EXISTENCE AND STABILITY OF A SPIKE IN THE CENTRAL COMPONENT FOR A CONSUMER CHAIN MODEL IN A TWO-DIMENSIONAL DOMAIN

WEIWEI AO, YUNJIE PENG, AND MATTHIAS WINTER

ABSTRACT. In this paper, we study a three-component consumer chain model which is based on Schnakenberg type kinetics in a two-dimensional domain. In the model there is one consumer feeding on the producer and a second consumer feeding on the first consumer. Through a rigorous analysis, we show that there exist two different single spike solutions if the feed rates are small. Further, we also establish the stability results: If the time-relaxation constants for both producer and the second consumer vanish, the large amplitude spike solution is stable and the small-amplitude spike solution is unstable. We also derive results on the stability of solutions when these two time-relaxation constants are positive. We show a new effect that if the time-relaxation constant of the second consumer is bounded, the large-amplitude spike solution is still stable while it is unstable in the one-dimensional case.

Keywords: Consumer chain model, Reaction-diffusion systems, Spiky solutions, Stability.

1. INTRODUCTION

We consider a reaction-diffusion system which serves as a cooperative consumer chain model. In the model, there are three components considered: One pure producer, one pure consumer and a central component who is both producer and consumer, which means that the central component consumes the pure producer and it is consumed by the second consumer. This model is an extension of the model introduced in [20] which considers the model in the one-dimensional case. For realistic consideration, we assume that the producer and the second consumer diffuse much faster than the first consumer. We also assume that cooperation of consumers is prevalent in the system, which has been proven to be correct in many types of consumer groups or populations.

The system can be written as follows:

$$\begin{cases} \tau \frac{\partial S}{\partial t} = D_1 \Delta S + \frac{1}{|\log \epsilon|} - \frac{a_1}{\epsilon^2 |\log \epsilon|} S u_1^2, & x \in \Omega, \quad t > 0 \\ \frac{\partial u_1}{\partial t} = \epsilon^2 \Delta u_1 - u_1 + S u_1^2 - a_2 u_1 u_2^2, & x \in \Omega, \quad t > 0 \\ \tau_1 \frac{\partial u_2}{\partial t} = D_2 \Delta u_2 - u_2 + \frac{1}{\epsilon^2 |\log \epsilon|} u_1 u_2^2, & x \in \Omega, \quad t > 0 \end{cases} \tag{1.1}$$

where S and u_i denote the concentrations of the producer and the two consumers, respectively. $0 < \epsilon^2 \ll 1$, D_1 and D_2 are three positive diffusion constants. The positive constants a_1 and a_2 are the feed rates, and τ, τ_1 (nonnegative) are the time relaxation constants.

We choose the domain Ω as the unit ball $B(0,1)$ in $\mathbb{R}^2$ and consider Neumann boundary conditions

$$\left.\frac{\partial S}{\partial \nu}\right|_{\partial \Omega} = 0, \quad \left.\frac{\partial u_1}{\partial \nu}\right|_{\partial \Omega} = 0, \quad \left.\frac{\partial u_2}{\partial \nu}\right|_{\partial \Omega} = 0. \tag{1.2}$$

It is necessary to mention that the choice of the coefficients of the nonlinear reaction terms in system (1.1) is to ensure that the spiky solutions for all three components have an amplitude of order $\mathcal{O}(1)$ as $\epsilon \to 0$. Further, to get profiles on the order unity scale, we need to have a very

The research of the first author is supported by NSFC 11801421 and 12071357. The third author thanks the Department of Pure Mathematics at Wuhan University for their kind hospitality.

small diffusion constant for the central component and much larger diffusion constants for the other two components.

Models involving a chain of components are significant in many fields, such as biology, social sciences and so on. Many useful works have been done by different authors. Our model is an extension of the Schnakenberg model introduced in [7] and [9]. Firstly, let us recall some related works. [8] and [10] studied the existence and stability of spiky patterns on a bounded interval. [18], [13] and [14] studied similar results for a two-dimensional domain. And there are also many useful results of Gray-Scott model, which is closely related to ours. [3], [4], [5], [6] studied the existence and stability of spike patterns on the real line. [11], [12], [15], [16], [19] studied the two-dimensional cases. The results we obtained in this paper generalize similar statements in the one-dimensional case in [20].

In the following sections, we first prove the existence of single spike solutions in a unit ball. We show that if the feed rates a_1 and a_2 are small enough, two such spiky solutions can be obtained.

We also show that in the case $\tau = \tau_1 = 0$, the large amplitude solution is stable while the small amplitude solution is unstable. What's more, we have shown that if the two time relaxation constants τ and τ_1 are small, the stability is the same as in the case $\tau = \tau_1 = 0$, and that is exactly what we expected.

Throughout the paper, the symbol C represents a constant independent of ϵ which may not be the same in different places. Denote $A = \mathcal{O}(B)$ as $|A| \le C|B|$ for some $C > 0$.

The structure of this paper is as follows. In Section 2 we will present the main results on existence and stability. The proof of the existence and Theorem 2.1 will be presented in Section 3 and 4. In Section 5 we derive a nonlocal eigenvalue problem (NLEP) and study large eigenvalues, and the study for small eigenvalues will be presented in Section 6. The linear theory and properties for the Green's function are given in the appendix.

2. Main Results

We first construct stationary spike solutions to (1.1), i.e. spike solution to the system

$$\begin{cases} D_1\Delta S + \frac{1}{|\log\epsilon|} - \frac{a_1}{\epsilon^2|\log\epsilon|}Su_1^2 = 0, & x \in \Omega, \\ \epsilon^2\Delta u_1 - u_1 + Su_1^2 - a_2u_1u_2^2 = 0, & x \in \Omega, \\ D_2\Delta u_2 - u_2 + \frac{1}{\epsilon^2|\log\epsilon|}u_1u_2^2 = 0, & x \in \Omega, \end{cases} \tag{2.1}$$

with Neumannn boundary conditions given in (1.2).

We will construct solutions of (2.1) as follows:

$$S = S(|x|) \in H_N^2(\Omega),$$

$$u_1 = u_1(|y|) \in H_N^2(\Omega_\epsilon), \quad y = \frac{x}{\epsilon},$$

$$u_2 = u_2(|x|) \in H_N^2(\Omega),$$

where

$$H_N^2(\Omega) = \{v \in H^2(\Omega) : \frac{\partial v}{\partial \nu}\Big|_{\partial\Omega} = 0\},$$

$$\Omega_\epsilon = B(0, \frac{1}{\epsilon}),$$

$$H_N^2(\Omega_\epsilon) = \{v \in H^2(\Omega_\epsilon) : \frac{\partial v}{\partial \nu}\Big|_{\partial\Omega_\epsilon} = 0\}.$$

Before stating the main results, we introduce some necessary notations and assumptions. Let w be the unique solution of the following problem

$$\begin{cases} \Delta w - w + w^2 = 0, \quad w > 0 \text{ in } \mathbb{R}^2, \\ w(0) = \max_{y \in \mathbb{R}^2} w(y), \quad w(y) \to 0 \text{ as } |y| \to \infty. \end{cases} \tag{2.2}$$

It's well known that $w(y) \sim |y|^{-1/2} e^{-|y|}$ as $|y| \to \infty$, we can also get that $\int_{\mathbb{R}^2} w^3 = \frac{3}{2} \int_{\mathbb{R}^2} w^2$. Our main result can be stated as follows:

Theorem 2.1. *Assume that*

$$D_1 = const, \quad \epsilon \ll 1, \quad D_2 = const. \tag{2.3}$$

Let G_{D_1} and G_{D_2} be the Green's functions defined in (7.13) and (7.17), respectively. Assume that

$$a_1^2 a_2 < \frac{|\Omega|^2 (\int_{\mathbb{R}^2} w(y) dy)^2}{16\pi^2 D_2^2 (\int_{\mathbb{R}^2} w^2(y) dy)^2} - \delta_0 \quad \textit{for some } \delta_0 > 0 \textit{ small.} \tag{2.4}$$

Then problem (2.1) admits two "single-spike" solutions $(S_\epsilon^s, u_{1,\epsilon}^s, u_{2,\epsilon}^s)$ and $(S_\epsilon^l, u_{1,\epsilon}^l, u_{2,\epsilon}^l)$ with the following properties:

(i) *all components are radially symmetric functions.*
(ii) *For $x \neq 0$,*

$$S_\epsilon^{s,l}(x) = c_{1,\epsilon}^{s,l} G_{D_1}(x, 0) + \mathcal{O}\left(\frac{1}{|\log \epsilon|}\right), \tag{2.5}$$

$$u_{1,\epsilon}^{s,l}(x) = \xi_\epsilon w\left(\frac{\sqrt{1 + \alpha_\epsilon^{s,l}}|x|}{\epsilon}\right) + \mathcal{O}\left(\frac{1}{|\log \epsilon|}\right), \tag{2.6}$$

$$u_{2,\epsilon}^{s,l}(x) = c_{2,\epsilon}^{s,l} G_{D_2}(x, 0) + \mathcal{O}\left(\frac{1}{|\log \epsilon|}\right), \tag{2.7}$$

where w is the unique solution of (2.2),

$$\xi_\epsilon = \frac{|\Omega|}{a_1 \int_{\mathbb{R}^2} w^2 dy} + \mathcal{O}\left(\frac{1}{|\log \epsilon|}\right), \tag{2.8}$$

$$c_{1,\epsilon}^{s,l} = \frac{2\pi D_1 (1 + \alpha_\epsilon^{s,l})}{\xi_\epsilon |\log \epsilon|}\left(1 + \mathcal{O}\left(\frac{1}{|\log \epsilon|}\right)\right), \tag{2.9}$$

$$c_{2,\epsilon}^{s,l} = \frac{(2\pi D_2)^2 (1 + \alpha_\epsilon^{s,l})}{\xi_\epsilon |\log \epsilon| \int_{\mathbb{R}^2} w dy}\left(1 + \mathcal{O}\left(\frac{1}{|\log \epsilon|}\right)\right), \tag{2.10}$$

and $\alpha_\epsilon^{s,l}$ is given in (3.9).

On the other hand, if ϵ is small enough and

$$a_1^2 a_2 > \frac{|\Omega|^2 (\int_{\mathbb{R}^2} w(y) dy)^2}{16\pi^2 D_2^2 (\int_{\mathbb{R}^2} w^2(y) dy)^2} + \delta_0$$

for some $\delta_0 > 0$ independent of ϵ, then there are no single-spike solutions which satisfy (i) and (ii).

