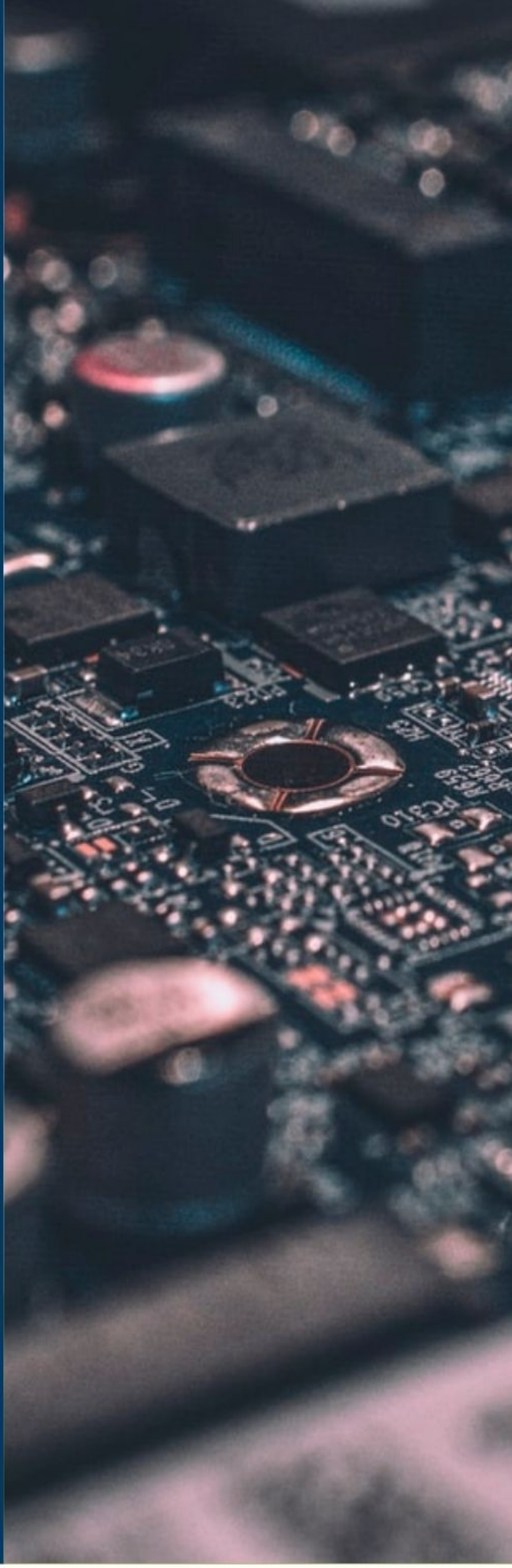

Simple PDF of the sum and maximum of non-identical fluctuating two-ray variates

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In this letter, probability density function (PDF) of the sum and maximum of non-identical fluctuating two-ray (FTR) variates are derived in terms of the power and exponential functions. To this effect, novel mathematically tractable expressions of the outage probability (OP) and average bit error probability (ABEP) for both the maximal ratio combining (MRC) and selection combining (SC) diversity receptions are provided. The numerical results are affirmed by the Monte-Carlo simulations for different scenarios.

Introduction: The sum and maximum of the random variables (RVs) have been widely used to analyze the performance of the wireless communication systems with maximal ratio combining (MRC) and selection combining (SC) diversity techniques, respectively [1]. Hence, the probability density function (PDF) of the sum of independent and non-identically distributed (i.n.d.) Nakagami-*m* variates was derived in [2] with applications to the outage probability (OP) and average bit error probability (ABEP) of the MRC receivers. The PDF, cumulative distribution function (CDF), and moment generating function (MGF) of the maximum of i.n.d. Nakagami-*m* RVs were given in [3].

