of two-dimensional systems over sensor networks with a given topology. Both the measurement degradations of the network outputs and the stochastic perturbations of network couplings are modeled to reflect engineering practice via introducing some random variables with given statistics. The goal of the addressed problem is to devise the distributed recursive filters capable of cooperatively estimating the true state in order to ensure locally minimal upper bound (UB) on the second-order moment of the filtering error (also viewed as the general error variance). For this purpose, the general error variance regarding the underlying target plant is first provided to facilitate the subsequent filter design, and then a certain UB on the error variance is constructed by exploiting the stochastic analysis and induction approach. Furthermore, in view of the inherent sparsity of the sensor network, the gain parameters of the desired distributed filters are determined and the proposed recursive filtering algorithm is shown to be scalable. Finally, an illustrative example is given to demonstrate the validity of the established filtering strategy.**

***Index Terms*—Two-dimensional systems, recursive distributed filtering, sensor networks, measurement degradations, random couplings.**

## I. Introduction

Sensor networks have drawn persistent research attention owing to their great application potentials in various areas such as intelligent transportation, military facility and environment monitoring [4], [7], [22], [34]. Broadly speaking, a representative sensor network consists of massive smart, inexpensive and low-power sensor nodes which are geographically deployed over a certain region [2], [9]. Equipped with a local filter, each sensor node is competent to sense/compute in the process of information collection. Unlike a single sensor that merely observes its own measurements, a sensor within the sensor network can collect data from not only itself but also its neighbors through wireless communication channels. Benefiting from such a distinguishing feature, sensors are empowered to collaborate with other nodes in a neighbor-wise manner to accomplish a common yet complicated task.

One crucial issue with sensor networks is the development of the *distributed* filtering problem, for which the core idea is to reconstruct the system state on the basis of available measurements observed from the local and neighboring sensors [3], [12], [49]. Comparing with the filtering scheme in a centralized setup, the distributed filtering algorithm possesses the merits of consuming less energy and incurring less computation cost (at the expense of sacrificing the estimation performance within an acceptable range), and is particularly attractive in a resource-constrained environment as evidenced by its widespread applications in engineering practice. Up to now, the distributed filtering issue has become a popular research topic drawing considerable interest from various communities, and a wealth of literature has been published [15], [18], [21], [31], [33], [50]. For example, the consensus-based distributed filtering problem has been studied in [33] where the asymptotic stability of the error dynamics and the optimization of the quadratic filtering cost have been analyzed.

The existing distributed filtering methods can be mainly classified into two categories. Methodologically, the first category shares the commonalty of suppressing the effect of external disturbances, optimizing the worst-case estimation performance, and then finding the filter gains by means of the feasibility of some linear matrix inequalities [37], [42], [47]. For instance, the distributed filtering scheme has been investigated in [37] for sector-bounded nonlinear systems with randomly varying sampling periods, where sufficient conditions have been provided to guarantee the $H_\infty$ performance. The main idea of the second category is to determine the distributed filters within the framework of Kalman (or Kalman-type) filters via recursive equations [5], [29], [35]. More specifically, the celebrated Kalman filter intends to minimize the estimation error variance for linear systems with exactly known parameters [20], [52]. The Kalman filter algorithms have been further extended to accommodate nonlinearities/uncertainties in system models through developing alternative *suboptimal* approaches [14], [36], where the basic ideology behind such a suboptimal strategy is to attain the tightest upper bound (UB) on the general error variance. Recently, the distributed filter design problem has been tackled in [29] for stochastic systems subject to energy constraints and Markovian switching topologies.

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So far, most available results concerning distributed filtering problems have been exclusively on the traditional one-dimensional models whose states broadcast along a *single* direction only [6], [10], [23], [26], [27], [30], [38], [40], [51]. Nonetheless, it is observed from the practical situations that evolutions of many system states follow two (or even more) directions with typical examples including, but are not limited to, multi-variable networks, chemical processes, and image processing. As such, the so-called two-dimensional (2-D) systems exhibit the fascinating property of two-directional transmissions and have thus been introduced to model those real-world systems with dynamics evolving along two horizons [16], [17], [19]. For decades, the estimation issues for 2-D systems have been gaining an ongoing research interest, see e.g. [1], [25], [43], [45] for some representative results. Unfortunately, a thorough literature review has disclosed that slight research effort has been devoted to the recursive filtering problem for 2-D systems over sensor networks where the error variance serves as a crucial criterion. The reason for such a problem to remain open yet challenging is mainly the essential difficulties resulting from the complicated system dynamics and the sparsity of the network topology. Therefore, in this paper, we are motivated to launch a systematic investigation on the design issue of 2-D distributed filters in a recursive structure.

Despite the low cost and high flexibility of the network communication, sensor networks may undergo coupling perturbations on account of potentially harsh and uncertain wireless environments [41], [48]. The occurrence of random couplings is fairly pervasive, which brings about complexities in stochastic analysis of the system dynamics. To date, some initial research attention has been paid to the filtering problem concerning sensor networks with random couplings [13], [42]. Another frequently encountered phenomenon in networked environments is degraded measurements which cover the packets dropout as a special case. In consideration of sensors aging/failure as well as transmission congestion, measurements may suffer from inevitable degradation that might lead to serious performance deterioration if not properly handled [24], [28], [32], [44]. So far, in presence of random couplings and measurement degradations, the 2-D filtering issue has not drawn much attention yet especially in the case where the error variance is also of concern.

To summarize the discussions made so far, it is of theoretical importance and practical interest to investigate the recursive filtering problem for 2-D systems with information degradations and network coupling perturbations. To address such an open problem, there appear to be some substantial challenges that should be overcome. The first challenge we are facing is the development of effective techniques that can be used to analyze the general error variance in the 2-D framework. Notice that it is literally impossible to acquire the analytical expression of the general error variance based on the minimum-variance filtering scheme, especially for systems with coupling perturbations. The second challenge is the determination of the tightest UB on the error variance by resorting to the Kalman-type strategy. It should be pointed out that both the degraded measurement and the network topology have major impacts on the filtering performance. The third challenge is, therefore, to propose effective approaches to cope with the considered network-induced phenomena and further parameterize suitable filter gains that optimize the UB in the trace sense.

In this paper, we focus our attention on the 2-D recursive filtering scheme over sensor networks. In particular, both the measurement degradations of the network outputs and the random communication links of sensor networks are taken into account in order to reflect the engineering phenomena caused by changeable networked environments. With aid of the inductive approach, the desired recursive filters are determined to ensure the guaranteed estimation performance. *The main contributions are emphasized as follows: 1) a distributed recursive filtering strategy is, for the first time, investigated for 2-D systems over sensor networks with degraded measurements and random couplings; 2) a delicate distributed filter is proposed which collects not only the innovation from the local sensor but also the complemental information from the neighboring nodes; 3) an elaborated design of the 2-D filter gains is provided by making full use of the topology information and the stochastic matrix analysis; and 4) a satisfactory state estimation is achieved by developing the locally minimal UB on the general error variance.*

The remainder of this paper is outlined as follows. Section II presents the target plant and the 2-D recursive filtering problem to be investigated. Section III shows some preliminaries, provides the algorithm for finding the locally minimal UB, and gives the filter design scheme. Section IV consists of simulation studies to confirm the efficiency of the developed filtering algorithm. Conclusion is drawn in Section V.

*Notations:* $\mathbb{R}^n$ signifies the $n$-dimensional Euclidean space, and $\mathbf{1}_n$ is the n-dimensional vector with all entries being 1. For a matrix $X$, $\|X\|$ represents the norm of $X$ with $\|X\| \triangleq \sqrt{\mathrm{tr}\{X^T X\}}$. For a real and symmetric matrix $Y$, $Y > 0$ means that $Y$ is real, symmetric and positive definite, whilst $Y \geq 0$ infers that $Y$ is positive semi-definite. The symbol $\mathrm{col}_{s=1}^{L}\{A_s\}$ stands for the matrix $[A_1^T \ A_1^T \ \ldots \ A_1^T]^T$. $I$ and $0$ are respectively the identity and zero matrices of compatible dimensions. For integers $k_1$, $k_2$ with $k_1 \leq k_2$, $[k_1 \ k_2]$ denotes the set $\{k_1, k_1+1, \ldots k_2\}$. $\mathrm{tr}\{\cdot\}$ is the trace of certain square matrix and $\mathrm{diag}\{\cdots\}$ means the block-diagonal matrix. '$\circ$' denotes the Hadamard product and '$\otimes$' is the Kronecker product. $\mathbb{E}\{\beta\}$ and $\mathrm{Var}\{\beta\}$ are respectively the mathematical expectation and the variance of random variable $\beta$.

## II. Problem formulation and preliminaries

### A. The plant description

Consider the following 2-D system over a finite horizon $j, k \in [0 \ \flat]$ with $\flat$ being a given positive integer:

$$
\begin{aligned}
x(j,k) =& f_1((j,k-1), x(j,k-1)) + w(j,k-1) \\
&+ f_2((j-1,k), x(j-1,k)) + w(j-1,k) \qquad (1)
\end{aligned}
$$

where $x(j,k) \in \mathbb{R}^n$ is the state vector, $w(j,k) \in \mathbb{R}^n$ is the process white noise obeying the Gaussian distribution with zero mean and variance $Q(j,k) > 0$. For $\hbar = 1, 2$, the nonlinear functions $f_\hbar((j,k),x(j,k))$ satisfy $f_\hbar((j,k),0)=0$ and

$$\begin{aligned}&\|f_\hbar((j,k),x_1)-f_\hbar((j,k),x_2)-A_\hbar(j,k)(x_1-x_2)\|\\&\le a_\hbar(j,k)\|x_1-x_2\|,\quad \forall x_1,x_2\in\mathbb{R}^n\end{aligned}\tag{2}$$

where $A_\hbar(j,k)$ are known shift-varying matrices and $a_\hbar(j,k)$ are given nonnegative scalars. The initial states $x(j,0)$ and $x(0,k)$ for system (1) are modeled by two white-noise sequences possessing $\mathbb{E}\{x(j,0)\}=\eta_1(j)$ and $\mathbb{E}\{x(0,k)\}=\eta_2(k)$, where $\eta_1(j)$ and $\eta_2(k)$ are known vectors with $\eta_1(0)=\eta_2(0)$.

*Remark 1:* In view of the practical need, 2-D systems have been a powerful tool to model many physical systems with two-directional information propagation. In the real world, nonlinearities occur frequently in some practical situations, especially in the man-made systems and maneuvering target modeling. Moreover, due to the system heterogeneity, system parameters may be shift-varying, and the transient behaviors of the underlying system are of significance. In this regard, the 2-D system with shift-varying parameters and nonlinear functions is considered here.