The proof of Theorem 2.1 will be given in Section 3 and 4.

We also study the stability properties of the single-spike solution constructed in Theorem 2.1. The following are the main results on stability.

Theorem 2.2. *Assume that (2.3) and (2.4) are satisfied. Suppose that $\tau = \tau_1 = 0$, then we have the following results:*

(i) *(Stability) The large-amplitude solution* $(S^l_\epsilon, u^l_{1,\epsilon}, u^l_{2,\epsilon})$ *is linearly stable. There is a small eigenvalue of exact order* $\mathcal{O}\left(\frac{\epsilon^2}{|\log \epsilon|}\right)$ *with negative real part which is given in (6.9).*

(ii) *(Instability) The small-amplitude solution* $(S^s_\epsilon, u^s_{1,\epsilon}, u^s_{2,\epsilon})$ *is linearly unstable. There is a large eigenvalue of* $\mathcal{O}(1)$ *with positive real part.*

For the case of τ and τ_1 positive, and τ small we have the following result:

Corollary 2.3. *Assume that (2.3) and (2.4) are satisfied.*

(i) *There exists a constant* $\tau_0 > 0$ *independent of* ϵ *such that for* $0 < \tau \le \tau_0$ *and* $0 < \tau_1 < \infty$ *the stability properties of the large-amplitude solution* $(S^l_\epsilon, u^l_{1,\epsilon}, u^l_{2,\epsilon})$ *and the small-amplitude solution* $(S^s_\epsilon, u^s_{1,\epsilon}, u^s_{2,\epsilon})$ *are the same as in the case* $\tau = \tau_1 = 0$.

(ii) *There is a small eigenvalue of exact order* $\mathcal{O}\left(\frac{\epsilon^2}{|\log \epsilon|}\right)$ *with negative real part which is given in (6.9).*

Remark 2.4. *To have this type of spiky solution, the feed rates* a_1 *and* a_2*, in particular their combination* $a_1^2 a_2$*, must be small enough. Otherwise the food source S and the hybrid* u_1 *will not be able to sustain* u_1 *and* u_2*, respectively.*

3. Existence: Calculating the Amplitudes

In this section, we will show the existence of spike solutions to (2.1) and prove Theorem 2.1. We begin by computing the amplitudes in leading order and will give a rigourous existence proof in the next section.

We will show the existence of spike solutions to (2.1) which in leading order are given by (2.5), (2.6) and (2.7). We choose the second component of the approximate solution as follows:

$$\tilde{u}_{1,\epsilon}(x) = \xi_\epsilon w\left(\frac{|x|\sqrt{1+\alpha_\epsilon}}{\epsilon}\right)\chi(|x|) \tag{3.1}$$

for some positive constants ξ_ϵ and α_ϵ. Here χ is a smooth, radially symmetric cut-off function which satisfies

$$\chi \in C_0^\infty(\mathbb{R}^2), \quad \chi(x) = 1 \text{ for } |x| \le \frac{5}{8} \quad \text{and} \quad \chi(x) = 0 \text{ for } |x| \ge \frac{3}{4}. \tag{3.2}$$

The main reason for using the cut-off function (3.2) in the ansatz (3.1) is that Neumann boundary conditions are satisfied exactly.

We set

$$y = \frac{x}{\epsilon},$$

and consider the limit

$$\epsilon \to 0.$$

By a simple computation, we know that $w(y\sqrt{1+\alpha_\epsilon})$ satisfies

$$\Delta_y w - (1+\alpha_\epsilon)w + (1+\alpha_\epsilon)w^2 = 0. \tag{3.3}$$

Comparing coefficients between the second equation and (3.3), we have

$$\alpha_\epsilon = a_2 u_{2,\epsilon}^2(0) + \mathcal{O}\left(\frac{1}{|\log \epsilon|}\right), \tag{3.4}$$

$$\xi_\epsilon = \frac{1+\alpha_\epsilon}{S_\epsilon(0)} + \mathcal{O}\left(\frac{1}{|\log \epsilon|}\right). \tag{3.5}$$

We remark that in leading order $S_\epsilon u_{1,\epsilon}^2$ agrees with $S_\epsilon(0) u_{1,\epsilon}^2$ since $u_{1,\epsilon}$ decays exponentially away from 0.

From the Green function G_{D_2} defined in (7.17), we get

$$u_{2,\epsilon}(x) = \frac{1}{\epsilon^2|\log\epsilon|}\int_\Omega G_{D_2}(x,z)u_{1,\epsilon}(z)u_{2,\epsilon}^2(z)dz$$
$$= \frac{1}{|\log\epsilon|}\int_{\Omega_\epsilon} G_{D_2}(x,\epsilon y)u_{1,\epsilon}(\epsilon y)u_{2,\epsilon}^2(\epsilon y)dy. \tag{3.6}$$

This implies

$$u_{2,\epsilon}(0) = \frac{\xi_\epsilon u_{2,\epsilon}^2(0)}{2\pi D_2(1+\alpha_\epsilon)}\int_{\mathbb{R}^2} w(y)dy + \mathcal{O}\left(\frac{1}{|\log\epsilon|}\right),$$

i.e.

$$u_{2,\epsilon}(0) = \frac{2\pi D_2(1+\alpha_\epsilon)}{\xi_\epsilon\int_{\mathbb{R}^2} w(y)dy} + \mathcal{O}\left(\frac{1}{|\log\epsilon|}\right). \tag{3.7}$$

Integrating the first equation in (2.1), using the Neumann boundary condition and balancing the last two terms, we can get

$$|\Omega| = \frac{a_1}{\epsilon^2}\int_\Omega S_\epsilon(x)u_{1,\epsilon}^2(x)dx$$
$$= \frac{a_1 S_\epsilon(0)\xi_\epsilon^2}{1+\alpha_\epsilon}\int_{\mathbb{R}^2} w^2(z)dz + \mathcal{O}\left(\frac{1}{|\log\epsilon|}\right)$$
$$= a_1\xi_\epsilon\int_{\mathbb{R}^2} w^2(z)dz + \mathcal{O}\left(\frac{1}{|\log\epsilon|}\right),$$

from which we can get that

$$\xi_\epsilon = \frac{|\Omega|}{a_1\int_{\mathbb{R}^2} w^2 dy} + \mathcal{O}\left(\frac{1}{|\log\epsilon|}\right). \tag{3.8}$$

Then by (3.4), (3.7) and (3.8), we can calculate that

$$\sqrt{\alpha_\epsilon} = \frac{|\Omega|\int_{\mathbb{R}^2} w(y)dy \pm \sqrt{|\Omega|^2(\int_{\mathbb{R}^2} w(y)dy)^2 - 16\pi^2 D_2^2 a_1^2 a_2(\int_{\mathbb{R}^2} w^2(y)dy)^2}}{4\pi D_2 a_1\sqrt{a_2}\int_{\mathbb{R}^2} w^2(y)dy} \tag{3.9}$$

under the condition

$$a_1^2 a_2 < \frac{|\Omega|^2(\int_{\mathbb{R}^2} w(y)dy)^2}{16\pi^2 D_2^2(\int_{\mathbb{R}^2} w^2(y)dy)^2}.$$

The last condition states that, all other constants being equal, the combination $a_1^2 a_2$ must be small enough.

This implies that under the condition

$$a_1^2 a_2 < \frac{|\Omega|^2(\int_{\mathbb{R}^2} w(y)dy)^2}{16\pi^2 D_2^2(\int_{\mathbb{R}^2} w^2(y)dy)^2} - \delta_0 \quad \text{for some } \delta_0 > 0,$$

there are two solutions for α_ϵ, denote α_ϵ^s and α_ϵ^l, respectively, and one solution for ξ_ϵ.
On the other hand, if

$$a_1^2 a_2 > \frac{|\Omega|^2(\int_{\mathbb{R}^2} w(y)dy)^2}{16\pi^2 D_2^2(\int_{\mathbb{R}^2} w^2(y)dy)^2} + \delta_0 \quad \text{for some } \delta_0 > 0,$$

there are no such solutions.

Now we show that

$$\alpha_\epsilon^s > 1 \quad \text{and} \quad \alpha_\epsilon^l < 1. \tag{3.10}$$

Since that

$$
\begin{aligned}
&|\Omega| \int_{\mathbb{R}^2} w(y)dy - \sqrt{|\Omega|^2(\int_{\mathbb{R}^2} w(y)dy)^2 - 16\pi^2 D_2^2 a_1^2 a_2 (\int_{\mathbb{R}^2} w^2(y)dy)^2} < 4\pi D_2 a_1 \sqrt{a_2} \int_{\mathbb{R}^2} w^2(y)dy \\
&\Leftrightarrow 4\pi D_2 a_1 \sqrt{a_2} \int_{\mathbb{R}^2} w^2(y)dy < |\Omega| \int_{\mathbb{R}^2} w(y)dy \\
&\Leftrightarrow a_1^2 a_2 < \frac{|\Omega|^2(\int_{\mathbb{R}^2} w(y)dy)^2}{16\pi^2 D_2^2 (\int_{\mathbb{R}^2} w^2(y)dy)^2},
\end{aligned}
$$

from which we can get that

$$
\sqrt{\alpha_\epsilon^l} = \frac{|\Omega| \int_{\mathbb{R}^2} w(y)dy - \sqrt{|\Omega|^2(\int_{\mathbb{R}^2} w(y)dy)^2 - 16\pi^2 D_2^2 a_1^2 a_2 (\int_{\mathbb{R}^2} w^2(y)dy)^2}}{4\pi D_2 a_1 \sqrt{a_2} \int_{\mathbb{R}^2} w^2(y)dy} < 1,
$$

which means that $\alpha_\epsilon^l < 1$. Similarly, we can prove that $\alpha_\epsilon^s > 1$.

Finally, this results in the two single-spike solutions $(S_\epsilon^l, u_{1,\epsilon}^l, u_{2,\epsilon}^l)$ and $(S_\epsilon^s, u_{1,\epsilon}^s, u_{2,\epsilon}^s)$ of (2.1). In the next section we will rigorously prove the existence of these two solutions.