Recently, the fluctuating two-ray (FTR) distribution has been widely employed in the literature. This is because, when it is used in modelling the fading channel of the millimeter wave (mmWave) communications, it gives more closer results to the practical measurements than the conventional models [4, 5]. Accordingly, the OP and ABEP of the MRC diversity reception were analysed in [6] via using the statistics of the sum of squared FTR RVs. However, both the PDF and CDF of [6] are evaluated numerically. This is because they include the confluent multivariate hypergeometric function, $\Phi_2^{(L)}(\cdot)$, that is not available in the popular software packages. Furthermore, the PDF of the maximum of arbitrarily distributed FTR RVs was derived in terms of the multivariate Fox's *H*-function [7], whereas the performance of identical dual-branch SC diversity was investigated in [8].

Motivated by the above considerations, the exact and asymptotic at high average SNR values of the PDF of both the sum and maximum of i.n.d. FTR variates are provided. In contrast to [6, 7], our expressions are given in terms of power and exponential functions. Based on the derived results, novel simple accurate mathematically tractable expressions of the exact and asymptotic of the OP, ABEP, and EC of multiple-input single-output (MISO) systems are obtained.

Statistics of ftr fading channel: The CDF of the instantaneous SNR at *i*th receiver, γ_i, over FTR fading channel is given by [5, equation (7)]

$$F_{\gamma_i}(\gamma) = \sum_{j_i=1}^{\infty} \frac{\Psi_{j_i}}{\Gamma(j_i)(2\sigma_i^2)^{-j_i}} \mathcal{G}\left(j_i, \frac{\gamma}{2\sigma_i^2}\right) \tag{1}$$

where $\Psi_{j_i} = \frac{m_i^{m_i} K_i^{j_i-1} d_{j_i}}{\Gamma(m_i)(2\sigma_i^2)^{j_i}\Gamma(j_i)}$, m_i is the fading severity index, K_i is the average power ratio between the dominant component and the scattering multipath, $2\sigma_i^2 = \bar{\gamma}_i/(1+K_i)$ with $\bar{\gamma}_i$ is the received average SNR, $\mathcal{G}(\cdot,\cdot)$ is the lower incomplete gamma function [9, equation (8.350.1)], and the coefficient d_{j_i} is expressed as [10, equation (19)]

$$d_{j_i} = \sum_{k_i=0}^{j_i-1} \binom{j_i-1}{k_i} \sum_{l_i=0}^{k_i} \binom{k_i}{l_i} \frac{\Gamma(j_i+m_i+2l_i-k_i-1)}{(m_i+K_i)^{j_i+m_i+2l_i-k_i-1}} \times \left(\frac{\Delta_i}{2}\right)^{2l_i} K_i^{2l_i-k_i} (-1)^{2l_i-k_i} R_{j_i+m_i-1}^{k_i-2l_i}\left(\left[\frac{K_i\Delta_i}{m_i+K_i}\right]^2\right) \tag{2}$$

where $\Delta_i \in [0, 1]$ is the similarity of two dominant waves and $R_\upsilon^\mu(z)$ is defined in [10], equation (20)].

The asymptotic of the CDF in (1) at high average SNR values is expressed as [11, equation (18)]

$$F_{\gamma_i}(\gamma) \simeq \frac{m_i(1+K_i)d_{j_i=1}}{\Gamma(m_i)\bar{\gamma}_i}\gamma \tag{3}$$

where $d_{j_i=1}$ is the value of d_{j_i} in (2) at $j_i = 1$.