### *B. The measurement over sensor network*

Consider a sensor network consisting of $L$ nodes to track the system states in a cooperative paradigm. The topology of the sensor network is represented by a directed graph $\mathcal{G}=(\mathcal{V},\mathcal{E},\mathcal{A})$, where $\mathcal{V}=\{1,2,\cdots,L\}$ is the index set of nodes, $\mathcal{E}\subseteq\mathcal{V}\times\mathcal{V}$ is the edge set, and $\mathcal{A}=(h_{st})_{L\times L}$ is the weighted adjacency matrix with nonnegative elements. To be specific, for any $s,t\in\mathcal{V}$, the case $h_{st}>0$ holds if and only if node $s$ receives information from node $t$ (that is to say, $(s,t)\in\mathcal{E}$), otherwise $h_{st}=0$. Besides, self-loops are not allowed here, and the neighbors of node $s$ is denoted by the set $\aleph_s\triangleq\{t\in\mathcal{V}|(s,t)\in\mathcal{E}\}$ for brevity.

The measurement model of sensor node $s$ is given by

$$y_s(j,k)=\gamma_s(j,k)C_s(j,k)x(j,k)+v_s(j,k),\quad s\in\mathcal{V}\tag{3}$$

where $y_s(j,k)\in\mathbb{R}^m$ is the measured output of the $s$-th sensor, $C_s(j,k)$ is a known shift-varying matrix, $v_s(j,k)\in\mathbb{R}^m$ is the measurement noise modeled by the zero-mean Gaussian white sequence with variance $R_s(j,k)>0$, and $\gamma_s(j,k)\in\mathbb{R}$ is a random variable taking value on the interval $[0,1]$ that governs the measurement degradation with known statistics $\mathbb{E}\{\gamma_s(j,k)\}=\bar\gamma_s(j,k)$ and $\mathrm{Var}\{\gamma_s(j,k)\}=\hat\gamma_s(j,k)$, in which $\bar\gamma_s(j,k)$ and $\hat\gamma_s(j,k)$ are known scalars.

### *C. The distributed filter*

To pursue a common task, nodes over sensor network share information with their neighbors through the communication channels. In this case, each sensor not only measures its own local signal but also shares the data with its adjacent sensors, thereby cooperatively tracking or monitoring the target states of interest in a distributed manner.

The following distributed filter concerning node $s$ ($s\in\mathcal{V}$) is adopted for system (1):

$$\begin{aligned}\hat x_s^-(j,k)=&f_1((j,k-1),\hat x_s(j,k-1))\\&+f_2((j-1,k),\hat x_s(j-1,k))\end{aligned}\tag{4a}$$

$$\begin{aligned}\hat x_s(j,k)=&\hat x_s^-(j,k)+G_s(j,k)\big(y_s(j,k)-\bar\gamma_s(j,k)C_s(j,k)\\&\times\hat x_s^-(j,k)\big)+\sum_{t\in\aleph_s}\bar h_{st}(j,k)K_{st}(j,k)\\&\times\big(y_t(j,k)-\bar\gamma_s(j,k)C_s(j,k)\hat x_s^-(j,k)\big)\end{aligned}\tag{4b}$$

where $\hat x_s^-(j,k)\in\mathbb{R}^n$ is the prediction of state $x(j,k)$ and $\hat x_s(j,k)\in\mathbb{R}^n$ is the relevant estimate. $G_s(j,k)$ and $K_{st}(j,k)$ are the filter gains to be designed. The coupling coefficient $\bar h_{st}(j,k)$, which is probably confined to some small variations, is denoted as $\bar h_{st}(j,k)\triangleq h_{st}+\Delta h_{st}(j,k)$. Here, $h_{st}$ is the nominal parameter of the adjacency matrix $\mathcal{A}$, whereas $\Delta h_{st}(j,k)$ represents certain random perturbation acting on $h_{st}$. Specifically, $\Delta h_{st}(j,k)$ satisfies $\mathbb{E}\{\Delta h_{st}(j,k)\}=0$ and $\mathbb{E}\{\Delta h_{st}(j,k)\Delta h_{st}^T(j,k)\}\le\pi_{st}(j,k)$, where $\pi_{st}(j,k)$ is a known positive scalar when $h_{st}>0$, elsewise $\Delta h_{st}(j,k)=0$ when $h_{st}=0$. The initial states related to (4) are set as $\hat x_s(j,0)=\hat x_s(0,k)=0$ for $j,k\in[0\ \flat]$.

*Remark 2:* For the considered sensor network, each sensor is equipped with a local filter and therefore has the capacities of sensing, computing and interacting with the neighboring nodes. Unlike the case for a single sensor where only its own measurement is utilized to estimate the internal state, in the sensor network scenario, the updated information of the proposed filter comes from both itself and the neighboring nodes in a distributed manner. Actually, the updated information in (4) is composed of two parts from different sources, one is the conventional innovation $y_s(j,k)-\bar\gamma_s(j,k)C_s(j,k)\hat x_s^-(j,k)$ from the sensor itself, and the other is the coupling data $\sum_{t\in\aleph_s}\bar h_{st}(j,k)(y_t(j,k)-\bar\gamma_s(j,k)C_s(j,k)\hat x_s^-(j,k))$ from the neighbors' measurements. Such a structure, which is shown in Fig. 1, is adopted in this paper for achieving the state estimation with an adequate accuracy.

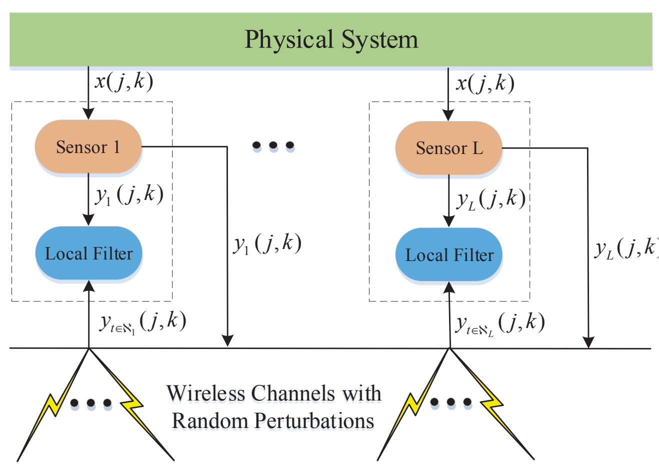


Fig. 1. Schematic diagram of the considered system over sensor network.

*Remark 3:* The random variable $\Delta h_{st}(j,k)$ is introduced in (4) to model the possible perturbations with known statistics. The rational for such an introduction is the practical need for reflecting the fact that the communication links might be randomly changeable due mainly to the unideal communication interferences. Also, there appear to be unavoidable fluctuations on the deterministic couplings, and hence the combined coupling coefficients (both stochastic and deterministic ones) are used in (4) to characterize the random coupling strengths.

The following assumptions are made for the convenience of later discussion.

*Assumption 1:* For $j_\jmath, k_\jmath \in [0\ \flat]$ with $\jmath \in \{1,2,3,4\}$ and $s,t \in \mathcal{V}$, the initial states $x(j_1,0)$, $x(0,k_1)$, the noises $w(j_2,k_2)$, $v_s(j_3,k_3)$ and the random parameter $\gamma_k(j_4,k_4)$ are mutually uncorrelated with each other.

*Assumption 2:* For $j,k \in [0\ \flat]$ and $s,t \in \mathcal{V}$, the stochastic perturbation $\Delta h_{st}(j,k)$ is a white-noise sequence with respect to all indices $j,k,s,t$ in the case of $h_{st} > 0$. Moreover, $\Delta h_{st}(j,k)$ is uncorrelated with all the other random variables involved in systems (1) and (3).

For notational simplicity, let us define $e_s^-(j,k) \triangleq x(j,k) - \hat{x}_s^-(j,k)$ as the prediction error and $e_s(j,k) \triangleq x(j,k) - \hat{x}_s(j,k)$ as the estimation error. Then, the following error dynamics can be attained from (1) and (3)-(4):

$$\begin{aligned} e_s^-(j,k) =& \tilde{f}_1((j,k-1), e_s(j,k-1)) + w(j,k-1) \\ &+ \tilde{f}_2((j-1,k), e_s(j-1,k)) + w(j-1,k) \quad (5a) \\ e_s(j,k) =& e_s^-(j,k) - G_s(j,k)\big(y_s(j,k) - \bar{\gamma}_s(j,k)C_s(j,k) \\ &\times \hat{x}_s^-(j,k)\big) - \sum_{t\in\aleph_s} \bar{h}_{st}(j,k)K_{st}(j,k) \\ &\times \big(y_t(j,k) - \bar{\gamma}_s(j,k)C_s(j,k)\hat{x}_s^-(j,k)\big) \quad (5b) \end{aligned}$$

where for $\hbar = 1,2$, $\tilde{f}_\hbar((j,k), e_s(j,k)) \triangleq f_\hbar((j,k), x_s(j,k)) - f_\hbar((j,k), \hat{x}_s(j,k))$. By further denoting $\bar{I} \triangleq \mathbf{1}_L \otimes I$ and

$$\begin{aligned} &K_s(j,k) \triangleq [K_{s1}(j,k) \quad K_{s2}(j,k) \quad \cdots \quad K_{sL}(j,k)] \\ &\bar{H}_s(j,k) \triangleq \mathrm{diag}\{\bar{h}_{s1}(j,k), \bar{h}_{s2}(j,k), \cdots, \bar{h}_{sL}(j,k)\} \otimes I \\ &\Gamma(j,k) \triangleq \mathrm{diag}\{\gamma_1(j,k), \gamma_2(j,k), \cdots, \gamma_L(j,k)\} \otimes I \\ &\bar{C}(j,k) \triangleq \mathrm{col}_{s=1}^L\{C_s(j,k)\}, \quad v(j,k) \triangleq \mathrm{col}_{s=1}^L\{v_s(j,k)\} \end{aligned}$$

we rewrite the error dynamics (5b) as

$$\begin{aligned} e_s(j,k) =& \big[I - \bar{\gamma}_s(j,k)\left(G_s(j,k) + K_s(j,k)\bar{H}_s(j,k)\bar{I}\right) \\ &\times C_s(j,k)\big]e_s^-(j,k) - K_s(j,k)\bar{H}_s(j,k)\big[\big(\Gamma(j,k) \\ &\times \bar{C}(j,k) - \bar{\gamma}_s(j,k)\bar{I}C_s(j,k)\big)x(j,k) + v(j,k)\big] \\ &- G_s(j,k)\big(\tilde{\gamma}_s(j,k)C_s(j,k)x(j,k) + v_s(j,k)\big) \end{aligned} \quad (6)$$

with $\tilde{\gamma}_s(j,k) \triangleq \gamma_s(j,k) - \bar{\gamma}_s(j,k)$.

In this paper, we are interested in designing the distributed recursive filter (4) with desirable gains, for the 2-D system (1) under consideration, to provide an UB on the general estimation error for each sensor node (namely, a bound on $\mathbb{E}\{e_s(j,k)e_s^T(j,k)\}$ with $s \in \mathcal{V}$), and further obtain the locally minimized UB in the trace sense.

## III. Main Results

This section provides certain UB on the error variance at each iteration by solving two sets of recursive difference equations. Moreover, the desirable distributed filter is acquired for each senor, which ensures the locally tightest UB in the trace sense. Prior to giving the main results, some preliminaries are first introduced.