4. Existence: Rigorous Proofs

In this section we show the existence of solutions of (2.1) for which the central component has a spike. As we have shown in the previous section, there are two such solutions $(S_\epsilon^l, u_{1,\epsilon}^l, u_{2,\epsilon}^l)$ and $(S_\epsilon^s, u_{1,\epsilon}^s, u_{2,\epsilon}^s)$ which differ by the size of the amplitude. We will not write the superscripts ^{s} and ^{l} in this section for the existence proof applies to both of them.

The second component of the approximate spike solution introduced in (3.1) is given by

$$
\tilde{u}_{1,\epsilon}(x) = \xi_\epsilon w\left(\frac{|x|\sqrt{1+\alpha_\epsilon}}{\epsilon}\right)\chi(|x|),
$$

where ξ_ϵ and α_ϵ have been computed in (3.8) and (3.9), and χ has been introduced in (3.2).

Further, $\tilde{S}_\epsilon$ and $\tilde{u}_{2,\epsilon}$ solve a partial defferential equation which depends on $\tilde{u}_{1,\epsilon}$ only. Therefore we denote $\tilde{S}_\epsilon = T_1[\tilde{u}_{1,\epsilon}]$ and $\tilde{u}_{2,\epsilon} = T_2[\tilde{u}_{1,\epsilon}]$, respectively. We insert this approximate spike solution into (2.1) and compute its error.

The left hand side of the second equation in (2.1) at $(\tilde{S}_\epsilon, \tilde{u}_{1,\epsilon}, \tilde{u}_{2,\epsilon}) = (T_1[\tilde{u}_{1,\epsilon}], \tilde{u}_{1,\epsilon}, T_2[\tilde{u}_{1,\epsilon}])$ is calculated as follows:

$$
\begin{aligned}
\Delta_y \tilde{u}_{1,\epsilon} - \tilde{u}_{1,\epsilon} + \tilde{S}_\epsilon \tilde{u}_{1,\epsilon}^2 - a_2 \tilde{u}_{1,\epsilon}\tilde{u}_{2,\epsilon}^2 &= \Delta_y \tilde{u}_{1,\epsilon} - \tilde{u}_{1,\epsilon} + \tilde{S}_\epsilon(0)\tilde{u}_{1,\epsilon}^2 - a_2 \tilde{u}_{1,\epsilon}\tilde{u}_{2,\epsilon}^2(0) + [\tilde{S}_\epsilon - \tilde{S}_\epsilon(0)]\tilde{u}_{1,\epsilon}^2 \\
&\quad - 2a_2 \tilde{u}_{1,\epsilon}(\tilde{u}_{2,\epsilon} - \tilde{u}_{2,\epsilon}(0))\tilde{u}_{2,\epsilon}(0) + \mathcal{O}(\frac{1}{|\log\epsilon|^2}) \\
&=: E_1 + E_2 + E_3 + \mathcal{O}(\frac{1}{|\log\epsilon|^2})
\end{aligned}
$$

in $L^2(\Omega_\epsilon)$, where $\Omega_\epsilon = B(0, \frac{1}{\epsilon})$.

We compute

$$
\begin{aligned}
E_1 &= \Delta_y \tilde{u}_{1,\epsilon} - \tilde{u}_{1,\epsilon} + \tilde{S_\epsilon}(0)\tilde{u}_{1,\epsilon}^2 - a_2\tilde{u}_{1,\epsilon}\tilde{u}_{2,\epsilon}^2(0) \\
&= \Delta_y \tilde{u}_{1,\epsilon} - (1+\alpha_\epsilon)\tilde{u}_{1,\epsilon} + \frac{1+\alpha_\epsilon}{\xi_\epsilon}\tilde{u}_{1,\epsilon}^2 + \mathcal{O}\left(\frac{1}{|\log\epsilon|}\right) \\
&= \xi_\epsilon[\Delta_y w - (1+\alpha_\epsilon)w + (1+\alpha_\epsilon)w^2] + \mathcal{O}\left(\frac{1}{|\log\epsilon|}\right) \\
&= \mathcal{O}\left(\frac{1}{|\log\epsilon|}\right).
\end{aligned}
$$

Computing $\tilde{S_\epsilon}(x)$, using the Green's function G_{D_1} defined in (7.13), we derive the following estimate:

$$
\begin{aligned}
E_2 &= [\tilde{S_\epsilon}(\epsilon y) - \tilde{S_\epsilon}(0)]\tilde{u}_{1,\epsilon}^2(\epsilon y) \\
&= -\tilde{u}_{1,\epsilon}^2(\epsilon y)\frac{a_1}{|\log\epsilon|}\int_{\Omega_\epsilon}[G_{D_1}(\epsilon y,\epsilon z) - G_{D_1}(0,\epsilon z)]\tilde{S_\epsilon}(\epsilon z)\tilde{u}_{1,\epsilon}^2(\epsilon z)dz \\
&= -\frac{a_1\xi_\epsilon^2\tilde{S}_\epsilon(0)}{|\log\epsilon|}\tilde{u}_{1,\epsilon}^2(\epsilon y)\int_{\mathbb{R}^2}[\frac{1}{2\pi D_1}\log\frac{|z|}{|y-z|} + H_{D_1}(0,\epsilon z) - H_{D_1}(\epsilon y,\epsilon z)]w^2(\sqrt{1+\alpha_\epsilon}z)dz \\
&\quad\times\left(1+\mathcal{O}\left(\frac{1}{|\log\epsilon|}\right)\right) \\
&= \mathcal{O}\left(\frac{1}{|\log\epsilon|}\right)\tilde{u}_{1,\epsilon}^2.
\end{aligned}
$$

Thus we have

$$
E_2 = \mathcal{O}\left(\frac{1}{|\log\epsilon|}\right) \quad \text{in } L^2(\Omega_\epsilon).
$$

Similarly, from (3.6), we compute

$$
\begin{aligned}
E_3 &= -2a_2\tilde{u}_{1,\epsilon}(\epsilon y)(\tilde{u}_{2,\epsilon}(\epsilon y) - \tilde{u}_{2,\epsilon}(0))\tilde{u}_{2,\epsilon}(0) \\
&= 2a_2\tilde{u}_{1,\epsilon}(\epsilon y)\tilde{u}_{2,\epsilon}(0)\int_{\Omega_\epsilon}[G_{D_2}(\epsilon y,\epsilon z) - G_{D_2}(0,\epsilon z)]\frac{1}{|\log\epsilon|}\tilde{u}_{2,\epsilon}^2(\epsilon z)\tilde{u}_{1,\epsilon}(\epsilon z)dz \\
&= 2a_2\tilde{u}_{1,\epsilon}(\epsilon y)\tilde{u}_{2,\epsilon}^3(0)\frac{1}{|\log\epsilon|}\int_{\mathbb{R}^2}[\frac{1}{2\pi D_2}\log\frac{|z|}{|y-z|} + H_{D_2}(\epsilon y,\epsilon z) - H_{D_2}(0,\epsilon z)]\tilde{u}_{1,\epsilon}(\epsilon z)dz \\
&\quad\times\left(1+\mathcal{O}\left(\frac{1}{|\log\epsilon|}\right)\right) \\
&= \mathcal{O}\left(\frac{1}{|\log\epsilon|}\right)\tilde{u}_{1,\epsilon}.
\end{aligned}
$$

Thus we have

$$
E_3 = \mathcal{O}\left(\frac{1}{|\log\epsilon|}\right) \quad \text{in } L^2(\Omega_\epsilon).
$$

By definition, the first and third equations of (2.1) are solved exactly and so do not contribute to the error.

Writing the system (2.1) in the form $R_\epsilon(S_\epsilon, u_{1,\epsilon}, u_{2,\epsilon}) = 0$, we have now shown the estimate

$$
\|R_\epsilon(T_1[\tilde{u}_{1,\epsilon}], \tilde{u}_{1,\epsilon}, T_2[\tilde{u}_{1,\epsilon}])\|_{L^2(\Omega_\epsilon)} \le \frac{C_1}{|\log\epsilon|} \tag{4.1}
$$

for some $C_1 > 0$ independent of ϵ small.

Next, we investigate the linearized operator $\tilde{\mathcal{L}}_\epsilon$ around the approximate solution $(\tilde{S}_\epsilon, \tilde{u}_{1,\epsilon}, \tilde{u}_{2,\epsilon})$. It is defined as follows:

$$\tilde{\mathcal{L}}_\epsilon \begin{pmatrix} \Psi_{1,\epsilon} \\ \Phi_\epsilon \\ \Psi_{2,\epsilon} \end{pmatrix} = \begin{pmatrix} D_1\Delta\Psi_{1,\epsilon} - \frac{2a_1}{\epsilon^2|\log\epsilon|}\tilde{S}_\epsilon\tilde{u}_{1,\epsilon}\Phi_\epsilon - \frac{a_1}{\epsilon^2|\log\epsilon|}\tilde{u}_{1,\epsilon}^2\Psi_{1,\epsilon} \\ \epsilon^2\Delta\Phi_\epsilon - \Phi_\epsilon + 2\tilde{S}_\epsilon\tilde{u}_{1,\epsilon}\Phi_\epsilon + \tilde{u}_{1,\epsilon}^2\Psi_{1,\epsilon} - a_2\tilde{u}_{2,\epsilon}^2\Phi_\epsilon - 2a_2\tilde{u}_{1,\epsilon}\tilde{u}_{2,\epsilon}\Psi_{2,\epsilon} \\ D_2\Delta\Psi_{2,\epsilon} - \Psi_{2,\epsilon} + \frac{1}{\epsilon^2|\log\epsilon|}\tilde{u}_{2,\epsilon}^2\Phi_\epsilon + \frac{2}{\epsilon^2|\log\epsilon|}\tilde{u}_{1,\epsilon}\tilde{u}_{2,\epsilon}\Psi_{2,\epsilon} \end{pmatrix}. \tag{4.2}$$

We will show that this operator leads to a uniformly invertible one for ϵ small enough.