The MGF of γ_i is derived as [6, equation (4)]

$$\mathcal{M}_{\gamma_i}(s) = \sum_{j_i=1}^{\infty} \frac{\Psi_{j_i}(2\sigma_i^2)^{j_i}}{(1-2\sigma_i^2 s)^{j_i}} \tag{4}$$

Substituting (3) into $\mathcal{M}_\gamma(s) = s\mathcal{L}[F_\gamma(x); s]$, where $\mathcal{L}[\cdot]$ stands for Laplace transform and recalling [9, equation (3.381.4)], the asymptotic of the MGF at $\bar{\gamma}_i \to \infty$ is deduced as

$$\mathcal{M}_{\gamma_i}(s) \simeq \frac{m_i(1+K_i)d_{j_i=1}}{\Gamma(m_i)\bar{\gamma}_i s} \tag{5}$$

Pdf of the sum of FTR variates: Let $\gamma_i \sim \mathcal{FTR}(m_i, K_i, \Delta_i, 2\sigma_i^2)$, for $i = 1, \ldots, L$ where L is the number of the RVs, following i.n.d. FTR distribution. Then, substituting (4) into $\mathcal{M}_{\gamma^{Sum}}(s) = \prod_{i=1}^{L} \mathcal{M}_{\gamma_i}(s)$, the MGF of $\gamma^{Sum} = \sum_{i=1}^{L}\gamma_i$ is written as

$$\mathcal{M}_{\gamma^{Sum}}(s) = \sum_{j_1,\ldots,j_L=1}^{\infty} \Theta_{j_i} \prod_{i=1}^{L}\left(s - \frac{1}{2\sigma_i^2}\right)^{-j_i} \tag{6}$$

Inserting (6) in $f_{\gamma^{Sum}}(\gamma) = \mathcal{L}^{-1}[\mathcal{M}_{\gamma^{Sum}}(s); \gamma]$, where $\mathcal{L}^{-1}[\cdot]$ denotes the inverse Laplace transform. Thereafter, following the same steps as in [12] to calculate $\mathcal{L}^{-1}[\prod_{i=1}^{L}(s-\frac{1}{2\sigma_i^2})^{-j_i}; \gamma]$, the PDF of $\gamma^{Sum} = \sum_{i=1}^{L}\gamma_i$ is obtained as

$$f_{\gamma^{Sum}}(\gamma) = \sum_{j_1,\ldots,j_L=1}^{\infty} \Theta_{j_i} \sum_{i=1}^{L}\sum_{l=1}^{j_i} \frac{(-1)^l \Omega_{i_l}}{\Gamma(l)} \gamma^{l-1} e^{-\frac{\gamma}{2\sigma_i^2}} \tag{7}$$

where $\Theta_{j_i} = \prod_{i=1}^{L} \frac{\Psi_{j_i}}{(-1)^{j_i}}$ and

$$\Omega_{i_l} = \left[\sum_{r_1=0}^{j_i-l-1}\binom{j_i-l-1}{r_1}\Xi_i^{(j_i-l-r_1-1)}\cdots \sum_{r_L=0}^{r_{L-1}-1}\binom{r_{L-1}-1}{r_L}\Xi_i^{(r_{L-1}-r_L-1)}\right]\frac{\chi_i}{(j_i-l)!} \tag{8}$$

with $\Xi_l^{(n)} = (-1)^{n+1} n! \sum_{k=1 k\neq i}^{L} j_i(\frac{\sigma_{j_k}^2-\sigma_{j_i}^2}{2\sigma_{j_i}^2\sigma_{j_k}^2})^{-(n+1)}$ and $\chi_i = \prod_{k=1 k\neq i}^{L}(\frac{\sigma_{j_k}^2-\sigma_{j_i}^2}{2\sigma_{j_i}^2\sigma_{j_k}^2})^{-j_k}$.