Hereafter, let us define $P_s(j,k) \triangleq \mathbb{E}\{e_s(j,k)e_s^T(j,k)\}$ and $P_s^-(j,k) \triangleq \mathbb{E}\{e_s^-(j,k)(e_s^-(j,k))^T\}$ which are, respectively, termed as the general estimation and prediction error variances concerning the $s$-th filter, and further let the second-order moment of the system state be $X(j,k) \triangleq \mathbb{E}\{x(j,k)x^T(j,k)\}$.

### A. Auxiliary lemmas

The following fundamental lemmas determine the dynamics of the general error variances.

*Lemma 1:* For the prediction error dynamics (5a), the evolution of $P_s^-(j,k)$ is given by

$$\begin{aligned} P_s^-(j,k) =& \Phi_{1,s}(j,k-1) + \Phi_{2,s}(j-1,k) \\ &+ \Psi_s((j,k-1),(j-1,k)) + Q(j,k-1) \\ &+ \Psi_s^T((j,k-1),(j-1,k)) + Q(j-1,k) \end{aligned} \quad (7)$$

where, for $\hbar = 1,2$,

$$\begin{aligned} &\Phi_{\hbar,s}(j,k) = \mathbb{E}\big\{\tilde{f}_\hbar((j,k), e_s(j,k))\tilde{f}_\hbar^T((j,k), e_s(j,k))\big\} \\ &\Psi_s((j,k-1),(j-1,k)) \\ &= \mathbb{E}\big\{\tilde{f}_1((j,k-1), e_s(j,k-1))\tilde{f}_2^T((j-1,k), e_s(j-1,k))\big\}. \end{aligned}$$

*Proof:* For all $j,k \in [0\ \flat]$, notice the uncorrelatedness between random variables $w(j,k)$ and $\tilde{f}_\hbar((\imath_0,\jmath_0), e_s(\imath_0,\jmath_0))$ with $\hbar = 1,2$, $s \in \mathcal{V}$ and $(\imath_0,\jmath_0) \in \{(\imath,\jmath)|\imath \in [0\ \ j-1], \jmath \in [0\ \flat]\} \cup \{(\imath,\jmath)|\imath \in [0\ \flat], \jmath \in [0\ \ k-1]\} \cup \{(j,k)\}$. It is concluded from (5a) that the statement of this lemma is correct. ■

*Lemma 2:* For the estimation error dynamics (6), the evolution of $P_s(j,k)$ obeys the following inequality constraint:

$$\begin{aligned} P_s(j,k) \leq& \big[I - \bar{\gamma}_s(j,k)\left(G_s(j,k) + K_s(j,k)H_s\bar{I}\right)C_s(j,k)\big] \\ &\times P_s^-(j,k)\big[I - \big(G_s(j,k) + K_s(j,k)H_s\bar{I}\big) \\ &\times \bar{\gamma}_s(j,k)C_s(j,k)\big]^T - \Pi_s(j,k) - \Pi_s^T(j,k) \\ &+ K_s(j,k)\Big[\bar{\gamma}_s^2(j,k)\hat{H}_s(j,k) \circ \big(\bar{I}C_s(j,k) \\ &\times P_s^-(j,k)C_s^T(j,k)\bar{I}^T\big) + \hat{H}_s(j,k) \circ \bar{\Theta}_s(j,k) \\ &+ H_s\bar{\Theta}_s(j,k)H_s^T\Big]K_s^T(j,k) + G_s(j,k)\big(R_s(j,k) \\ &+ \hat{\gamma}_s(j,k)C_s(j,k)X(j,k)C_s^T(j,k)\big)G_s^T(j,k) \end{aligned} \quad (8)$$

where

$$\begin{aligned} &H_s = \mathrm{diag}\{h_{s1}, h_{s2}, \cdots, h_{sL}\} \otimes I \\ &\hat{H}_s(j,k) = \mathrm{diag}\{\pi_{s1}(j,k), \pi_{s2}(j,k), \cdots, \pi_{sL}(j,k)\} \otimes I \\ &\hat{\Gamma}(j,k) = \mathrm{diag}\{\hat{\gamma}_1(j,k), \hat{\gamma}_2(j,k), \cdots, \hat{\gamma}_L(j,k)\} \otimes I \\ &\bar{\Theta}_s(j,k) = \Theta_s(j,k) + R(j,k) \\ &\qquad + \hat{\Gamma}(j,k) \circ \big(\bar{C}(j,k)X(j,k)\bar{C}^T(j,k)\big) \\ &\Theta_s(j,k) = \big(\bar{\Gamma}(j,k)\bar{C}(j,k) - \bar{\gamma}_s(j,k)\bar{I}C_s(j,k)\big)X(j,k) \\ &\qquad \times \big(\bar{\Gamma}(j,k)\bar{C}(j,k) - \bar{\gamma}_s(j,k)\bar{I}C_s(j,k)\big)^T \\ &R(j,k) \triangleq \mathrm{diag}\{R_1(j,k), R_2(j,k), \cdots, R_L(j,k)\} \otimes I \\ &\bar{\Gamma}(j,k) \triangleq \mathrm{diag}\{\bar{\gamma}_1(j,k), \bar{\gamma}_2(j,k), \cdots, \bar{\gamma}_L(j,k)\} \otimes I \\ &\Pi_s(j,k) = \mathbb{E}\Big\{\big[I - \bar{\gamma}_s(j,k)\left(G_s(j,k) + K_s(j,k)\bar{H}_s(j,k)\bar{I}\right) \\ &\qquad \times C_s(j,k)\big]e_s^-(j,k)x^T(j,k)\big(\Gamma(j,k)\bar{C}(j,k) \\ &\qquad - \bar{\gamma}_s(j,k)\bar{I}C_s(j,k)\big)^T\bar{H}_s^T(j,k)K_s^T(j,k)\Big\}. \end{aligned}$$

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*Proof:* The statistics of $\Delta h_{st}(j,k)$ indicate that

$$\mathbb{E}\left\{\Delta h_{st}(j,k)\Delta h_{st}^T(j,k)\right\}\leq \pi_{st}(j,k)$$
$$\mathbb{E}\left\{\Delta h_{st_1}(j,k)\Delta h_{st_2}^T(j,k)\right\}=0,\quad t_1\neq t_2.$$

Moreover, based on the expression of $\tilde{\gamma}_s(j,k)$, one has $\mathbb{E}\{\tilde{\gamma}_s(j,k)\}=0$ and $\mathbb{E}\{\tilde{\gamma}_{s_1}(j,k)\tilde{\gamma}_{s_2}(j,k)\}=\hat{\gamma}_{s_1}(j,k)$ for $s_1=s_2$, while $\mathbb{E}\{\tilde{\gamma}_{s_1}(j,k)\tilde{\gamma}_{s_2}(j,k)\}=0$ otherwise.

For a given deterministic matrix $Z\geq 0$, it follows from the property of Hadamard product that

$$\begin{aligned}
&\mathbb{E}\big\{\tilde{H}_s(j,k)Z\tilde{H}_s^T(j,k)\big\}\\
&=\mathbb{E}\left\{\mathrm{diag}\{\Delta h_{s1}^2(j,k)I,\Delta h_{s2}^2(j,k)I,\cdots,\Delta h_{sL}^2(j,k)I\}\circ Z\right\}\\
&\leq \mathrm{diag}\{\pi_{s1}(j,k)I,\pi_{s2}(j,k)I,\cdots,\pi_{sL}(j,k)I\}\circ Z\\
&=\hat{H}_s(j,k)\circ Z
\end{aligned}$$

and

$$\mathbb{E}\big\{\tilde{\Gamma}(j,k)Z\tilde{\Gamma}^T(j,k)\big\}=\hat{\Gamma}(j,k)\circ Z$$

where

$$\begin{aligned}
\tilde{H}_s(j,k)&\triangleq\mathrm{diag}\{\Delta h_{s1}(j,k),\Delta h_{s2}(j,k),\cdots,\Delta h_{sL}(j,k)\}\otimes I\\
\tilde{\Gamma}(j,k)&\triangleq\mathrm{diag}\{\tilde{\gamma}_1(j,k),\tilde{\gamma}_2(j,k),\cdots,\tilde{\gamma}_L(j,k)\}\otimes I
\end{aligned}$$

with $\mathbb{E}\{\tilde{\Gamma}(j,k)\}=0$ and $\mathbb{E}\{\tilde{\Gamma}(j,k)\tilde{\Gamma}^T(j,k)\}=\hat{\Gamma}(j,k)$. Consequently, it is not difficult to verify the validity of the following relationships

$$\begin{aligned}
&\mathbb{E}\Big\{\bar{\gamma}_s^2(j,k)K_s(j,k)\tilde{H}_s(j,k)\bar{I}C_s(j,k)e_s^-(j,k)\\
&\times\left(K_s(j,k)\tilde{H}_s(j,k)\bar{I}C_s(j,k)e_s^-(j,k)\right)^T\Big\}\\
&\quad\leq\bar{\gamma}_s^2(j,k)K_s(j,k)\big[\hat{H}_s(j,k)\circ\big(\bar{I}C_s(j,k)P_s^-(j,k)\\
&\qquad\times C_s^T(j,k)\bar{I}^T\big)\big]K_s^T(j,k),\\
&\mathbb{E}\Big\{\big[\big(\Gamma(j,k)\bar{C}(j,k)-\bar{\gamma}_s(j,k)\bar{I}C_s(j,k)\big)x(j,k)+v(j,k)\big]\\
&\times\big[\big(\Gamma(j,k)\bar{C}(j,k)-\bar{\gamma}_s(j,k)\bar{I}C_s(j,k)\big)x(j,k)+v(j,k)\big]^T\Big\}\\
&\quad=\mathbb{E}\Big\{\big(\Gamma(j,k)\bar{C}(j,k)-\bar{\gamma}_s(j,k)\bar{I}C_s(j,k)\big)X(j,k)\\
&\qquad\times\big(\Gamma(j,k)\bar{C}(j,k)-\bar{\gamma}_s(j,k)\bar{I}C_s(j,k)\big)^T\Big\}+R(j,k)\\
&\quad=\big(\bar{\Gamma}(j,k)\bar{C}(j,k)-\bar{\gamma}_s(j,k)\bar{I}C_s(j,k)\big)X(j,k)\\
&\qquad\times\big(\bar{\Gamma}(j,k)\bar{C}(j,k)-\bar{\gamma}_s(j,k)\bar{I}C_s(j,k)\big)+R(j,k)\\
&\qquad+\mathbb{E}\big\{\tilde{\Gamma}(j,k)\bar{C}(j,k)X(j,k)\bar{C}^T(j,k)\tilde{\Gamma}^T(j,k)\big\}\\
&\quad=\Theta_s(j,k)+R(j,k)+\hat{\Gamma}(j,k)\circ\big(\bar{C}(j,k)X(j,k)\bar{C}^T(j,k)\big)\\
&\quad=\bar{\Theta}_s(j,k).
\end{aligned}$$

On account of Assumptions 1-2, the definition of $P_s(j,k)$ and equation (6) result in the validity of (8), where part of the detailed technical analysis is omitted for conciseness. ∎

This subsection derives the standard recursions of the general error variances. Nevertheless, in the considered framework of (7)-(8), wherein not only the nonlinearities but also the random couplings affect the 2-D evolutions, calculation of the exact error variances is inaccessible, which complicates the design of the *optimal* filter gains. In view of this, alternative *suboptimal* filtering strategies should be exploited, and a frequently used strategy is to devise the filter gains that achieve the locally tightest bounds on the error variances.