To study the kernel of $\tilde{\mathcal{L}}_\epsilon$, we first solve its first and third components. Therefore, we have $\Psi_{1,\epsilon} = T_1'[\tilde{u}_{1,\epsilon}]\Phi_\epsilon$ and $\Psi_{2,\epsilon} = T_2'[\tilde{u}_{1,\epsilon}]\Phi_\epsilon$, where $T_1'[\tilde{u}_{1,\epsilon}]$ and $T_2'[\tilde{u}_{1,\epsilon}]$ are linearized operators which can be expressed by the Green's functions G_{D_1} and G_{D_2} defined in (7.13) and (7.17), respectively. Substituting these expressions into $\tilde{\mathcal{L}}_\epsilon$, the first and third components vanish and it only remains to consider the second component. We obtain the following operator:

$$\hat{\mathcal{L}}_\epsilon:\ H_N^2(\Omega_\epsilon)\ \to L^2(\Omega_\epsilon),$$

$$\hat{\mathcal{L}}_\epsilon(\Phi_\epsilon) = \Delta_y\Phi_\epsilon - \Phi_\epsilon + 2\tilde{S}_\epsilon\tilde{u}_{1,\epsilon}\Phi_\epsilon + \tilde{u}_{1,\epsilon}^2(T_1'[\tilde{u}_{1,\epsilon}]\Phi_\epsilon) - a_2\tilde{u}_{2,\epsilon}^2\Phi_\epsilon - 2a_2\tilde{u}_{1,\epsilon}\tilde{u}_{2,\epsilon}(T_2'[\tilde{u}_{1,\epsilon}]\Phi_\epsilon). \tag{4.3}$$

In order to introduce a uniformly invertible operator, we define approximate kernel and the approximate cokernel as follows:

$$\mathcal{K}_\epsilon = \text{span}\{\frac{\partial\tilde{u}_{1,\epsilon}}{\partial y_1}, \frac{\partial\tilde{u}_{1,\epsilon}}{\partial y_2}\} \subset H_N^2(\Omega_\epsilon),$$

$$\mathcal{C}_\epsilon = \text{span}\{\frac{\partial\tilde{u}_{1,\epsilon}}{\partial y_1}, \frac{\partial\tilde{u}_{1,\epsilon}}{\partial y_2}\} \subset L^2(\Omega_\epsilon).$$

Then the linear operator $\mathcal{L}_\epsilon$ is defined by

$$\mathcal{L}_\epsilon:\ \mathcal{K}_\epsilon^\perp \to \mathcal{C}_\epsilon^\perp,$$

$$\mathcal{L}_\epsilon:\ \pi_\epsilon \circ \hat{\mathcal{L}}_\epsilon, \tag{4.4}$$

where $\mathcal{C}_\epsilon^\perp$ and $\mathcal{K}_\epsilon^\perp$ denote the orthogonal complement with the scalar product of $L^2(\Omega_\epsilon)$ to $\mathcal{C}_\epsilon$ and $\mathcal{K}_\epsilon$ in $H_N^2(\Omega_\epsilon)$ and $L^2(\Omega_\epsilon)$, respectively, and π_ϵ denotes the projection in $L^2(\Omega_\epsilon)$ onto $\mathcal{C}_\epsilon^\perp$. Here $\pi_\epsilon = id$ since we consider the radially symmetric case.

The next proposition will show that this operator is uniformly invertible for ϵ small enough.

Proposition 4.1. *There exist positive constants ϵ' and λ such that for all $\epsilon \in (0, \epsilon')$,*

$$\|\mathcal{L}_\epsilon\Phi\|_{L^2(\Omega_\epsilon)} \geq \lambda\|\Phi\|_{H_N^2(\Omega_\epsilon)} \quad \textit{for all } \Phi \in \mathcal{K}_\epsilon^\perp. \tag{4.5}$$

Further, the linear operator $\mathcal{L}_\epsilon$ is surjective.

Proof. The details of the proof will be shown in the Appendix. □

Remark 4.2. *So if we consider in radial function space, the operator $\mathcal{L}_\epsilon$ is invertible.*

Finally, we solve the system (2.1). It can be written as:

$$R_\epsilon(\tilde{S}_\epsilon + \Psi_1, \tilde{u}_{1,\epsilon} + \Phi, \tilde{u}_{2,\epsilon} + \Psi_2) = R_\epsilon(U_\epsilon + \Phi) = 0, \tag{4.6}$$

where $U_\epsilon = (\tilde{S}_\epsilon, \tilde{u}_{1,\epsilon}, \tilde{u}_{2,\epsilon})$, $\Phi = (\Psi_1, \Phi, \Psi_2)$. Since $\mathcal{L}_\epsilon$ is uniformly invertible if ϵ is small enough, we can write (4.6) in function space with Φ as

$$\Phi = -\mathcal{L}_\epsilon^{-1}R_\epsilon(U_\epsilon) - \mathcal{L}_\epsilon^{-1}N_\epsilon(\Phi) =: M_\epsilon(\Phi), \tag{4.7}$$

where $\mathcal{L}_\epsilon^{-1}$ is the inverse of $\mathcal{L}_\epsilon$ and

$$N_\epsilon(\Phi) = R_\epsilon(U_\epsilon + \Phi) - R_\epsilon(U_\epsilon) - R'_\epsilon(U_\epsilon)\Phi. \tag{4.8}$$

Note that the operator M_ϵ is defined by (4.7) for $\Phi \in H^2_{N,r}(\Omega) \times H^2_{N,r}(\Omega_\epsilon) \times H^2_{N,r}(\Omega)$, where $H^2_{N,r}(\Omega) = \{u \in H^2_N(\Omega),\ u \text{ is radial }\}$. We are going to show that the operator M_ϵ is a contraction on

$$B_\epsilon \equiv \{\Phi \in H^2_{N,r}(\Omega) \times H^2_{N,r}(\Omega_\epsilon) \times H^2_{N,r}(\Omega) : \|\Phi\|_{H^2(\Omega)\times H^2(\Omega_\epsilon)\times H^2(\Omega)} < \frac{C_0 C_1}{|\log\epsilon|}\} \tag{4.9}$$

if ϵ is small enough and C_0 is chosen properly large. We have by (4.1) and Proposition 4.1 that

$$\begin{aligned}\|M_\epsilon(\Phi)\|_{H^2(\Omega)\times H^2(\Omega_\epsilon)\times H^2(\Omega)} &\le \lambda^{-1}\left(\|N_\epsilon(\Phi)\|_{L^2(\Omega)\times L^2(\Omega_\epsilon)\times L^2(\Omega)} + \|R_\epsilon(U_\epsilon)\|_{L^2(\Omega)\times L^2(\Omega_\epsilon)\times L^2(\Omega)}\right)\\ &\le \lambda^{-1}\left(c_\epsilon\frac{C_0C_1}{|\log\epsilon|} + \frac{C_1}{|\log\epsilon|}\right),\end{aligned}$$

where $c_\epsilon \to 0$ as $\epsilon \to 0$. Similarly, we show

$$\|M_\epsilon(\Phi_1) - M_\epsilon(\Phi_2)\|_{H^2(\Omega)\times H^2(\Omega_\epsilon)\times H^2(\Omega)} \le c_\epsilon\|\Phi_1 - \Phi_2\|_{H^2(\Omega)\times H^2(\Omega_\epsilon)\times H^2(\Omega)},$$

where $c_\epsilon \to 0$ as $\epsilon \to 0$. If we choose C_0 large, then M_ϵ is a contraction mapping in B_ϵ. The existence of a fixed point $\Phi_\epsilon \in B_\epsilon$ follows from the Contraction Mapping Principle, and Φ_ϵ is a solution of (4.7).

We have thus proved:

Lemma 4.3. *There exists $\epsilon' > 0$ such that for every $\epsilon \in (0, \epsilon')$ there is a $\Phi_\epsilon \in H^2_{N,r}(\Omega) \times H^2_{N,r}(\Omega_\epsilon) \times H^2_{N,r}(\Omega)$ satisfying $R_\epsilon(U_\epsilon + \Phi_\epsilon) = 0$. Further, we have the estimate*

$$\|\Phi_\epsilon\|_{H^2(\Omega)\times H^2(\Omega_\epsilon)\times H^2(\Omega)} \le \frac{C}{|\log\epsilon|}. \tag{4.10}$$

In this section we have constructed two exact spike solutions of the form $U_\epsilon + \Phi_\epsilon = (S_\epsilon, u_{1,\epsilon}, u_{2,\epsilon})$. In the next sections, we are going to study their stability.

5. Stability I: Study of a Nonlocal Eigenvalue problem

We study a small perturbation of a single-spike steady state $(S_\epsilon, u_{1,\epsilon}, u_{2,\epsilon})$ which could be either the small-amplitude solution $(S^s_\epsilon, u^s_{1,\epsilon}, u^s_{2,\epsilon})$ or the large-amplitude solution $(S^l_\epsilon, u^l_{1,\epsilon}, u^l_{2,\epsilon})$.

We linearize (1.1) around the single-spike solution derived in leading order $S_\epsilon + \Psi_{1,\epsilon}e^{\lambda_\epsilon t}$, $u_{1,\epsilon} + \Phi_\epsilon e^{\lambda_\epsilon t}$, $u_{2,\epsilon} + \Psi_{2,\epsilon}e^{\lambda_\epsilon t}$, where the three perturbation $\Psi_{1,\epsilon} \in H^2_N(\Omega)$, $\Phi_\epsilon \in H^2_N(\Omega_\epsilon)$, $\Psi_{2,\epsilon} \in H^2_N(\Omega)$ are small in their respective norms. Then the perturbations in leading order satisfy the eigenvalue problem

$$\tilde{\mathcal{L}}_\epsilon \begin{pmatrix} \Psi_{1,\epsilon} \\ \Phi_\epsilon \\ \Psi_{2,\epsilon} \end{pmatrix} = \begin{pmatrix} \tau\lambda_\epsilon\Psi_{1,\epsilon} \\ \lambda_\epsilon\Phi_\epsilon \\ \tau_1\lambda_\epsilon\Psi_{2,\epsilon} \end{pmatrix}, \tag{5.1}$$

where $\tilde{\mathcal{L}}_\epsilon$ denotes the linearized operator around the steady state $(S_\epsilon, u_{1,\epsilon}, u_{2,\epsilon})$ which has the domain $H^2_N(\Omega) \times H^2_N(\Omega_\epsilon) \times H^2_N(\Omega)$. Here we have $\lambda_\epsilon \in \mathbb{C}$, the set of complex numbers. To show the stability, we first introduce a necessary definition.