When $\bar{\gamma}_i \to \infty$, the asymptotic of (7) is obtained via inserting (5) in $f_{\gamma^{Sum}}(\gamma) = \mathcal{L}^{-1}[\mathcal{M}_{\gamma^{Sum}}(s); \gamma]$ and invoking [13, equation (5.4.1)]

$$f_{\gamma^{Sum}}(\gamma) \simeq \left(\prod_{i=1}^{L} \frac{m_i(1+K_i)d_{j_i=1}}{\Gamma(m_i)\bar{\gamma}_i}\right)\frac{\gamma^{L-1}}{\Gamma(L)} \tag{9}$$

Pdf of the maximum of FTR variates: The CDF of $\gamma^{Max} = \max\{\gamma_1, \gamma_2, \ldots, \gamma_L\}$ of i.n.d. FTR variates can be computed via plugging (1) in $F_{\gamma^{Max}}(\gamma) = \prod_{i=1}^{L} F_{\gamma_i}(\gamma)$ and utilizing [9, equation (8.352.6)]. Thus, this yields

$$F_{\gamma^{Max}}(\gamma) = \sum_{j_1,\ldots,j_L=1}^{\infty} \Lambda_{j_i} \prod_{i=1}^{L} \left[1 - e^{-\frac{\gamma}{2\sigma_i^2}} \sum_{n_i=0}^{j_i-1} \frac{\gamma^{n_i}}{(2\sigma_i^2)^{n_i} n_i!} \right] \tag{10}$$

Employing equation (30) from [3], (10) can be rewritten as

$$F_{\gamma^{Max}}(\gamma) = \sum_{j_1,\ldots,j_L=1}^{\infty} \Lambda_{j_i} \mho_{j_i} \gamma^{\vartheta_{j_i}} e^{-\varphi_t \gamma} \tag{11}$$

where $\Lambda_{j_i} = \prod_{i=1}^{L} \frac{\Psi_{j_i}}{(2\sigma_i^2)^{-j_i}}$, $\vartheta_{j_i} = \sum_{t=1}^{L} n_{r_t}$, $\varphi_t = \sum_{t=1}^{L} \frac{1}{2\sigma_t^2}$, and

$$\mho_{j_i} = \sum_{l=0}^{L-1} \frac{(-1)^l}{l!} \underbrace{\sum_{r_1=1}^{L} \cdots \sum_{r_l=1}^{L}}_{r_1 \neq r_2 \neq \cdots \neq r_l \neq i} \sum_{n_{r_1}=0}^{j_{r_1}-1} \cdots \sum_{n_{r_l}=0}^{j_{r_l}-1} \left(\prod_{t=1}^{L} \frac{\varphi_t}{n_{r_t}!} \right) \tag{12}$$

Applying $\mathcal{M}_{\gamma^{Max}}(s) = s\mathcal{L}[F_{\gamma^{Max}}(\gamma); s]$ for (11) and making use of equation (3.478.1) from [9], the MGF of γ^{Max} is derived as

$$\mathcal{M}_{\gamma^{Max}}(s) = \sum_{j_1,\ldots,j_L=1}^{\infty} \frac{\Lambda_{j_i} \mho_{j_i} \Gamma(1+\vartheta_{j_i}) s}{(s+\varphi_t)^{1+\vartheta_{j_i}}} \tag{13}$$

Using $f_{\gamma^{Max}}(\gamma) = \mathcal{L}^{-1}[\mathcal{M}_{\gamma^{Max}}(s); \gamma]$ for (13) and with help of [13, equation (5.4.4)] and [9, equation (8.970.1)], the PDF of γ^{Max} is expressed as

$$f_{\gamma^{Max}}(\gamma) = \sum_{j_1,\ldots,j_L=1}^{\infty} \Lambda_{j_i} \mho_{j_i} \varphi_t \gamma^{\vartheta_{j_i}} e^{-\varphi_t \gamma} \tag{14}$$

The asymptotic of (14) can be obtained after inserting (3) in $F_{\gamma^{Max}}(\gamma) \simeq \prod_{i=1}^{L} F_{\gamma_i}(\gamma)$ and taking the partial derivative with respect to γ, namely, $f_{\gamma^{Max}}(\gamma) = \partial F_{\gamma^{Max}}(\gamma)/\partial\gamma$, to obtain

$$f_{\gamma^{Max}}(\gamma) \simeq L \left(\prod_{i=1}^{L} \frac{m_i(1+K_i) d_{j_i=1}}{\Gamma(m_i)\bar{\gamma}_i} \right) \gamma^{L-1} \tag{15}$$

Truncating the derived pdfs: Assume each infinite series is truncated for N terms with truncating error, $\varepsilon(N)$, that is given by [, equation (5)]

$$\varepsilon(N) = \int_0^{\infty} f_{\gamma}(\gamma) d\gamma - \int_0^{\infty} \hat{f}_{\gamma}(\gamma) d\gamma \tag{16}$$

where $\hat{f}_{\gamma}(\gamma)$ is the approximate PDF.