## B. Upper bounds (UBs)

It is observed from (7)-(8) that the involvement of the cross terms, the nonlinearities and the second-order moment of the state in the general error variances adds much difficulty to the filter design, which are to be handled in the following.

First, the cross term $\Pi_s(j,k)$ will be addressed by using the elementary inequality. For an arbitrary positive scalar $\mu$, it is derived from the expression of $\Pi_s(j,k)$ in Lemma 2 that

$$\begin{aligned}
&-\Pi_s(j,k)-\Pi_s^T(j,k)\\
&=-\mathbb{E}\Big\{\big[I-\bar{\gamma}_s(j,k)\left(G_s(j,k)+K_s(j,k)\bar{H}_s(j,k)\bar{I}\right)C_s(j,k)\big]\\
&\quad\times e_s^-(j,k)x^T(j,k)\big(\bar{\Gamma}(j,k)\bar{C}(j,k)-\bar{\gamma}_s(j,k)\bar{I}C_s(j,k)\big)^T\\
&\quad\times\bar{H}_s^T(j,k)K_s^T(j,k)+K_s(j,k)\bar{H}_s(j,k)\\
&\quad\times\big(\bar{\Gamma}(j,k)\bar{C}(j,k)-\bar{\gamma}_s(j,k)\bar{I}C_s(j,k)\big)x(j,k)(e_s^-(j,k))^T\\
&\quad\times\big[I-\bar{\gamma}_s(j,k)\left(G_s(j,k)+K_s(j,k)\bar{H}_s(j,k)\bar{I}\right)C_s(j,k)\big]^T\Big\}\\
&\leq\mu\Big\{\big[I-\bar{\gamma}_s(j,k)\left(G_s(j,k)+K_s(j,k)H_s\bar{I}\right)C_s(j,k)\big]\\
&\quad\times P_s^-(j,k)\big[I-\bar{\gamma}_s(j,k)\left(G_s(j,k)+K_s(j,k)H_s\bar{I}\right)\\
&\quad\times C_s(j,k)\big]^T+\bar{\gamma}_s^2(j,k)K_s(j,k)\big[\hat{H}_s(j,k)\circ\big(\bar{I}C_s(j,k)\\
&\quad\times P_s^-(j,k)C_s^T(j,k)\bar{I}^T\big)\big]K_s^T(j,k)\Big\}+\mu^{-1}K_s(j,k)\\
&\quad\times\big(\hat{H}_s(j,k)\circ\Theta_s(j,k)+H_s\Theta_s(j,k)H_s^T\big)K_s^T(j,k).
\end{aligned}$$

Consequently, the obtained evolution of $P_s(j,k)$ (as shown in (8)) obeys:

$$\begin{aligned}
P_s(j,k)\leq&(1+\mu)\Big\{\big[I-\bar{\gamma}_s(j,k)\left(G_s(j,k)+K_s(j,k)H_s\bar{I}\right)\\
&\times C_s(j,k)\big]P_s^-(j,k)\big[I-\big(G_s(j,k)+K_s(j,k)\\
&\times H_s\bar{I}\big)\bar{\gamma}_s(j,k)C_s(j,k)\big]^T+\bar{\gamma}_s^2(j,k)K_s(j,k)\\
&\times\big[\hat{H}_s(j,k)\circ\big(\bar{I}C_s(j,k)P_s^-(j,k)C_s^T(j,k)\bar{I}^T\big)\big]\\
&\times K_s^T(j,k)\Big\}+K_s(j,k)\big(\hat{H}_s(j,k)\circ\hat{\Theta}_s(j,k)\\
&+H_s\hat{\Theta}_s(j,k)H_s^T\big)K_s^T(j,k)+G_s(j,k)\big(R_s(j,k)\\
&+\hat{\gamma}_s(j,k)C_s(j,k)X(j,k)C_s^T(j,k)\big)G_s^T(j,k)
\end{aligned}\tag{9}$$

with

$$\begin{aligned}
\hat{\Theta}_s(j,k)=&(1+\mu^{-1})\Theta_s(j,k)+R(j,k)\\
&+\hat{\Gamma}(j,k)\circ\big(\bar{C}(j,k)X(j,k)\bar{C}^T(j,k)\big).
\end{aligned}\tag{10}$$

Next, to deal with the nonlinearities, one obtains from (2) and the elementary inequality that

$$\begin{aligned}
&\mathbb{E}\big\{\tilde{f}_\hbar((j,k),e_s(j,k))\tilde{f}_\hbar^T((j,k),e_s(j,k))\big\}\\
&=\mathbb{E}\Big\{\Big(\tilde{f}_\hbar((j,k),e_s(j,k))-(A_\hbar(j,k)-A_\hbar(j,k))e_s(j,k)\Big)\\
&\quad\times\Big(\tilde{f}_\hbar((j,k),e_s(j,k))-(A_\hbar(j,k)-A_\hbar(j,k))e_s(j,k)\Big)^T\Big\}\\
&\leq(1+\epsilon)\mathbb{E}\Big\{\Big(\tilde{f}_\hbar((j,k),e_s(j,k))-A_\hbar(j,k)e_s(j,k)\Big)\\
&\quad\times\Big(\tilde{f}_\hbar((j,k),e_s(j,k))-A_\hbar(j,k)e_s(j,k)\Big)^T\Big\}\\
&\quad+(1+\epsilon^{-1})A_\hbar(j,k)P_s(j,k)A_\hbar^T(j,k)\\
&\leq(1+\epsilon^{-1})A_\hbar(j,k)P_s(j,k)A_\hbar^T(j,k)\\
&\quad+(1+\epsilon)a_\hbar^2(j,k)\mathrm{tr}\{P_s(j,k)\}I\triangleq\hat{P}_{\hbar,s}(j,k)
\end{aligned}\tag{11}$$

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for $\hbar = 1,2$ and a given scalar $\epsilon > 0$. Analogously, it is evident to see that

$$\begin{aligned}&\mathbb{E}\left\{f_\hbar((j,k),x(j,k))f_\hbar^T((j,k),x(j,k))\right\}\\&\le(1+\epsilon)a_\hbar^2(j,k)\mathrm{tr}\{X(j,k)\}I\\&\quad+(1+\epsilon^{-1})A_\hbar(j,k)X(j,k)A_\hbar^T(j,k)\triangleq X_\hbar(j,k). \quad (12)\end{aligned}$$

Based on (11), matrix $P_s^-(j,k)$ in (7) satisfies

$$\begin{aligned}P_s^-(j,k)\le&(1+\mu)\hat{P}_{1s}(j,k-1)+(1+\mu^{-1})\hat{P}_{2s}(j-1,k)\\&+Q(j,k-1)+Q(j-1,k). \quad (13)\end{aligned}$$

Furthermore, a bound function of matrix $X(j,k)$ is derived by using (12) in the following lemma.

*Lemma 3:* Let $\mu$ and $\epsilon$ be given positive scalars. For system (1), matrix $X(j,k)$ is bounded by the solution of the following recursion:

$$\begin{aligned}\bar{X}(j,k)=&(1+\mu)\hat{X}_1(j,k-1)+(1+\mu^{-1})\hat{X}_2(j-1,k)\\&+Q(j,k-1)+Q(j-1,k) \quad (14)\end{aligned}$$

whose initial states are set as

$$\bar{X}(j,0)=X(j,0), \quad \bar{X}(0,k)=X(0,k)$$

with

$$\begin{aligned}\hat{X}_\hbar(j,k)\triangleq&(1+\epsilon)a_\hbar^2(j,k)\mathrm{tr}\left\{\bar{X}(j,k)\right\}I\\&+(1+\epsilon^{-1})A_\hbar(j,k)\bar{X}(j,k)A_\hbar^T(j,k), \quad \hbar=1,2.\end{aligned}$$

*Proof:* Recalling the system dynamics (1) and the statistical property of $w(j,k)$, one has

$$\begin{aligned}X(j,k)\le&(1+\mu)X_1(j,k-1)+(1+\mu^{-1})X_2(j-1,k)\\&+Q(j,k-1)+Q(j-1,k)\end{aligned}$$

where (12) has been used to dispose the considered nonlinear functions. Denote $\tilde{X}(j,k)\triangleq X(j,k)-\bar{X}(j,k)$ for simplicity. Then, the dynamics of $\tilde{X}(j,k)$ satisfies

$$\begin{aligned}\tilde{X}(j,k)\le&(1+\mu)\big[(1+\epsilon)a_1^2(j,k-1)\mathrm{tr}\big\{\tilde{X}(j,k-1)\big\}I\\&+(1+\epsilon^{-1})A_1(j,k-1)\tilde{X}(j,k-1)A_1^T(j,k-1)\big]\\&+(1+\mu^{-1})\big[(1+\epsilon)a_2^2(j-1,k)\mathrm{tr}\big\{\tilde{X}(j-1,k)\big\}I\\&+(1+\epsilon^{-1})A_2(j-1,k)\tilde{X}(j-1,k)A_2^T(j-1,k)\big].\end{aligned}$$

According to the above inequality and the property of trace for positive semi-definite matrix, for certain given integers $j,k\in[1\ \flat]$, one easily confirms that $X(j,k)\le\bar{X}(j,k)$ is true under the conditions $X(j,k-1)\le\bar{X}(j,k-1)$ and $X(j-1,k)\le\bar{X}(j-1,k)$. Bearing this fact in mind, it is not difficult to check that

$$X(j,k)\le\bar{X}(j,k)$$

holds for all $j,k\in[0\ \flat]$ by employing the initial states of (14) and the induction. Therefore, $\bar{X}(j,k)$ expressed by the recursion (14) is indeed a bound function on $X(j,k)$. ■

Now, let us consider the state estimation issue of the 2-D system (1) over sensor network (3). It is worth mentioning that the addressed distributed recursive filtering problem is much more complicated than the one-dimensional standard one or the conventional one with a sole sensor, it is by no means trivial to design proper gain matrices for each local filter with consideration of the topology information of the sensor networks. In this case, we would like to apply the stochastic analysis techniques to acquire certain UBs on the general error variances. The following result is readily accessible on the existence of UBs.