Definition 5.1. *A spike solution is* ***linearly stable*** *if the spectrum of $\mathcal{L}_\epsilon$ lies in the left half plane $\{\lambda \in \mathbb{C} :\ Re(\lambda) \le -c_0\}$ for some $c_0 > 0$. A spike solution is* ***linearly unstable*** *if there exists a eigenvalue λ_ϵ of $\mathcal{L}_\epsilon$ such that $Re(\lambda_\epsilon) > 0$.*

We first consider the case $\tau = 0$ and $\tau_1 = 0$ and show its stability. Writing down $\tilde{\mathcal{L}}_\epsilon$ explicitly and expressing $\Psi_{i,\epsilon} = T_i'[\tilde{u}_{i,\epsilon}]\Phi_\epsilon$, $i = 1,2$, using the Green's functions G_{D_i} defined in (7.13) and (7.17), respectively, we can rewrite (5.1) as

$$\epsilon^2\Delta\Phi_\epsilon - \Phi_\epsilon + 2\tilde{S_\epsilon}\tilde{u}_{1,\epsilon}\Phi_\epsilon + \tilde{u}_{1,\epsilon}^2(T_1'[\tilde{u}_{1,\epsilon}]\Phi_\epsilon) - a_2\tilde{u}_{2,\epsilon}^2\Phi_\epsilon - 2a_2\tilde{u}_{1,\epsilon}\tilde{u}_{2,\epsilon}(T_2'[\tilde{u}_{1,\epsilon}]\Phi_\epsilon) = \lambda_\epsilon\Phi_\epsilon. \quad (5.2)$$

Then, arguing as in the proof of Proposition 4.1, a subsequence of the sequence Φ_ϵ converges to a limit which we denote by Φ. Next we derive an eigenvalue problem for Φ.

Integrating the first equation of (5.1), we get

$$\int_\Omega u_{1,\epsilon}^2(x)\Psi_{1,\epsilon}(x)dx = -2\int_\Omega S_\epsilon(x)u_{1,\epsilon}(x)\Phi_\epsilon(x)dx.$$

Letting $y = \frac{x}{\epsilon}$ and $\epsilon \to 0$, we have

$$\Psi_{1,\epsilon}(0)\xi_\epsilon^2\int_{\mathbb{R}^2} w^2dy = -2S_\epsilon(0)\xi_\epsilon\int_{\mathbb{R}^2} w\Phi dy + \mathcal{O}\left(\frac{1}{|\log\epsilon|}\right),$$

which implies that

$$\Psi_{1,\epsilon}(0) = -\frac{2S_\epsilon(0)}{\xi_\epsilon}\frac{\int_{\mathbb{R}^2} w\Phi dy}{\int_{\mathbb{R}^2} w^2dy}\left(1 + \mathcal{O}\left(\frac{1}{|\log\epsilon|}\right)\right). \quad (5.3)$$

Then by (3.1) and (3.5) we have

$$\begin{aligned}\Psi_{1,\epsilon}(0)u_{1,\epsilon}^2(\epsilon y) &= -\frac{2S_\epsilon(0)}{\xi_\epsilon}\xi_\epsilon^2 w^2\frac{\int_{\mathbb{R}^2} w\Phi dy}{\int_{\mathbb{R}^2} w^2dy}\left(1 + \mathcal{O}\left(\frac{1}{|\log\epsilon|}\right)\right)\\ &= -2(1+\alpha_\epsilon)\frac{\int_{\mathbb{R}^2} w\Phi dy}{\int_{\mathbb{R}^2} w^2dy}w^2\left(1 + \mathcal{O}\left(\frac{1}{|\log\epsilon|}\right)\right). \quad (5.4)\end{aligned}$$

We also compute

$$\begin{aligned}\Psi_{2,\epsilon}(0) &= \frac{1}{|\log\epsilon|}\int_{\Omega_\epsilon} G_{D_2}(0,\epsilon z)[u_{2,\epsilon}^2(\epsilon z)\Phi_\epsilon(\epsilon z) + 2u_{1,\epsilon}(\epsilon z)u_{2,\epsilon}(\epsilon z)\Psi_{2,\epsilon}(\epsilon z)]dz\\ &= \frac{u_{2,\epsilon}(0)}{2\pi D_2(1+\alpha_\epsilon)}[u_{2,\epsilon}(0)\int_{\mathbb{R}^2}\Phi dy + 2\xi_\epsilon\Psi_{2,\epsilon}(0)\int_{\mathbb{R}^2} wdy]\left(1 + \mathcal{O}\left(\frac{1}{|\log\epsilon|}\right)\right).\end{aligned}$$

We know by (3.7)

$$u_{2,\epsilon}(0) = \frac{2\pi D_2(1+\alpha_\epsilon)}{\xi_\epsilon\int_{\mathbb{R}^2} w(y)dy} + \mathcal{O}(\frac{1}{|\log\epsilon|}),$$

which means that

$$\Psi_{2,\epsilon}(0) = \frac{1}{\xi_\epsilon\int_{\mathbb{R}^2} wdy}\left(u_{2,\epsilon}(0)\int_{\mathbb{R}^2}\Phi dy + 2\xi_\epsilon\Psi_{2,\epsilon}(0)\int_{\mathbb{R}^2} wdy\right)\left(1 + \mathcal{O}\left(\frac{1}{|\log\epsilon|}\right)\right).$$

Finally we get

$$\begin{aligned}\Psi_{2,\epsilon}(0) &= -\frac{u_{2,\epsilon}(0)}{\xi_\epsilon\int_{\mathbb{R}^2} wdy}\int_{\mathbb{R}^2}\Phi dy\left(1 + \mathcal{O}\left(\frac{1}{|\log\epsilon|}\right)\right)\\ &= -\frac{2\pi D_2(1+\alpha_\epsilon)}{\xi_\epsilon^2\int_{\mathbb{R}^2} wdy}\frac{\int_{\mathbb{R}^2}\Phi dy}{\int_{\mathbb{R}^2} wdy}\left(1 + \mathcal{O}\left(\frac{1}{|\log\epsilon|}\right)\right). \quad (5.5)\end{aligned}$$

Therefore we compute

$$-2a_2u_{1,\epsilon}u_{2,\epsilon}\Psi_{2,\epsilon} = 2\alpha_\epsilon\frac{\int_{\mathbb{R}^2}\Phi dy}{\int_{\mathbb{R}^2} wdy}w\left(1 + \mathcal{O}\left(\frac{1}{|\log\epsilon|}\right)\right). \quad (5.6)$$

Putting the expressions (3.4), (3.5), (5.4) and (5.6) into (5.2) and let $\epsilon \to 0$, we derive the nonlocal eigenvalue problem(NLEP)

$$\mathcal{L}\Phi = \Delta_y \Phi - (1+\alpha)\Phi + 2(1+\alpha)w\Phi - 2(1+\alpha)\frac{\int_{\mathbb{R}^2} w\Phi dy}{\int_{\mathbb{R}^2} w^2 dy} w^2 + 2\alpha \frac{\int_{\mathbb{R}^2} \Phi dy}{\int_{\mathbb{R}^2} w dy} w = \lambda\Phi, \qquad (5.7)$$

where $\alpha = \lim_{\epsilon \to 0} \alpha_\epsilon$.

Although this derivation has been only made formally, we can rigorously prove the following separation of eigenvalues.

Theorem 5.2. *let λ_ϵ be an eigenvalue of (5.2) such that $Re(\lambda_\epsilon) > -a_0$ for some suitable constant a_0 fixed independent of ϵ.*

(1) *Suppose that (for suitable sequences $\epsilon_n \to 0$), we have $\lambda_{\epsilon_n} \to \lambda_0 \neq 0$, then λ_0 is an eigenvalue of the NLEP given in (5.7).*
(2) *Let $\lambda_0 \neq 0$ be an eigenvalue of the NLEP given in (5.7), then for all ϵ sufficiently small, there is an eigenvalue λ_ϵ of (5.2) with $\lambda_\epsilon \to \lambda_0$ as $\epsilon \to 0$.*

Proof. Part (1) follows by an asymptotic analysis combined with passing to the limit as $\epsilon \to 0$ which is similar to the proof of Proposition 4.1.

Part (2) follows from a compactness argument by Dancer introduced in Section 2 of [2].

Let $\lambda_0 \neq 0$ be an eigenvalue of (5.7) with $Re(\lambda_0) > 0$. We rewrite (5.2) as follows:

$$\Phi_\epsilon = -R_\epsilon(\lambda_\epsilon)\left[2\tilde{S}_\epsilon \tilde{u}_{1,\epsilon}\Phi_\epsilon + \tilde{u}_{1,\epsilon}^2 (T_1'[\tilde{u}_{1,\epsilon}]\Phi_\epsilon) - a_2 \tilde{u}_{2,\epsilon}^2 \Phi_\epsilon - 2a_2 \tilde{u}_{1,\epsilon}\tilde{u}_{2,\epsilon}(T_2'[\tilde{u}_{1,\epsilon}]\Phi_\epsilon)\right], \qquad (5.8)$$

where $R_\epsilon(\lambda_\epsilon)$ is the inverse operator of $-\Delta + (1+\lambda_\epsilon)$ in $H^2(\Omega_\epsilon)$ (which exists if $Re(\lambda_\epsilon) > -1$ or $Im(\lambda_\epsilon) \neq 0$). The important thing is that $R_\epsilon(\lambda_\epsilon)$ is a compact operator if ϵ is small enough. The rest of the argument follows in the same way as in [2], we omit the details. □

Remark 5.3. *From Theorem 5.2 we see that the eigenvalue problem (5.2) is reduced to the study of the NLEP (5.7).*

The stability or instability of the large eigenvalues follows from the following results:

Theorem 5.4. [19] *Consider the eigenvalue problem*

$$\Delta\Phi - \Phi + 2w\Phi - \gamma \frac{\int_{\mathbb{R}^2} w\Phi}{\int_{\mathbb{R}^2} w^2} w^2 = \lambda\Phi, \quad \Phi \in H^2(\mathbb{R}^2), \qquad (5.9)$$

where w is a solution of (2.2) and γ is real.