Truncating each infinite series of (7) up to N terms, we obtain

$$\hat{f}_{\gamma^{Sum}}(\gamma) = \sum_{j_1,\ldots,j_L=1}^{N_1,\ldots,N_L} \Theta_{j_i} \sum_{i=1}^{L} \sum_{l=1}^{j_i} \frac{\Omega_{i_l} \gamma^{l-1} e^{-\frac{\gamma}{2\sigma_i^2}}}{(-1)^l \Gamma(l)} \tag{17}$$

Plugging (7) and (17) in (16) and using the fact that $\int_0^{\infty} f_{\gamma}(\gamma) d\gamma \triangleq 1$ and [9, equation (3.478.1)], this yields

$$\varepsilon^{Sum}(N) = 1 - \sum_{j_1,\ldots,j_L=1}^{N_1,\ldots,N_L} \Theta_{j_i} \sum_{i=1}^{L} \sum_{l=1}^{j_i} (-2\sigma_i^2)^l \Omega_{i_l} \tag{18}$$

Similarly, $\varepsilon^{Max}(N)$ for the maximum of FTR RVs is obtained as

$$\varepsilon^{Max}(N) = 1 - \sum_{j_1,\ldots,j_L=1}^{N_1,\ldots,N_L} \frac{\Lambda_{j_i} \mho_{j_i} \Gamma(1+\vartheta_{j_i})}{\varphi_t^{\vartheta_{j_i}}} \tag{19}$$

Table 1 explains the number of terms, N, that is required to satisfy $\varepsilon(N) \leq 10^{-5}$ for different L and channel parameters. It is noticed that 28 and 33 terms are sufficient to obtain the required accuracy for (18) and (19), respectively. Additionally, Table 1 shows that (18) and (19) converge faster than [6, equation (9)/equation (10)] and [7], respectively.

Table 1. Required N of (18), [6] , (19), and [7] to Satisfy $\varepsilon \leq 10^{-5}$

L	m_i	K_i	Δ_i	N (18)	N [6]	N (19)	N [7]
1	8.5	5	0.35	23	28	30	32
2	8.5	5	0.35	25	29	33	35
2	5	3	0.5	28	31	34	37
3	25.5	3	0.48	16	18	23	25

Outage probability: The OP, P_o, can be computed by [1, equation (1.4)]

$$P_o = \int_0^{\Upsilon} f_{\gamma}(\gamma) d\gamma = F_{\gamma}(\Upsilon) \tag{20}$$

where Υ is a certain threshold value.

According to (20), the OP of the MRC scheme, P_o^{MRC}, is derived as

$$P_o^{MRC} = \sum_{j_1,\ldots,j_L=1}^{\infty} \Theta_{j_i} \sum_{i=1}^{L} \sum_{l=1}^{j_i} \frac{\Omega_{i_l} \mathcal{G}\left(l, \frac{\Upsilon}{2\sigma_i^2}\right)}{\Gamma(l)(-2\sigma_i^2)^{-l}} \tag{21}$$

The asymptotic expression of the OP is given as

$$P_o^{MRC} \simeq \left(\prod_{i=1}^{L} \frac{m_i(1+K_i) d_{j_i=1}}{\Gamma(m_i)\bar{\gamma}_i} \right) \frac{\Upsilon^L}{\Gamma(1+L)} \tag{22}$$

The diversity gain, G_d, that shows the increase in the slope of the OP versus $\bar{\gamma}$, can de deduced from $P_o^{\infty} \simeq \bar{\gamma}^{-G_d}$. Hence, one can notice from (22) that G_d is proportional to L.