*Theorem 1:* Let $\mu$, $\epsilon$ be given positive scalars and $\bar{X}(j,k)$ be the solution to (14). For $j,k\in[0\ \flat]$ and $s\in\mathcal{V}$, consider system (1) with filter (4). The general error variances are bounded by

$$P_s^-(j,k)\le\Omega_s(j,k), \quad P_s(j,k)\le M_s(j,k) \quad (15)$$

where the matrix sequences $\Omega_s(j,k)$ and $M_s(j,k)$ with initial states $M_s(j,0)=P_s(j,0)$ and $M_s(0,k)=P_s(0,k)$ are the solutions of the following recursions

$$\begin{aligned}\Omega_s(j,k)=&(1+\mu)\hat{M}_{1,s}(j,k-1)+(1+\mu^{-1})\hat{M}_{2,s}(j-1,k)\\&+Q(j,k-1)+Q(j-1,k), \quad (16)\\M_s(j,k)=&(1+\mu)\Big\{\big[I-\bar{\gamma}_s(j,k)\left(G_s(j,k)+K_s(j,k)H_s\bar{I}\right)\\&\times C_s(j,k)\big]\Omega_s(j,k)\big[I-\bar{\gamma}_s(j,k)\big(G_s(j,k)\\&+K_s(j,k)H_s\bar{I}\big)C_s(j,k)\big]^T+\bar{\gamma}_s^2(j,k)K_s(j,k)\\&\times\left[\hat{H}_s(j,k)\circ\left(\bar{I}C_s(j,k)\Omega_s(j,k)C_s^T(j,k)\bar{I}^T\right)\right]\\&\times K_s^T(j,k)\Big\}+K_s(j,k)\big(\hat{H}_s(j,k)\circ\Upsilon_s(j,k)\\&+H_s\Upsilon_s(j,k)H_s^T\big)K_s^T(j,k)+G_s(j,k)\big(R_s(j,k)\\&+\hat{\gamma}_s(j,k)C_s(j,k)\bar{X}(j,k)C_s^T(j,k)\big)G_s^T(j,k) \quad (17)\end{aligned}$$

where

$$\begin{aligned}\hat{M}_{\hbar,s}(j,k)=&(1+\epsilon)a_\hbar^2(j,k)\mathrm{tr}\{M_s(j,k)\}I\\&+(1+\epsilon^{-1})A_\hbar(j,k)M_s(j,k)A_\hbar^T(j,k)\\\Upsilon_s(j,k)=&(1+\mu^{-1})\bar{\Upsilon}_s(j,k)+R(j,k)\\&+\hat{\Gamma}(j,k)\circ\left(\bar{C}(j,k)\bar{X}(j,k)\bar{C}^T(j,k)\right)\\\bar{\Upsilon}_s(j,k)=&\left(\bar{\Gamma}(j,k)\bar{C}(j,k)-\bar{\gamma}_s(j,k)\bar{I}C_s(j,k)\right)\bar{X}(j,k)\\&\times\left(\bar{\Gamma}(j,k)\bar{C}(j,k)-\bar{\gamma}_s(j,k)\bar{I}C_s(j,k)\right)^T.\end{aligned}$$

*Proof:* Subtracting (16) from (13) results in

$$\begin{aligned}&P_s^-(j,k)-\Omega_s(j,k)\\&\le(1+\mu)\big(\hat{P}_{1,s}(j,k-1)-\hat{M}_{1,s}(j,k-1)\big)\\&\quad+(1+\mu^{-1})\big(\hat{P}_{2,s}(j-1,k)-\hat{M}_{2,s}(j-1,k)\big). \quad (18)\end{aligned}$$

On the other hand, it is known from Lemma 3 that $X(j,k)$ is bounded by $\bar{X}(j,k)$. Then, according to the expressions of $\hat{\Theta}_s(j,k)$ in (10) and $\Upsilon_s(j,k)$, one has

$$\begin{aligned}&\hat{\Theta}_s(j,k)-\Upsilon_s(j,k)\\&\le\left(\bar{\Gamma}(j,k)\bar{C}(j,k)-\bar{\gamma}_s(j,k)\bar{I}C_s(j,k)\right)\left(X(j,k)-\bar{X}(j,k)\right)\\&\quad\times(1+\mu^{-1})\big(\bar{\Gamma}(j,k)\bar{C}(j,k)-\bar{\gamma}_s(j,k)\bar{I}C_s(j,k)\big)^T\\&\quad+\hat{\Gamma}(j,k)\circ\left(\bar{C}(j,k)(X(j,k)-\bar{X}(j,k))\bar{C}^T(j,k)\right)\le0.\end{aligned}$$

It follows from (9) and (17) that

$$\begin{aligned}&P_s(j,k)-M_s(j,k)\\&\le(1+\mu)\Big\{\left[I-\bar{\gamma}_s(j,k)\left(G_s(j,k)+K_s(j,k)H_s\bar{I}\right)C_s(j,k)\right]\end{aligned}$$

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$$
\begin{aligned}
&\times \big(P_s^-(j,k)-\Omega_s(j,k)\big)\big[I-\big(G_s(j,k)+K_s(j,k)H_s\bar{I}\big)\\
&\times \bar{\gamma}_s(j,k)C_s(j,k)\big]^T+\bar{\gamma}_s^2(j,k)K_s(j,k)\big[\hat{H}_s(j,k)\circ\big(\bar{I}\\
&\times C_s(j,k)\left(P_s^-(j,k)-\Omega_s(j,k)\right)C_s^T(j,k)\bar{I}^T\big)\big]K_s^T(j,k)\Big\}\\
&+K_s(j,k)\big[\hat{H}_s(j,k)\circ\big(\hat{\Theta}_s(j,k)-\Upsilon_s(j,k)\big)\\
&+H_s\big(\hat{\Theta}_s(j,k)-\Upsilon_s(j,k)\big)H_s^T\big]K_s^T(j,k)+\hat{\gamma}_s(j,k)\\
&\times G_s(j,k)C_s(j,k)\big(X(j,k)-\bar{X}(j,k)\big)C_s^T(j,k)G_s^T(j,k).
\end{aligned} \tag{19}
$$

To validate the conclusion of this theorem, we assume that (15) holds for $(j,k)\in\{(\imath,\jmath)|\imath,\jmath\in[1\ \flat];\ \imath+\jmath=\ell\}$ with $\ell\in[2\quad 2\flat-1]$. Then, the validity of (15) can be checked for $(j,k)\in\{(\imath,\jmath)|\imath,\jmath\in[1\ \flat];\ \imath+\jmath=\ell+1\}$ based on the induction. Actually, the introduced hypothesis brings about

$$\hat{P}_{\hbar,s}(j,k)-\hat{M}_{\hbar,s}(j,k)\le 0$$

for $(j,k)\in\{(\imath,\jmath)|\imath,\jmath\in[1\ \flat];\ \imath+\jmath=\ell\}$ with $\ell\in[2\quad 2\flat-1]$. This fact, in combination with (18), results in

$$P_s^-(j,k)-\Omega_s(j,k)\le 0$$

for $(j,k)\in\{(\imath,\jmath)|\imath,\jmath\in[1\ \flat];\ \imath+\jmath=\ell+1\}$, which further yields

$$P_s(j,k)-M_s(j,k)\le 0$$

with the help of (19). The proof is now complete. ■

### C. Design of the distributed filter

Note that the UBs of the error variances are presented in Theorem 1 in a recursive form. Once the initial states $M_s(j,0)$ and $M_s(0,k)$ are given, the analytical solutions to (16)-(17) can be iteratively calculated with aid of the filter gains, which can be appropriately designed so as to optimize the derived bound at each iteration. As for the distributed filter (4), its key characteristic lies in that the combined coupling data (collected from the neighboring nodes) are used to update the estimate value which, in turn, makes the design of the distributed filter challenging because of the sparsity of the connectivity for the sensor network. Keeping the above discussions in mind, the filtering problem needs to be effectively solved through applying some intriguing techniques.

For convenience of the subsequent developments, we introduce the following notations:

$$
\begin{aligned}
\gamma_{j,k}^{(s)}\triangleq&(1+\mu)\bar{\gamma}_s^2(j,k)\\
\mathcal{A}_s(j,k)\triangleq&(1+\mu)\bar{\gamma}_s(j,k)\big[I-\bar{\gamma}_s(j,k)K_s(j,k)H_s\bar{I}C_s(j,k)\big]\\
&\times\Omega_s(j,k)C_s^T(j,k)\\
\mathcal{B}_s(j,k)\triangleq&\gamma_{j,k}^{(s)}C_s(j,k)\Omega_s(j,k)C_s^T(j,k)+R_s(j,k)\\
&+\hat{\gamma}_s(j,k)C_s(j,k)\bar{X}(j,k)C_s^T(j,k)\\
\mathcal{C}_s(j,k)\triangleq&\big[I-\gamma_{j,k}^{(s)}\Omega_s(j,k)C_s^T(j,k)\mathcal{B}_s^{-1}(j,k)C_s(j,k)\big]\\
&\times(1+\mu)\bar{\gamma}_s(j,k)\Omega_s(j,k)C_s^T(j,k)\bar{I}^TH_s^T\\
\mathcal{D}_s(j,k)\triangleq&\gamma_{j,k}^{(s)}\Big[\hat{H}_s(j,k)\circ\big(\bar{I}C_s(j,k)\Omega_s(j,k)C_s^T(j,k)\bar{I}^T\big)\\
&+H_s\bar{I}C_s(j,k)\Omega_s(j,k)C_s^T(j,k)\bar{I}^TH_s^T\Big]-\mathcal{E}_s(j,k)\\
&+H_s\Upsilon_s(j,k)H_s^T+\hat{H}_s(j,k)\circ\Upsilon_s(j,k)\\
\mathcal{E}_s(j,k)\triangleq&\big(\gamma_{j,k}^{(s)}\big)^2H_s\bar{I}C_s(j,k)\Omega_s(j,k)C_s^T(j,k)\\
&\times\mathcal{B}_s^{-1}(j,k)C_s(j,k)\Omega_s(j,k)C_s^T(j,k)\bar{I}^TH_s^T.
\end{aligned}
$$

The following theorem provides an algorithm to explicitly design the distributed filter gains.