(1) *If $\gamma > 1$, there exists a positive constant c_0 such that $Re(\lambda) \leq -c_0$ for any nonzero eigenvalue λ of (5.9).*
(2) *If $\gamma < 1$, there exists a positive eigenvalue λ of (5.9).*
(3) *If $\gamma \neq 1$ and $\lambda = 0$, then $\Phi \in span\{\frac{\partial w}{\partial y_1}, \frac{\partial w}{\partial y_2}\}$.*
(4) *If $\gamma = 1$ and $\lambda = 0$, then $\Phi \in span\{w, \frac{\partial w}{\partial y_1}, \frac{\partial w}{\partial y_2}\}$.*

In our applications to the case when $\tau > 0$ or $\tau_1 > 0$, we have to deal with the situation when the coefficient γ is a complex function of $\tau\lambda$. Let us suppose that

$$\gamma(0) \in \mathbb{R}, \quad |\gamma(\tau\lambda)| \leq C \text{ for } \lambda_R \geq 0, \quad \tau \geq 0, \qquad (5.10)$$

where C is a generic constant independent of τ, λ. Then we have

Theorem 5.5. [17] *Consider the nonlocal eigenvalue problem*

$$\Delta\Phi - \Phi + 2w\Phi - \gamma(\tau\lambda)\frac{\int_{\mathbb{R}^2} w\Phi}{\int_{\mathbb{R}^2} w^2} w^2 = \lambda\Phi, \quad \Phi \in H^2(\mathbb{R}^2), \qquad (5.11)$$

where $\gamma(\tau\lambda)$ satisfies (5.10). Then there exists $\tau_0 > 0$ such that for all $0 \le \tau < \tau_0$,
(1) if $\gamma(0) < 1$, then there is a positive eigenvalue to (5.11);
(2) if $\gamma(0) > 1$, then for any nonzero eigenvalue λ of (5.11), we have

$$Re(\lambda) \le -c_0 < 0.$$

Proof. Theorem 5.5 follows from Theorem 5.4 by a perturbation argument. To make sure that the perturbation argument works, we have to show that if $\lambda_R \ge 0$ and $0 < \tau < 1$, then $|\lambda| \le C$, where C is a generic constant. In fact, multiplying (5.11) by $\bar{\Phi}$ (the conjugate of Φ) and integrating by parts, we obtain that

$$\int_{\mathbb{R}^2}(|\nabla\Phi|^2 + |\Phi|^2 - 2w|\Phi|^2) = -\lambda\int_{\mathbb{R}^2}|\Phi|^2 - \gamma(\tau\lambda)\frac{\int_{\mathbb{R}^2} w\Phi}{\int_{\mathbb{R}^2} w^2}\int_{\mathbb{R}^2} w^2\bar{\Phi}. \tag{5.12}$$

From the imaginary part of (5.12), we obtain that

$$\lambda_I\int_{\mathbb{R}^2}|\Phi|^2 = Im\left(-\gamma(\tau\lambda)\frac{\int_{\mathbb{R}^2} w\Phi}{\int_{\mathbb{R}^2} w^2}\int_{\mathbb{R}^2} w^2\bar{\Phi}\right),$$

hence we have

$$|\lambda_I| \le C_1|\gamma(\tau\lambda)|,$$

where $\lambda = \lambda_R + \sqrt{-1}\lambda_I$ and C_1 is a positive constant. By assumption (5.10), $|\gamma(\tau\lambda)| \le C$ and so $|\lambda_I| \le C$. Taking the real part of (5.12) and noting that

$$l.h.s\ of\ (5.12) \ge C\int_{\mathbb{R}^2}|\Phi|^2 \text{ for some } C \in \mathbb{R},$$

we obtain that $\lambda_R \le C_2$, where C_2 is a positive constant. Therefore, $|\lambda|$ is uniformly bounded and hence a perturbation argument gives the desired conclusion. □

Now we consider the large eigenvalue problem (5.7).

Lemma 5.6. (1) *If $\alpha < 1$, for any nonzero eigenvalue of (5.7), we have*

$$Re(\lambda) \le -c_0 < 0.$$

If $\alpha > 1$, the eigenvalue problem (5.7) has an eigenvalue with $Re(\lambda) > 0$.
(2) *If $\alpha \ne 1$ and $\lambda = 0$, then $\Phi \in span\{\frac{\partial w}{\partial y_1}, \frac{\partial w}{\partial y_2}\}$.*

Proof. Integrating (5.7), we derive

$$(\alpha - \lambda - 1)\int_{\mathbb{R}^2}\Phi dy = 0.$$

Then for all the eigenvalues we have (i) $\alpha - \lambda - 1 = 0$, or (ii) the corresponding eigenfunction satisfies $\int_{\mathbb{R}^2}\Phi dy = 0$.

We first consider case (i). If $\alpha < 1$, then $\lambda = \alpha - 1 < 0$ and this eigenvalue λ is stable.

If $\alpha > 1$, then $\lambda = \alpha - 1 > 0$, we construct an eigenvalue Φ with eigenvalue λ as follows and the eigenvalue problem (5.7) is unstable: first we set

$$\Phi = (L + 1 - \alpha)^{-1}[c_1w^2 + c_2w], \tag{5.13}$$

where

$$L:\ K^\perp \to C^\perp, \quad L\Phi := \Delta\Phi - (1+\alpha)\Phi + 2(1+\alpha)w\Phi,$$

$$K^\perp = \{v \in H^2(\mathbb{R}^2) : \int_{\mathbb{R}^2} v\nabla w dy = 0\}, \quad C^\perp = \{v \in L^2(\mathbb{R}^2) : \int_{\mathbb{R}^2} v\nabla w dy = 0\},$$

$$c_1 = 2(1+\alpha)\frac{\int_{\mathbb{R}^2} w\Phi dy}{\int_{\mathbb{R}^2} w^2 dy}, \quad c_2 = -2\alpha \frac{\int_{\mathbb{R}^2} \Phi dy}{\int_{\mathbb{R}^2} w dy}.$$

Then we multiply (5.13) by w and 1, respectively, and integrating we get a linear system for the coefficients $(\int_{\mathbb{R}^2} w\Phi dy, \int_{\mathbb{R}^2} \Phi dy)$ which has a unique nontrivial solution. Solving this system, using the identities

$$Lw = (1+\alpha)w^2, \quad L\left(\frac{y\sqrt{\alpha+1}}{2}w_y + w\right) = (1+\alpha)w,$$

we can eliminate Φ in the definition of c_1 and c_2. We can finally get

$$c_1 = \int_{\mathbb{R}^2} w(L+1-\alpha)^{-1} w dy, \quad c_2 = -\int_{\mathbb{R}^2} w(L+1-\alpha)^{-1} w^2 dy + \frac{\int_{\mathbb{R}^2} w^2 dy}{2(1+\alpha)}.$$

Thus the eigenvalue problem is unstable for $\alpha > 1$.

Next we consider case (ii).

Rescaling the spatial variable, the NLEP (5.7) reduces to the familiar NLEP considered in Theorem 5.4 with $\gamma = 2$ which implies that the real parts of all eigenvalues are strictly negative and thus we get the stability. Then we get the proof for (1).

As for (2), integrating (5.7), we derive

$$\int_{\mathbb{R}^2} \Phi \, dy = 0.$$

Rescaling the spatial variable, The NLEP (5.7) reduces to the familiar NLEP considered in Theorem 5.4 with $\gamma = 2$ and we thus get that $\Phi \in \text{span}\{\frac{\partial w}{\partial y_1}, \frac{\partial w}{\partial y_2}\}$. □

Proof of Theorem 2.2. By (3.10) we know that $\alpha_\epsilon^l < 1$ and $\alpha_\epsilon^s > 1$. Then the theorem follows by combining the results of Theorem 5.2 and Lemma 5.6.

Next we extend the case $\tau = 0$ and $\tau_1 = 0$ to the case $\tau \geq 0$ or $\tau_1 \geq 0$ and show their stability.

Proof of Corollary 2.3. To emphasize the possible different behavior if $\tau > 0$ or $\tau_1 > 0$, we consider the cases separately:

(1): Consider $0 < \tau \leq \tau_0$ for some $\tau_0 > 0$ and $\tau_1 = 0$.

From the first equation of (5.1), using (3.5) and (7.15),

$$\begin{aligned}
\Psi_{1,\epsilon}(0) &= -\frac{a_1}{\epsilon^2 |\log \epsilon|} \int_\Omega G_{D_1,\tau\lambda}(0,z)[\Psi_{1,\epsilon} u_{1,\epsilon}^2 + 2S_\epsilon u_{1,\epsilon}\Phi_\epsilon](z) dz \\
&= -\frac{a_1}{\epsilon^2 |\log \epsilon|} \int_\Omega \left(\frac{1}{|\Omega|\tau\lambda} + G_{D_1}(0,z) + \mathcal{O}(\tau\lambda)\right)[\Psi_{1,\epsilon} u_{1,\epsilon}^2 + 2S_\epsilon u_{1,\epsilon}\Phi_\epsilon](z) dz \\
&= -a_1 \int_{\Omega_\epsilon} \left(\frac{1}{|\log\epsilon||\Omega|\tau\lambda} + \frac{G_{D_1}(0,\epsilon z)}{|\log \epsilon|} + \mathcal{O}\left(\frac{\tau\lambda}{|\log\epsilon|}\right)\right)[\Psi_{1,\epsilon} u_{1,\epsilon}^2 + 2S_\epsilon u_{1,\epsilon}\Phi_\epsilon](\epsilon z) dz \\
&= -a_1 \int_{\mathbb{R}^2} \left(\frac{1}{|\log\epsilon||\Omega|\tau\lambda} + \frac{1}{2\pi D_1}\right)(\Psi_{1,\epsilon} u_{1,\epsilon}^2 + 2S_\epsilon u_{1,\epsilon}\Phi_\epsilon)(\epsilon z) dz \left(1 + \mathcal{O}\left(\frac{1}{|\log\epsilon|}\right)\right).
\end{aligned} \tag{5.14}$$

We solve the equation above in three cases:

case 1: $\tau\lambda|\log\epsilon| \to 0$:

Then from (5.14), we get

$$\Psi_{1,\epsilon}(0) = -a_1 \int_{\mathbb{R}^2} \frac{1}{|\log\epsilon||\Omega|\tau\lambda}(\Psi_{1,\epsilon} u_{1,\epsilon}^2 + 2S_\epsilon u_{1,\epsilon}\Phi_\epsilon)(\epsilon z) dz \left(1 + \mathcal{O}\left(\frac{1}{|\log\epsilon|}\right)\right),$$

which implies that

$$\Psi_{1,\epsilon}(0) = \frac{-2a_1}{\frac{|\log\epsilon|\tau\lambda|\Omega|}{\int_{\mathbb{R}^2} w^2 dy} + \frac{a_1\xi_\epsilon^2}{1+\alpha_\epsilon}} \frac{\int_{\mathbb{R}^2} w\Phi_\epsilon dy}{\int_{\mathbb{R}^2} w^2 dy}\left(1+\mathcal{O}\left(\frac{1}{|\log\epsilon|}\right)\right),$$

then we compute

$$\Psi_{1,\epsilon}(0)u_{1,\epsilon}^2 = \frac{-2a_1}{\frac{|\log\epsilon|\tau\lambda|\Omega|}{\xi_\epsilon^2\int_{\mathbb{R}^2} w^2 dy} + \frac{a_1}{1+\alpha_\epsilon}} \frac{\int_{\mathbb{R}^2} w\Phi_\epsilon dy}{\int_{\mathbb{R}^2} w^2 dy} w^2\left(1+\mathcal{O}\left(\frac{1}{|\log\epsilon|}\right)\right).$$