The OP of the SC, P_o^{SC}, can be computed by (11) and its asymptotic can be obtained after plugging (3) in $P_o^{SC} = \prod_{i=1}^{L} F_{\gamma_i}(\Upsilon)$. It is obvious that the G_d of SC scheme also depends on L.

Average bit error probability: The ABEP can be calculated by [1, equation (9.11)]

$$P_e = \frac{1}{2\Gamma(b)} \int_0^{\infty} \Gamma(b, a\gamma) f_{\gamma}(\gamma) d\gamma \tag{23}$$

where $(a, b) = (1, 0.5)$ for binary phase shift keying (BPSK) and $(a, b) = (1, 1)$ for differential BPSK (DBPSK).

Inserting (7) in (23) and using [11, equation (6.455.1)], the ABEP of MRC, P_e^{MRC}, is given as

$$P_e^{MRC} = \sum_{j_1,\ldots,j_L=1}^{\infty} \Theta_{j_i} \sum_{i=1}^{L} \sum_{l=1}^{j_i} \frac{\Omega_{i_l}(-1)^l (2\sigma_i^2)^{l+b} a^b (b)_l}{2(1+2\sigma_i^2 a)^{l+b} \Gamma(l+1)} \times {}_2F_1\left(1, l+b; l+1; \frac{1}{1+2\sigma_i^2 a}\right) \tag{24}$$

where $(\cdot)_a$ is the Pochhammer symbol and ${}_2F_1(\cdot; \cdot, .; \cdot)$ is the Gauss hypergeometric function [9, equation (9.14.2)]. It can be noted that, in [9], the expression of the ABEP includes a multivariate Lauricella hypergeometric function $F_D^{(L)}(\cdot)$ that cannot be evaluated for DBPSK.

Substituting (9) into (23) and making use of [14, equation (2.10.2.1)], the asymptotic of P_e^{MRC} is obtained as

$$P_e^{MRC} \simeq \left(\prod_{i=1}^{L} \frac{m_i(1+K_i) d_{j_i=1}}{\Gamma(m_i)\bar{\gamma}_i} \right) \frac{(L)_b}{2L\Gamma(b)a^L} \tag{25}$$

From (25), one can see that the diversity gain of P_e^{MRC} is also proportional to L.