*Theorem 2:* For $j,k\in[1\ \flat]$ and $s,t\in\mathcal{V}$, the filter gains that minimize the UB $M_s(j,k)$ in the trace sense are given by

$$G_s(j,k)=\mathcal{A}_s(j,k)\mathcal{B}_s^{-1}(j,k) \tag{20}$$

$$K_{st}(j,k)=\begin{cases}(\bar{\mathcal{C}}_s(j,k)\bar{\mathcal{D}}_s^{-1}(j,k))^\sharp, & h_{st}>0\\ 0, & h_{st}=0\end{cases} \tag{21}$$

where $\bar{\mathcal{C}}_s(j,k)$ is the *simplified* matrix of $\mathcal{C}_s(j,k)$ by removing its zero columns, and $\bar{\mathcal{D}}_s(j,k)$ is the *simplified* one of $\mathcal{D}_s(j,k)$ by removing both its zero columns and zero rows. Here, the symbol $Z^\sharp$ denotes the corresponding sub-matrix extracted from a given matrix $Z$. In this case, the optimal UB is expressed by

$$
\begin{aligned}
M_s(j,k)=&(1+\mu)\big(\Omega_s(j,k)-\Omega_s(j,k)C_s^T(j,k)\mathcal{B}_s^{-1}(j,k)\\
&\times\gamma_{j,k}^{(s)}C_s(j,k)\Omega_s(j,k)\big)-K_s(j,k)\mathcal{C}_s^T(j,k).
\end{aligned} \tag{22}
$$

*Proof:* Recalling the derived bound in (17) and then taking the partial derivatives of $\mathrm{tr}\{M_s(j,k)\}$ with regard to $G_s(j,k)$ and $K_s(j,k)$, we arrive at

$$
\begin{aligned}
&\frac{\partial\mathrm{tr}\{M_s(j,k)\}}{\partial G_s(j,k)}\\
&=-2(1+\mu)\bar{\gamma}_s(j,k)\big[I-\bar{\gamma}_s(j,k)\left(G_s(j,k)+K_s(j,k)H_s\bar{I}\right)\\
&\quad\times C_s(j,k)\big]\Omega_s(j,k)C_s^T(j,k)+2G_s(j,k)\big(R_s(j,k)\\
&\quad+\hat{\gamma}_s(j,k)C_s(j,k)\bar{X}(j,k)C_s^T(j,k)\big)
\end{aligned} \tag{23}
$$

and

$$
\begin{aligned}
&\frac{\partial\mathrm{tr}\{M_s(j,k)\}}{\partial K_s(j,k)}\\
&=-2(1+\mu)\bar{\gamma}_s(j,k)\big[I-\bar{\gamma}_s(j,k)\left(G_s(j,k)+K_s(j,k)H_s\bar{I}\right)\\
&\quad\times C_s(j,k)\big]\Omega_s(j,k)C_s^T(j,k)\bar{I}^TH_s^T+2K_s(j,k)\big\{(1+\mu)\\
&\quad\times\bar{\gamma}_s^2(j,k)\hat{H}_s(j,k)\circ\big(\bar{I}C_s(j,k)\Omega_s(j,k)C_s^T(j,k)\bar{I}^T\big)\\
&\quad+H_s\Upsilon_s(j,k)H_s^T+\hat{H}_s(j,k)\circ\Upsilon_s(j,k)\big\}.
\end{aligned} \tag{24}
$$

To obtain the filter parameter $G_s(j,k)$, we set (23) to be zero and then have

$$
\begin{aligned}
&G_s(j,k)\left[\bar{\gamma}_s^2(j,k)(1+\mu)C_s(j,k)\Omega_s(j,k)C_s^T(j,k)\right.\\
&\quad\left.+\hat{\gamma}_s(j,k)C_s(j,k)\bar{X}(j,k)C_s^T(j,k)+R_s(j,k)\right]\\
&=(1+\mu)\bar{\gamma}_s(j,k)\left(I-\bar{\gamma}_s(j,k)K_s(j,k)H_s\bar{I}C_s(j,k)\right)\\
&\quad\times\Omega_s(j,k)C_s^T(j,k)
\end{aligned}
$$

which ascertains the design of $G_s(j,k)$ as in (20).

Similarly, letting (24) be zero and further applying the relationship (20) to (24), one has

$$
\begin{aligned}
&K_s(j,k)\Big\{\gamma_{j,k}^{(s)}\hat{H}_s(j,k)\circ\big(\bar{I}C_s(j,k)\Omega_s(j,k)C_s^T(j,k)\bar{I}^T\big)\\
&\quad+H_s\Upsilon_s(j,k)H_s^T+\hat{H}_s(j,k)\circ\Upsilon_s(j,k)\\
&\quad+\gamma_{j,k}^{(s)}H_s\bar{I}C_s(j,k)\Omega_s(j,k)\big(\Omega_s^{-1}(j,k)-\gamma_{j,k}^{(s)}C_s^T(j,k)\\
&\quad\times\mathcal{B}_s^{-1}(j,k)C_s(j,k)\big)\Omega_s(j,k)C_s^T(j,k)\bar{I}^TH_s^T\Big\}
\end{aligned}
$$

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$$= \left[I - \gamma_{j,k}^{(s)}\Omega_s(j,k)C_s^T(j,k)\mathcal{B}_s^{-1}(j,k)C_s(j,k)\right] \times (1+\mu)\bar{\gamma}_s(j,k)\Omega_s(j,k)C_s^T(j,k)\bar{I}^T H_s^T$$

which can be rewritten as

$$K_s(j,k)\mathcal{D}_s(j,k) = \mathcal{C}_s(j,k). \quad (25)$$

It should be pointed out that the structures of $H_s$ and $\hat{H}_s(j,k)$ (reflecting, respectively, the information of the deterministic couplings and the statistics of the random coupling fluctuations associated with sensor $s$) may lead to the singularity of matrix $\mathcal{D}_s(j,k)$. In this scenario, $K_s(j,k)$ cannot be obtained directly from (25). To deal with such kind of sparsity issue of network communication, the so-called *matrix simplification* technique [28] is utilized here to simplify $\mathcal{C}_s(j,k)$ and $\mathcal{D}_s(j,k)$, and it can then be derived that

$$K_{st}(j,k) = \begin{cases} (\bar{\mathcal{C}}_s(j,k)\bar{\mathcal{D}}_s^{-1}(j,k))^\sharp, & h_{st} > 0 \\ 0, & h_{st} = 0. \end{cases}$$

The invertibility of matrix $\bar{\mathcal{D}}_s(j,k)$ is confirmed in Appendix A, which shows that (21) is well-defined. Finally, substituting (20)-(21) into (17), it is calculated via some routine matrix manipulations that

$$M_s(j,k) = (1+\mu)\big(\Omega_s(j,k) - \Omega_s(j,k)C_s^T(j,k)\mathcal{B}_s^{-1}(j,k) \times \gamma_{j,k}^{(s)}C_s^T(j,k)\Omega_s(j,k)\big) - K_s(j,k)\mathcal{C}_s^T(j,k)$$

which accords with (22) (for which the detailed derivations can be found in Appendix B). The proof is completed. ■

*Remark 4:* Due to the sparsity of the network topology, matrix $\mathcal{D}_s(j,k)$ may be singular, and this gives rise to certain obstacle in directly calculating the filer gain $K_s(j,k)$. To cope with such an issue, in Theorem 2, a matrix simplification technique has been adopted by taking advantages of the topological structure of the sensor network.

*Remark 5:* By far, the distributed recursive filtering problem has been addressed for the considered 2-D shift-varying systems over sensor networks. Both the degraded measurements and the random couplings have been introduced for depicting the possible sensor failures and random couplings. By means of two sets of recursive matrix equalities, an UB of the general error covariance has been established in Theorem 1 for each sensor node. Then, the filter gains have been determined in Theorem 2 to optimize the obtained UB at each step. It is apparent from (22) that all the information concerning the system model (i.e., the degradation coefficients, the sensor network topology, and the coupling perturbations) is reflected in the filter design algorithm that has direct influences on the filtering performance.

## IV. Numerical Example

This section provides an illustrated example to show the effectiveness of the filtering method proposed in the main results.

Consider the 2-D system (1) defined on a finite horizon $[0\ \flat]$ with $\flat = 40$. The nonlinear functions in (1) are given by:

$$f_\hbar((j,k),x(j,k)) = A_\hbar(j,k)x(j,k) + F_\hbar((j,k),x(j,k))$$

with $\hbar = 1,2$, where

$$A_1(j,k) = \begin{bmatrix} 0.28 & 0.1-0.1\sin(j) \\ -0.1 & 0.3 \end{bmatrix},$$
$$A_2(j,k) = \begin{bmatrix} 0.2 & 0.1-0.1\cos(k) \\ -0.2 & 0.25 \end{bmatrix},$$
$$F_1((j,k),x(j,k)) = 0.016\sin(x(j,k)),$$
$$F_2((j,k),x(j,k)) = 0.015\sin(x(j,k)).$$

It is obvious to know that the above nonlinear functions meet condition (2) with $a_1(j,k) = 0.016$ and $a_2(j,k) = 0.015$.

The sensor network is described by a directed and weighted graph $\mathcal{G} = (\mathcal{V},\mathcal{E},\mathcal{A})$, where $\mathcal{V} = \{1,2,3,4,5,6,7,8\}$, $\mathcal{E} = \{(1,6),(1,8),(2,4),(3,6),(4,7),(5,1),(6,2),(7,5),(8,3)\}$, and $\mathcal{A} = (h_{st})_{8\times 8}$ with $h_{st} = 1$ if and only if $(s,t) \in \mathcal{E}$. In the case of $h_{st} = 1$ $(s,t \in [1\ 8])$, the random variable $\Delta h_{st}(j,k)$ is assumed to obey a uniform distribution over the interval $[-0.6, 0.6]$, and the second-order moment of $\Delta h_{st}(j,k)$ is thus confined to a scalar $\pi_{st}(j,k) = 0.12$. Moreover, the matrix parameters in (3) are given by

$$C_1(j,k) = [0.3 \quad 1.2], \quad C_2(j,k) = [0.8 \quad 0.15\sin(j)],$$
$$C_3(j,k) = [0.45 \quad 1.6], \quad C_4(j,k) = [0.8+0.1e^{-j} \quad -1],$$
$$C_5(j,k) = [1 \quad 1.5], \quad C_6(j,k) = [0.5-0.1\sin(k) \quad 0.35],$$
$$C_7(j,k) = [-0.6 \quad 0.1\cos(2k)], \quad C_8(j,k) = [2 \quad 0.2\cos(j)].$$

The measurement degradation is depicted by the random variables $\gamma_s(j,k)$ $(s \in [1\ 8])$ whose probability mass functions are set to be

$$p_l(j,k) = \begin{cases} 0.05, & l = 0 \\ 0.1, & l = 0.5 \\ 0.85, & l = 1. \end{cases}$$

It is easy to compute $\bar{\gamma}_s(j,k) = 0.9$ and $\hat{\gamma}_s(j,k) = 0.065$. The process and measurement noises $w(j,k)$ and $v_s(j,k)$ are modeled by mutually uncorrelated Gaussian sequences with respective variances $Q(j,k) = 0.09I$ and $R_s(j,k) = 0.16$. For simulation purpose, the initial conditions of (1) are given as $\eta_1(j) = \eta_2(k) = [0\ 0]^T$ and $X(j,0) = X(0,k) = 0.01I$, which result in $P_s(j,0) = P_s(0,k) = 0.01I$ for all $s \in [1\ 8]$. The scaling scalars are chosen to be $\mu = 0.5$ and $\epsilon = 2$.

For notational brevity, denote $e_s^{(\imath)}(j,k)$ as the $\imath$-th element of $e_s(j,k)$ and $\mathrm{MSE}_s(j,k)$ as the mean square error with

$$\mathrm{MSE}_s(j,k) \triangleq \frac{1}{M}\sum_{\ell=1}^{M} e_s^T(j,k)e_s(j,k).$$

where the index $\ell$ infers the $\ell$-th individual run, and the index $M$ is the number of the independent repeated runs.