It is easy to see that the factor

$$\frac{-2a_1}{\frac{|\log\epsilon|\tau\lambda|\Omega|}{\int_{\mathbb{R}^2} w^2 dy} + \frac{a_1\xi_\epsilon^2}{1+\alpha_\epsilon}}$$

is bounded.

case 2: $\tau\lambda|\log\epsilon| \to C_0$, where C_0 is a constant:

Then from (5.14), we get

$$\Psi_{1,\epsilon}(0) = -a_1\int_{\mathbb{R}^2}\left(\frac{1}{|\log\epsilon||\Omega|\tau\lambda} + \frac{1}{2\pi D_1}\right)(\Psi_{1,\epsilon}u_{1,\epsilon}^2 + 2S_\epsilon u_{1,\epsilon}\Phi_\epsilon)(\epsilon z)dz\left(1+\mathcal{O}\left(\frac{1}{|\log\epsilon|}\right)\right),$$

which implies that

$$\Psi_{1,\epsilon}(0) = \frac{-2a_1\left(\frac{1}{C_0|\Omega|} + \frac{1}{2\pi D_1}\right)}{\frac{1}{\int_{\mathbb{R}^2} w^2 dy} + \frac{a_1\xi_\epsilon^2}{1+\alpha_\epsilon}\left(\frac{1}{C_0|\Omega|} + \frac{1}{2\pi D_1}\right)} \frac{\int_{\mathbb{R}^2} w\Phi_\epsilon dy}{\int_{\mathbb{R}^2} w^2 dy}\left(1+\mathcal{O}\left(\frac{1}{|\log\epsilon|}\right)\right),$$

then we compute

$$\Psi_{1,\epsilon}(0)u_{1,\epsilon}^2 = \frac{-2a_1}{\frac{1}{\xi_\epsilon^2\int_{\mathbb{R}^2} w^2 dy\left(\frac{1}{C_0|\Omega|} + \frac{1}{2\pi D_1}\right)} + \frac{a_1}{1+\alpha_\epsilon}} \frac{\int_{\mathbb{R}^2} w\Phi_\epsilon dy}{\int_{\mathbb{R}^2} w^2 dy} w^2\left(1+\mathcal{O}\left(\frac{1}{|\log\epsilon|}\right)\right).$$

It is easy to see that the factor

$$\frac{-2a_1}{\frac{1}{\xi_\epsilon^2\int_{\mathbb{R}^2} w^2 dy\left(\frac{1}{C_0|\Omega|} + \frac{1}{2\pi D_1}\right)} + \frac{a_1}{1+\alpha_\epsilon}}$$

is bounded for $Re(\lambda) > 0$.

case 3: $\tau\lambda|\log\epsilon| \to \infty$:

Then from (5.14), we get

$$\Psi_{1,\epsilon}(0) = -a_1\int_{\mathbb{R}^2}\frac{1}{2\pi D_1}(\Psi_{1,\epsilon}u_{1,\epsilon}^2 + 2S_\epsilon u_{1,\epsilon}\Phi_\epsilon)(\epsilon z)dz\left(1+\mathcal{O}\left(\frac{1}{|\log\epsilon|}\right)\right),$$

which implies that

$$\Psi_{1,\epsilon}(0) = \frac{-2(1+\alpha_\epsilon)}{\frac{2\pi D_1(1+\alpha_\epsilon)}{a_1\int_{\mathbb{R}^2} w^2 dy} + \xi_\epsilon^2} \frac{\int_{\mathbb{R}^2} w\Phi_\epsilon dy}{\int_{\mathbb{R}^2} w^2 dy}\left(1+\mathcal{O}\left(\frac{1}{|\log\epsilon|}\right)\right),$$

then we compute

$$\Psi_{1,\epsilon}(0)u_{1,\epsilon}^2 = \frac{-2(1+\alpha_\epsilon)}{\frac{2\pi D_1(1+\alpha_\epsilon)}{a_1\xi_\epsilon^2\int_{\mathbb{R}^2} w^2 dy} + 1} \frac{\int_{\mathbb{R}^2} w\Phi_\epsilon dy}{\int_{\mathbb{R}^2} w^2 dy} w^2\left(1+\mathcal{O}\left(\frac{1}{|\log\epsilon|}\right)\right).$$

It is easy to see that the factor

$$\frac{-2(1+\alpha_\epsilon)}{\frac{2\pi D_1(1+\alpha_\epsilon)}{a_1\xi_\epsilon^2\int_{\mathbb{R}^2} w^2dy}+1}$$

is bounded.

Combining the three cases above, we can know by Theorem 5.5 that both the stability and instability result extend from $\tau = 0$ to $0 < \tau \le \tau_0$ for some $\tau_0 > 0$.

(2): Consider $\tau = 0$ and $0 < \tau_1 < \infty$.

Similar to the derivation of (5.14), we have

$$\begin{aligned}
\Psi_{2,\epsilon}(0) &= \frac{1}{\epsilon^2|\log\epsilon|}\int_\Omega G_{D_2,\tau_1\lambda}(0,z)[u_{2,\epsilon}^2\Phi_\epsilon + 2u_{1,\epsilon}u_{2,\epsilon}\Psi_\epsilon](z)dz \\
&= \frac{1}{\epsilon^2|\log\epsilon|}\int_\Omega (\frac{1}{|\Omega|(1+\tau_1\lambda)} + G_{D_2}(0,z) + \mathcal{O}(1+\tau_1\lambda))[u_{2,\epsilon}^2\Phi_\epsilon + 2u_{1,\epsilon}u_{2,\epsilon}\Psi_\epsilon](z)dz \\
&= \int_{\Omega_\epsilon}(\frac{1}{|\log\epsilon||\Omega|(1+\tau_1\lambda)} + \frac{G_{D_2}(0,\epsilon z)}{|\log\epsilon|} + \mathcal{O}(\frac{1+\tau_1\lambda}{|\log\epsilon|}))[u_{2,\epsilon}^2\Phi_\epsilon + 2u_{1,\epsilon}u_{2,\epsilon}\Psi_\epsilon](\epsilon z)dz \\
&= \frac{1}{2\pi D_2(1+\alpha_\epsilon)}\int_{\mathbb{R}^2}(u_{2,\epsilon}^2(0)\Phi_\epsilon + 2\xi_\epsilon u_{2,\epsilon}(0)\Psi_{2,\epsilon}(0)w)dy\left(1+\mathcal{O}\left(\frac{1}{|\log\epsilon|}\right)\right),
\end{aligned}$$

which implies

$$\Psi_{2,\epsilon}(0) = \frac{u_{2,\epsilon}^2(0)\int_{\mathbb{R}^2} wdy}{2\pi D_2(1+\alpha_\epsilon) - 2\xi_\epsilon u_{2,\epsilon}(0)\int_{\mathbb{R}^2} wdy}\frac{\int_{\mathbb{R}^2}\Phi_\epsilon dy}{\int_{\mathbb{R}^2} wdy}\left(1+\mathcal{O}\left(\frac{1}{|\log\epsilon|}\right)\right). \tag{5.15}$$

Finally we get by (3.7) that

$$-2a_2u_{1,\epsilon}u_{2,\epsilon}(0)\Psi_{2,\epsilon}(0) = 2\alpha_\epsilon\frac{\int_{\mathbb{R}^2}\Phi_\epsilon dy}{\int_{\mathbb{R}^2} wdy}w\left(1+\mathcal{O}\left(\frac{1}{|\log\epsilon|}\right)\right) \quad \text{in } H^2(\Omega_\epsilon).$$

Thus we have for ϵ small enough, putting all the expressions into the second equation of (5.1), we know that both the stability and instability are the same as in the case when $\tau = 0$ and $\tau_1 = 0$.

(3): In this stage we assume that $0 < \tau \le \tau_0$ for some $\tau_0 > 0$ and $0 < \tau_1 < \infty$.

Combining the formulas in the proofs of (1) and (2), it follows that both the stability and instability result extend from $\tau = 0$ and $\tau_1 = 0$ to $0 < \tau \le \tau_0$ for some $\tau_0 > 0$ and $0 < \tau_1 < \infty$.

6. Stability II: Computation of the Small Eigenvalues

We now compute the small eigenvalues of the eigenvalue problem (5.1), i.e., we assume that $\lambda_\epsilon \to 0$ as $\epsilon \to 0$. We emphasize that the analysis in this section applies to both $(S_\epsilon^s, u_{1,\epsilon}^s, u_{2,\epsilon}^s)$ and $(S_\epsilon^l, u_{1,\epsilon}^l, u_{2,\epsilon}^l)$. Further, we assume that $0 \le \tau \le \tau_0$, where $\tau_0 > 0$ is a constant which is small enough and may be chosen independently of ϵ, and $0 \le \tau_1 < \infty$. Let us define

$$\tilde{u}_{1,\epsilon}(x) = u_{1,\epsilon}(x)\chi(|x|).$$

Then it follows easily that

$$u_{1,\epsilon}(x) = \tilde{u}_{1,\epsilon}(x) + \text{e.s.t.} \quad \text{in } H^2(\Omega_\epsilon),$$

where e.s.t. denote the exponentially small term of $\mathcal{O}(e^{-d/\epsilon})$ for some $d > 0$ in the corresponding term.