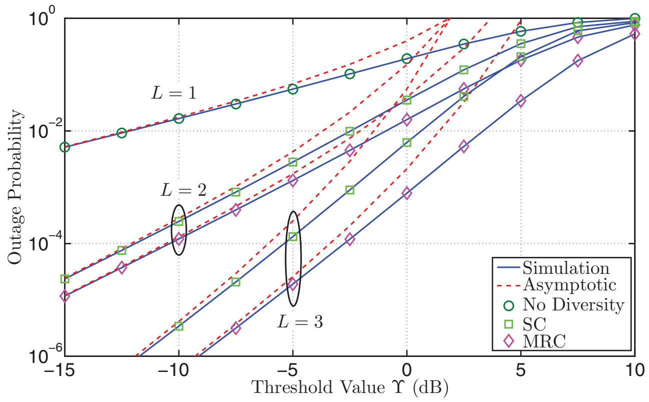

Fig. 1 *OP against* Υ *with* $\bar{\gamma} = 5$ *dB*

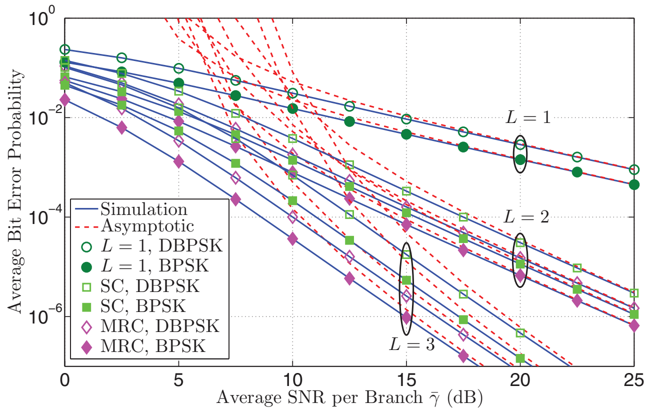

Fig. 2 *ABEP against* $\bar{\gamma}$ *for BPSK and DBPSK*

Plugging (14) in (23) and recalling [9, equation (6.455.1)], the ABEP of SC, P_e^{SC}, is provided as

$$P_e^{SC} = \sum_{j_1,\ldots,j_L=1}^{\infty} \frac{\Lambda_{j_i}\mho_{j_i}\varphi_t a^b (b)_{1+\vartheta_{j_i}}}{2(1+\vartheta_{j_i})(a+\varphi_t)^{1+\vartheta_{j_i}+b}} \times {}_2F_1\left(1, 1+\vartheta_{j_i}+b; \vartheta_{j_i}+2; \frac{\varphi_t}{a+\varphi_t}\right) \quad (26)$$

In comparison with [7, equation (17)], (26) is written in simple functions.

When $\bar{\gamma}_i \to \infty$, the ABEP of SC is derived via inserting (19) in (25) and using [14], equation (2.10.2.1)]. Thus, this obtains

$$P_e^{SC} \simeq \left(\prod_{i=1}^{L} \frac{m_i(1+K_i)d_{j_i=1}}{\Gamma(m_i)\bar{\gamma}_i}\right)\frac{\Gamma(L+b)}{a^L} \quad (27)$$

It is evident from (27) that G_d is related to L.

Analytical and simulation results: In this section, the results for the derived performance metrics are demonstrated for different scenarios. A comparison is carried out between $L = 1$, MRC and SC schemes with $L = 2$ and $L = 3$ branches with the fading parameters $m_1 = 3.5$, $m_2 = 4.5$, $m_3 = 5.5$, $K_1 = K_2 = K_3 = 3$, and $\Delta_1 = \Delta_2 = \Delta_3 = 0.5$. To obtain $\varepsilon \leq 10^{-6}$, N is chosen to be 40 for all scenarios.

Figure 1 plots the OP versus Υ for $\bar{\gamma}_i = 5$ dB, whereas Figure 2 shows the ABEP for BPSK and DBPSK modulations versus $\bar{\gamma}_i$. As expected, the results improve when the combining techniques are employed. This refers to the increase in the average SNR or/and diversity gain which is related to L. In addition, as anticipated, the MRC outperforms the SC for both cases of the number of the branches. However, the MRC has higher implementation intricacy than the SC. For example, at fixed $\bar{\gamma} = 5$ dB for all branches, the ABEP of DBPSK for MRC diversity reception with $L = 3$ is nearly 78.1% and 96.4% less than that for SC receivers with $L = 3$ and $L = 1$, respectively.

In all figures, the numerical results and their simulations that are obtained by 10^6 iterations are consistent which confirms the correctness of our analysis. In the same context, the asymptotic approximations at high γ_i perfectly coincided with the exact results.

Conclusions: In this letter, simple mathematically tractable exact and asymptotic expressions of the PDF of the sum and maximum of i.n.d. FTR variates were derived. Then, the OP, and ABEP of the MRC and SC receivers were analysed. The results explained that the system performance can be highly improved via using diversity techniques. The provided PDFs of this work can be used for a wide range of applications, such as the secrecy analysis of the PHY of MISO system.

Conflict of interest: The authors have declared no conflict of interest.

Data availability statement: The data that support the findings of this study are available from the corresponding author upon reasonable request.

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