With the above parameters, the UB and the filter gains can be iteratively calculated from Theorems 1 and 2. Only part of the simulation results are presented for the first sensor node due to space limit. Specifically, Figs. 2-3 plot the error trajectories $e_1^{(1)}(j,k)$ and $e_1^{(2)}(j,k)$, respectively. Fig. 4 shows the trace of the UB $M_1(j,k)$ and Fig. 5 displays the mean square error $\mathrm{MSE}_1(j,k)$ averaged over 500 independent runs. Figs. 2-5 illustrate that the developed recursive filter works well.

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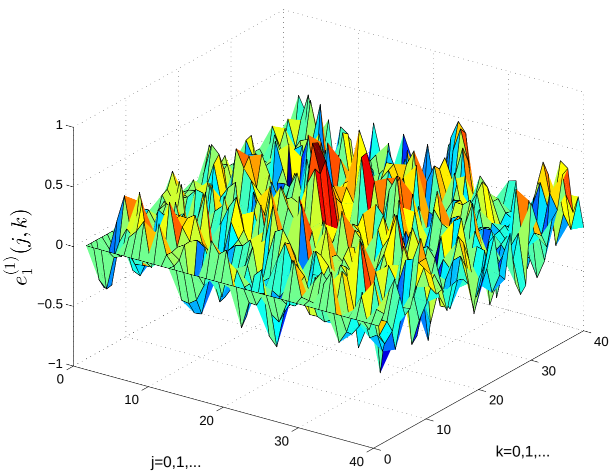


Fig. 2. Trajectory of the estimation error $e_1^{(1)}(j,k)$.

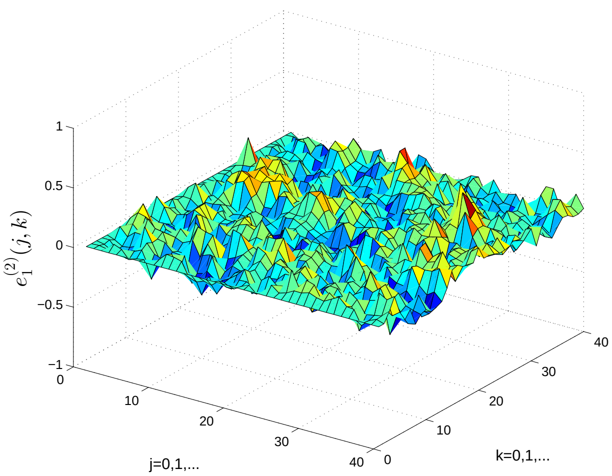


Fig. 3. Trajectory of the estimation error $e_1^{(2)}(j,k)$.

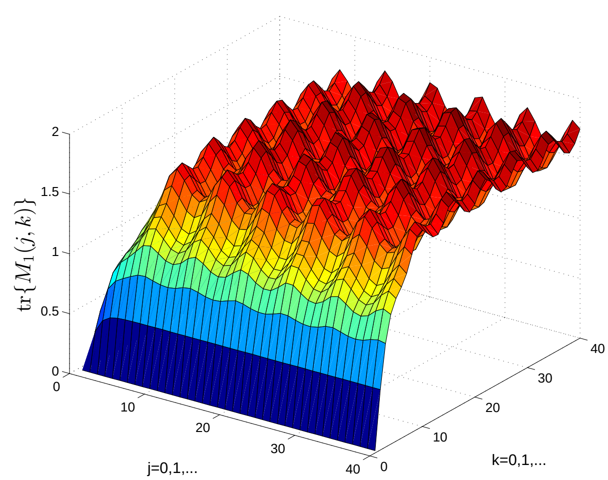


Fig. 4. Trace of the UB $M_1(j,k)$.

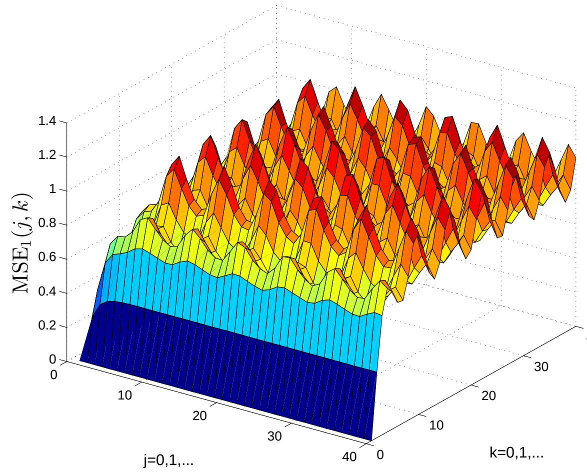


Fig. 5. The mean square error $\mathrm{MSE}_1(j,k)$.

To reflect the effect of the noise amplitudes, the noise covariances are reset as $Q(j,k) = 0.16$ and $R_s(j,k) = 0.81$ without changing other parameters. The corresponding simulation result regarding the trace of the UB $M_1(j,k)$ is given in Fig. 6. Comparing the trajectory of $\mathrm{tr}\{M_1(j,k)\}$ with the one in Fig. 4, we can see that the obtained UB becomes bigger with larger noise amplitudes.

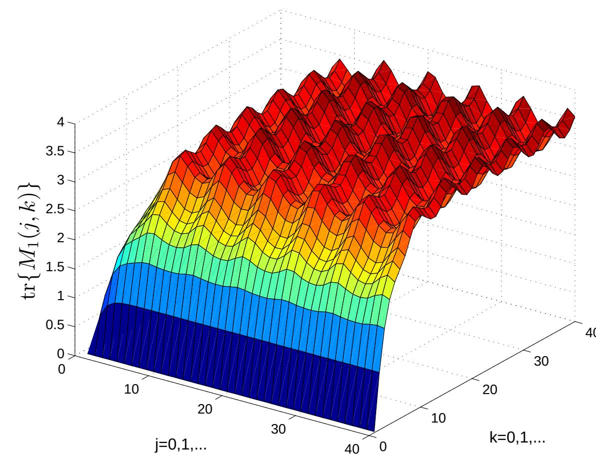


Fig. 6. Trace of $M_1(j,k)$ with $Q(j,k) = 0.16$ and $R_s(j,k) = 0.81$.

To evaluate the influence of the measurement degradation, the probability mass function is reset as:

$$p_l(j,k) = \begin{cases} 0.13, & l = 0 \\ 0.74, & l = 0.5 \\ 0.13, & l = 1. \end{cases}$$

In this case, it is easily calculated that $\bar{\gamma}_s(j,k) = 0.5$ and $\hat{\gamma}_s(j,k) = 0.065$. Accordingly, Fig. 7 presents the simulation result regarding the trace of the UB $M_1(j,k)$ with $\bar{\gamma}_s(j,k) = 0.5$. In comparison with Fig. 4, it is concluded that a larger value of $\bar{\gamma}_s(j,k)$ contributes to the tighter trace of the UB,

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namely, a better filtering performance because more useful measurements are available in the statistical sense.

Apart from the examination on the measurement degradation, the effect of the random perturbation couplings on the filtering performance is also evaluated. Let us first introduce the ideal case where only the deterministic couplings contribute to the process of data exchange between sensors without considering the random coupling perturbations (namely, $\Delta h_{st}(j,k) \equiv 0$ for all $s,t \in [1\ 8]$). Remaining all the other parameters, the corresponding UB is denoted as $M_s^{\dagger}(j,k)$. The comparison between the traces of UBs $M_1(j,k)$ and $M_1^{\dagger}(j,k)$ is depicted in Fig. 8. Next, without changing the remainder parameters, the random variable $\Delta h_{st}(j,k)$ is further reset to be uniformly distributed over the interval $[-0.9, 0.9]$, and the bound on the second-order moment of $\Delta h_{st}(j,k)$ is given as $\pi_{st}(j,k) = 0.27$. In this case, the obtained UB is denoted as $M_s^{\ddagger}(j,k)$, and the difference between $\mathrm{tr}\{M_1^{\ddagger}(j,k)\}$ and $\mathrm{tr}\{M_1(j,k)\}$ is shown in Fig. 9. It is revealed from Figs. 8-9 that the random perturbation couplings will degrade the filtering performance. To be more specific, the best filtering performance is achieved without any coupling perturbations, and the increased intensity of random perturbation couplings leads to the degradation of the filtering performance.

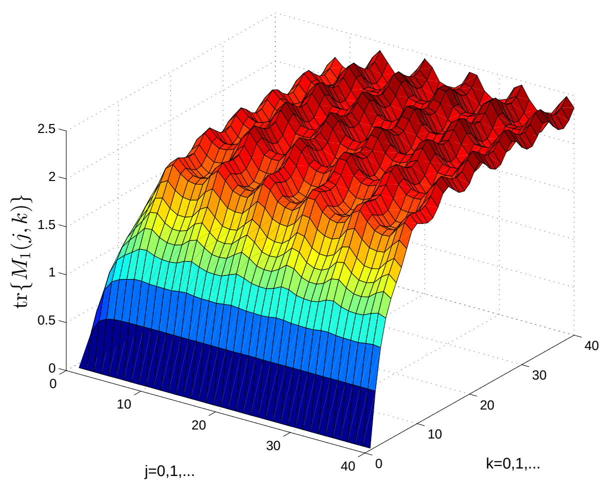


Fig. 7. Trace of $M_1(j,k)$ with $\bar{\gamma}_s(j,k) = 0.5$.

## V. Conclusion

This paper has discussed the recursive filtering problem for 2-D nonlinear system over sensor network in the presence of degraded measurements and coupling perturbations. Phenomena of measurement degradations and random couplings are governed by random variables satisfying certain probability distributions. A novel distributed filter has been proposed in 2-D case to estimate the system state with a guaranteed filtering performance. Theoretical results have been established to construct certain UBs on the general error variances. In addition, the explicit expression of the tightest bound has been derived and the desired gain matrices have been carefully designed by matrix analysis techniques and mathematical induction.

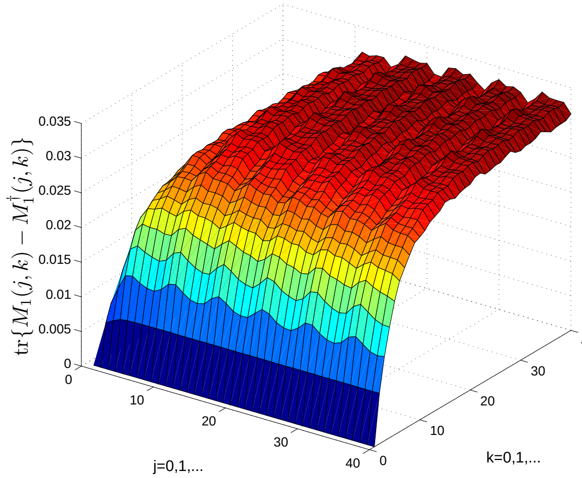


Fig. 8. The difference between $\mathrm{tr}\{M_1(j,k)\}$ and $\mathrm{tr}\{M_1^{\dagger}(j,k)\}$.

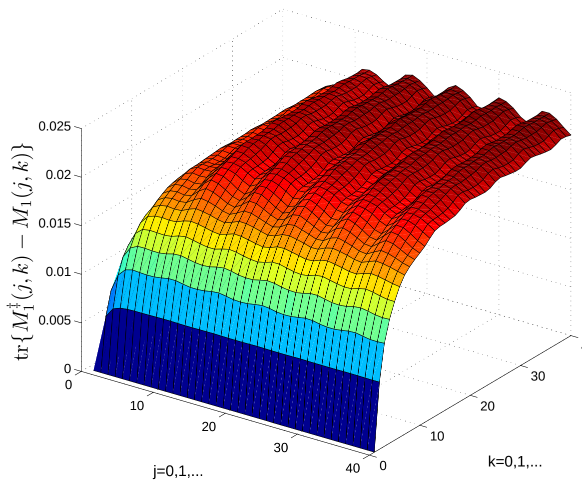


Fig. 9. The difference between $\mathrm{tr}\{M_1^{\ddagger}(j,k)\}$ and $\mathrm{tr}\{M_1(j,k)\}$.

The simulation example has also been presented to clarify the feasibility of our proposed filtering strategy. Our future research topics would be the extensions of the main results derived here to other 2-D systems with more complicated dynamics, such as the state-saturated filtering problem with cyber-attacks [11], [39] and the protocol-based state estimation problems [8], [46].

## Appendix

### A. Invertibility of $\bar{\mathcal{D}}_s(j,k)$

Noting that matrices $Q(j,k)$ and $R_s(j,k)$ are positive definite, one obtains $\Omega_s(j,k) > 0$ based on the evolution of $\Omega_s(j,k)$ in (16). In this case, the following matrix

$$\bar{\Omega}_s(j,k) \triangleq \Omega_s^{-1}(j,k) + \gamma_{j,k}^{(s)} C_s^T(j,k)\big(R_s(j,k) + \hat{\gamma}_s(j,k) C_s(j,k) \bar{X}(j,k) C_s^T(j,k)\big)^{-1} C_s(j,k)$$

is well-posed and positive definite. According to the well-known matrix inversion lemma, it can be derived that

$$\begin{aligned}\bar{\Omega}_s^{-1}(j,k) =&\Omega_s(j,k)-\gamma_{j,k}^{(s)}\Omega_s(j,k)C_s^T(j,k)\Big[R_s(j,k)\\&+\hat{\gamma}_s(j,k)C_s(j,k)\bar{X}(j,k)C_s^T(j,k)+\gamma_{j,k}^{(s)}\\&\times C_s(j,k)\Omega_s(j,k)C_s^T(j,k)\Big]^{-1}C_s(j,k)\Omega_s(j,k)\\=&\Omega_s(j,k)-\gamma_{j,k}^{(s)}\Omega_s(j,k)C_s^T(j,k)\mathcal{B}_s^{-1}(j,k)\\&\times C_s(j,k)\Omega_s(j,k)>0.\end{aligned}\tag{26}$$

Combing (26) with the definition of $\mathcal{E}_s(j,k)$, one has

$$\begin{aligned}&\gamma_{j,k}^{(s)}H_s\bar{I}C_s(j,k)\Big[\Omega_s(j,k)-\gamma_{j,k}^{(s)}\Omega_s(j,k)C_s^T(j,k)\mathcal{B}_s^{-1}(j,k)\\&\quad\times C_s(j,k)\Omega_s(j,k)\Big]C_s^T(j,k)\bar{I}^TH_s^T\\&=\gamma_{j,k}^{(s)}H_s\bar{I}C_s(j,k)\Omega_s(j,k)C_s^T(j,k)\bar{I}^TH_s^T-\mathcal{E}_s(j,k)\geq 0\end{aligned}$$

which means that the sum of the second and third terms of $\mathcal{D}_s(j,k)$ are positive semi-definite. Recall the fact that $\bar{\mathcal{D}}_s(j,k)$ is the simplified one of $\mathcal{D}_s(j,k)$ by removing both its zero columns and zero rows. The invertibility of $\bar{\mathcal{D}}_s(j,k)$ can now be verified from (26) and the sparsity of matrices $H_s$ and $\hat{H}_s(j,k)$.

## B. Proof of (22)

Recall the definitions of $\mathcal{A}_s(j,k)$, $\mathcal{B}_s(j,k)$, $\mathcal{C}_s(j,k)$ and $\mathcal{E}_s(j,k)$. We derive from the filter gain $G_s(j,k)$ determined in (20) that

$$\begin{aligned}&\mathcal{A}_s(j,k)\mathcal{B}_s^{-1}(j,k)\mathcal{A}_s^T(j,k)\\&=(1+\mu)\gamma_{j,k}^{(s)}\Omega_s(j,k)C_s^T(j,k)\mathcal{B}_s^{-1}(j,k)C_s(j,k)\Omega_s(j,k)\\&\quad+K_s(j,k)\mathcal{E}_s(j,k)K_s^T(j,k)-(1+\mu)^2\bar{\gamma}_s^3(j,k)K_s(j,k)H_s\\&\quad\times\bar{I}C_s(j,k)\Omega_s(j,k)C_s^T(j,k)\mathcal{B}_s^{-1}(j,k)C_s(j,k)\Omega_s(j,k)\\&\quad-(1+\mu)^2\bar{\gamma}_s^3(j,k)\Omega_s(j,k)C_s^T(j,k)\mathcal{B}_s^{-1}(j,k)\\&\quad\times\big(K_s(j,k)H_s\bar{I}C_s(j,k)\Omega_s(j,k)C_s^T(j,k)\big)^T,\end{aligned}\tag{27}$$

$$\begin{aligned}&(1+\mu)\bar{\gamma}_s(j,k)\Omega_s(j,k)\big[\big(G_s(j,k)+K_s(j,k)H_s\bar{I}\big)C_s(j,k)\big]^T\\&=(1+\mu)\bar{\gamma}_s(j,k)\Omega_s(j,k)\big[\mathcal{A}_s(j,k)\mathcal{B}_s^{-1}(j,k)C_s(j,k)\\&\quad+K_s(j,k)H_s\bar{I}C_s(j,k)\big]^T\\&=(1+\mu)\gamma_{j,k}^{(s)}\Omega_s(j,k)C_s^T(j,k)\mathcal{B}_s^{-1}(j,k)C_s(j,k)\Omega_s(j,k)\\&\quad+\mathcal{C}_s(j,k)K_s^T(j,k),\end{aligned}\tag{28}$$

$$\begin{aligned}&\gamma_{j,k}^{(s)}G_s(j,k)C_s(j,k)\Omega_s(j,k)(K_s(j,k)H_s\bar{I}C_s(j,k))^T\\&=\gamma_{j,k}^{(s)}\mathcal{A}_s(j,k)\mathcal{B}_s^{-1}(j,k)C_s(j,k)\Omega_s(j,k)\\&\quad\times(K_s(j,k)H_s\bar{I}C_s(j,k))^T\\&=(1+\mu)^2\bar{\gamma}_s^3(j,k)\Omega_s(j,k)C_s^T(j,k)\mathcal{B}_s^{-1}(j,k)\\&\quad\times\big(K_s(j,k)H_s\bar{I}C_s(j,k)\Omega_s(j,k)C_s^T(j,k)\big)^T\\&\quad-K_s(j,k)\mathcal{E}_s(j,k)K_s^T(j,k).\end{aligned}\tag{29}$$

Based on (17), (25) and (27)-(29), the following relationship for $M_s(j,k)$ can be established:

$$\begin{aligned}&M_s(j,k)\\&=(1+\mu)\Omega_s(j,k)+G_s(j,k)\mathcal{B}_s(j,k)G_s^T(j,k)\\&\quad+K_s(j,k)\Big\{\gamma_{j,k}^{(s)}H_s\bar{I}C_s(j,k)\Omega_s(j,k)C_s^T(j,k)\bar{I}^TH_s^T\\&\quad+\gamma_{j,k}^{(s)}\hat{H}_s(j,k)\circ\big(\bar{I}C_s(j,k)\Omega_s(j,k)C_s^T(j,k)\bar{I}^T\big)+H_s\\&\quad\times\Upsilon_s(j,k)H_s^T+\hat{H}_s(j,k)\circ\Upsilon_s(j,k)\Big\}K_s^T(j,k)-(1+\mu)\\&\quad\times\bar{\gamma}_s(j,k)\Big\{\Omega_s(j,k)\big[\big(G_s(j,k)+K_s(j,k)H_s\bar{I}\big)C_s(j,k)\big]^T\\&\quad+\big[\big(G_s(j,k)+K_s(j,k)H_s\bar{I}\big)C_s(j,k)\big]\Omega_s(j,k)\Big\}\\&\quad+\gamma_{j,k}^{(s)}G_s(j,k)C_s(j,k)\Omega_s(j,k)\big(K_s(j,k)H_s\bar{I}C_s(j,k)\big)^T\\&\quad+\gamma_{j,k}^{(s)}K_s(j,k)H_s\bar{I}C_s(j,k)\Omega_s(j,k)C_s^T(j,k)G_s^T(j,k)\\&=(1+\mu)\Omega_s(j,k)+\mathcal{A}_s(j,k)\mathcal{B}_s^{-1}(j,k)\mathcal{A}_s^T(j,k)+K_s(j,k)\\&\quad\times(\mathcal{D}_s(j,k)+\mathcal{E}_s(j,k))K_s^T(j,k)-(1+\mu)\bar{\gamma}_s(j,k)\\&\quad\times\Big\{\Omega_s(j,k)\big[\big(G_s(j,k)+K_s(j,k)H_s\bar{I}\big)C_s(j,k)\big]^T\\&\quad+\big[\big(G_s(j,k)+K_s(j,k)H_s\bar{I}\big)C_s(j,k)\big]\Omega_s(j,k)\Big\}\\&\quad+\gamma_{j,k}^{(s)}G_s(j,k)C_s(j,k)\Omega_s(j,k)\big(K_s(j,k)H_s\bar{I}C_s(j,k)\big)^T\\&\quad+\gamma_{j,k}^{(s)}K_s(j,k)H_s\bar{I}C_s(j,k)\Omega_s(j,k)C_s^T(j,k)G_s^T(j,k)\\&=(1+\mu)\Omega_s(j,k)-(1+\mu)\gamma_{j,k}^{(s)}\Omega_s(j,k)C_s^T(j,k)\mathcal{B}_s^{-1}(j,k)\\&\quad\times C_s^T(j,k)\Omega_s(j,k)+K_s(j,k)\mathcal{D}_s(j,k)K_s^T(j,k)\\&\quad-K_s(j,k)\mathcal{C}_s^T(j,k)-\mathcal{C}_s(j,k)K_s^T(j,k)\\&=(1+\mu)\big(\Omega_s(j,k)-\gamma_{j,k}^{(s)}\Omega_s(j,k)C_s^T(j,k)\mathcal{B}_s^{-1}(j,k)\\&\quad\times C_s^T(j,k)\Omega_s(j,k)\big)-K_s(j,k)\mathcal{C}_s^T(j,k).\end{aligned}$$

Consequently, the desired UB is given as (22) under the designed filter gains (20) and (21).

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