Taking the derivation of the system (2.1) w.r.t. y we compute

$$\Delta_y \nabla_y \tilde{u}_{1,\epsilon} - \nabla_y \tilde{u}_{1,\epsilon} + 2S_\epsilon u_{1,\epsilon} \nabla_y \tilde{u}_{1,\epsilon} + \epsilon \nabla_x S_\epsilon u_{1,\epsilon}^2 - a_2 \nabla_y \tilde{u}_{1,\epsilon} u_{2,\epsilon}^2 - 2a_2 u_{1,\epsilon} u_{2,\epsilon} \epsilon \nabla_x u_{2,\epsilon} = \text{e.s.t.}. \quad (6.1)$$

Let us now decompose the eigenfunction $(\Psi_{1,\epsilon}, \Phi_\epsilon, \Psi_{2,\epsilon})$ as follows:

$$\Phi_\epsilon = a^\epsilon \nabla_y \tilde{u}_{1,\epsilon} + \Phi_\epsilon^\perp,$$

where $a^\epsilon = \begin{pmatrix} a_{1,\epsilon} \\ a_{2,\epsilon} \end{pmatrix}$, $a_{1,\epsilon}$ and $a_{2,\epsilon}$ are complex numbers to be determined and

$$\Phi_\epsilon^\perp \perp \mathcal{K}_\epsilon = \text{span}\{\frac{\partial \tilde{u}_{1,\epsilon}}{\partial y_1}, \frac{\partial \tilde{u}_{1,\epsilon}}{\partial y_2}\} \subset H_N^2(\Omega_\epsilon).$$

We decompose the eigenfunction $\Psi_{1,\epsilon}$ as follows:

$$\Psi_{1,\epsilon} = a^\epsilon \Psi_{1,\epsilon}^0 + \Psi_{1,\epsilon}^\perp,$$

where $\Psi_{1,\epsilon}^0$ satisfies

$$\begin{cases} D_1 \Delta \Psi_{1,\epsilon}^0 - \frac{a_1}{\epsilon^2 |\log \epsilon|} \Psi_{1,\epsilon}^0 u_{1,\epsilon}^2 - \frac{2a_1}{\epsilon^2 |\log \epsilon|} S_\epsilon u_{1,\epsilon} \nabla_y \tilde{u}_{1,\epsilon} = \tau \lambda_\epsilon \Psi_{1,\epsilon}^0, \\ \frac{\partial}{\partial \nu} \Psi_{1,\epsilon}^0 \Big|_{\partial\Omega} = 0, \end{cases}$$

and $\Psi_{1,\epsilon}^\perp$ is given by

$$\begin{cases} D_1 \Delta \Psi_{1,\epsilon}^\perp - \frac{a_1}{\epsilon^2 |\log \epsilon|} \Psi_{1,\epsilon}^\perp u_{1,\epsilon}^2 - \frac{2a_1}{\epsilon^2 |\log \epsilon|} S_\epsilon u_{1,\epsilon} \Phi_\epsilon^\perp = \tau \lambda_\epsilon \Psi_{1,\epsilon}^\perp, \\ \frac{\partial}{\partial \nu} \Psi_{1,\epsilon}^\perp \Big|_{\partial\Omega} = 0. \end{cases} \quad (6.2)$$

Similarly, we decompose the eigenfunction $\Psi_{2,\epsilon}$ as follows:

$$\Psi_{2,\epsilon} = a^\epsilon \Psi_{2,\epsilon}^0 + \Psi_{2,\epsilon}^\perp,$$

where $\Psi_{2,\epsilon}^0$ satisfies

$$\begin{cases} D_2 \Delta \Psi_{2,\epsilon}^0 - \Psi_{2,\epsilon}^0 + \frac{2}{\epsilon^2 |\log \epsilon|} u_{1,\epsilon} u_{2,\epsilon} \Psi_{2,\epsilon}^0 - \frac{1}{\epsilon^2 |\log \epsilon|} u_{2,\epsilon}^2 \nabla_y \tilde{u}_{1,\epsilon} = \tau_1 \lambda_\epsilon \Psi_{2,\epsilon}^0, \\ \frac{\partial}{\partial \nu} \Psi_{2,\epsilon}^0 \Big|_{\partial\Omega} = 0, \end{cases}$$

and $\Psi_{2,\epsilon}^\perp$ is given by

$$\begin{cases} D_2 \Delta \Psi_{2,\epsilon}^\perp - \Psi_{2,\epsilon}^\perp + \frac{2}{\epsilon^2 |\log \epsilon|} u_{1,\epsilon} u_{2,\epsilon} \Psi_{2,\epsilon}^\perp - \frac{1}{\epsilon^2 |\log \epsilon|} u_{2,\epsilon}^2 \Phi_\epsilon^\perp = \tau_1 \lambda_\epsilon \Psi_{2,\epsilon}^\perp, \\ \frac{\partial}{\partial \nu} \Psi_{2,\epsilon}^\perp \Big|_{\partial\Omega} = 0. \end{cases} \quad (6.3)$$

Note that $\Psi_{1,\epsilon}$ and $\Psi_{2,\epsilon}$ can be uniquely expressed in terms of Φ_ϵ by solving the first and third equation using the Green's function $G_{D_1,\tau\lambda_\epsilon}$ and $G_{D_2,\tau\lambda_\epsilon}$ defined in (7.15) and (7.19), respectively:

$$\Psi_{1,\epsilon} = a^\epsilon \Psi_{1,\epsilon}^0 + \Psi_{1,\epsilon}^\perp = a^\epsilon T'_{1,\tau\lambda_\epsilon} \nabla_y \tilde{u}_{1,\epsilon} + T'_{1,\tau\lambda_\epsilon} \Phi_\epsilon^\perp,$$

$$\Psi_{2,\epsilon} = a^\epsilon \Psi_{2,\epsilon}^0 + \Psi_{2,\epsilon}^\perp = a^\epsilon T'_{2,\tau_1\lambda_\epsilon} \nabla_y \tilde{u}_{1,\epsilon} + T'_{2,\tau_1\lambda_\epsilon} \Phi_\epsilon^\perp.$$

Putting all the expressions into (5.2) we can decompose (5.2) by (6.1) as follows:

$$\begin{aligned} I_1 + I_2 + I_3 =:& \, a^\epsilon u_{1,\epsilon}^2 (\Psi_{1,\epsilon}^0 - \epsilon \nabla_x S_\epsilon) - 2a^\epsilon a_2 u_{1,\epsilon} u_{2,\epsilon} (\Psi_{2,\epsilon}^0 - \epsilon \nabla_x u_{2,\epsilon}) \\ &+ \Delta \Phi_\epsilon^\perp - \Phi_\epsilon^\perp + 2u_{1,\epsilon} S_\epsilon \Phi_\epsilon^\perp + u_{1,\epsilon}^2 \Psi_{1,\epsilon}^\perp - 2a_2 u_{1,\epsilon} u_{2,\epsilon} \Psi_{2,\epsilon}^\perp - 2a_2 u_{2,\epsilon}^2 \Phi_\epsilon^\perp - \lambda_\epsilon \Phi_\epsilon^\perp \\ =& \, \lambda_\epsilon a^\epsilon \nabla_y \tilde{u}_{1,\epsilon}. \end{aligned} \quad (6.4)$$

Using the Green's function G_{D_1} and $G_{D_1,\tau\lambda_\epsilon}$ defined in (7.13) and (7.15), we compute $\epsilon\nabla_x S_\epsilon$ and $\Psi^0_{1,\epsilon}$ near zero, we get

$$\begin{aligned}\epsilon\nabla_x S_\epsilon(\epsilon y) &= -\frac{a_1}{\epsilon^2|\log\epsilon|}\int_\Omega \nabla_y G_{D_1}(\epsilon y, z)S_\epsilon(z)u^2_{1,\epsilon}(z)dz\\ &= -\frac{a_1}{|\log\epsilon|}\int_{\Omega_\epsilon}\nabla_y\left(\frac{1}{2\pi D_1}\log\frac{1}{\epsilon|y-z|}-H_{D_1}(\epsilon y,\epsilon z)\right)S_\epsilon(\epsilon z)u^2_{1,\epsilon}(\epsilon z)dz\\ &= -\frac{a_1 S_\epsilon(0)}{|\log\epsilon|}\int_{\mathbb{R}^2}\nabla_y\left(\frac{1}{2\pi D_1}\log\frac{1}{\epsilon|y-z|}-H_{D_1}(\epsilon y,\epsilon z)\right)u^2_{1,\epsilon}(\epsilon z)dz\left(1+\mathcal{O}\left(\frac{1}{|\log\epsilon|}\right)\right),\end{aligned}$$

denote $z=\epsilon\tilde{z}$,

$$\begin{aligned}\Psi^0_{1,\epsilon}(\epsilon y) &= -\frac{2a_1}{\epsilon^2|\log\epsilon|}\int_\Omega G_{D_1,\tau\lambda_\epsilon}(\epsilon y,z)S_\epsilon(z)u_{1,\epsilon}(z)\nabla_{\tilde z}u_{1,\epsilon}(z)dz\\ &= -\frac{a_1}{\epsilon^2|\log\epsilon|}\int_\Omega G_{D_1,\tau\lambda_\epsilon}(\epsilon y,z)S_\epsilon(z)\nabla_{\tilde z}(u^2_{1,\epsilon}(z))dz\\ &= -\frac{a_1 S_\epsilon(0)}{|\log\epsilon|}\int_{\Omega_\epsilon}G_{D_1,\tau\lambda_\epsilon}(\epsilon y,\epsilon z)\nabla_z(u^2_{1,\epsilon}(\epsilon z))dz\left(1+\mathcal{O}\left(\frac{1}{|\log\epsilon|}\right)\right)\\ &= -\frac{a_1 S_\epsilon(0)}{|\log\epsilon|}\int_{\mathbb{R}_2}\left(\frac{1}{|\Omega|\tau\lambda_\epsilon}+\frac{1}{2\pi D_1}\log\frac{1}{\epsilon|y-z|}-H_{D_1}(\epsilon y,\epsilon z)\right)\\ &\quad\times\nabla_z(u^2_{1,\epsilon}(\epsilon z))dz\left(1+\mathcal{O}\left(\frac{1}{|\log\epsilon|}+\tau|\lambda_\epsilon|\right)\right).\end{aligned}$$

Using the fact that

$$\nabla_y\left(\frac{1}{2\pi D_1}\log\frac{1}{|\epsilon y-\epsilon z|}\right)+\nabla_z\left(\frac{1}{2\pi D_1}\log\frac{1}{|\epsilon y-\epsilon z|}\right)=0,\quad \text{for } y\neq z,$$

then we have by integrating by parts that

$$\begin{aligned}&\Psi^0_{1,\epsilon}(\epsilon y)-\epsilon\nabla_x S_\epsilon(\epsilon y)\\ &= -\frac{a_1 S_\epsilon(0)}{|\log\epsilon|}\int_{\mathbb{R}_2}\Bigg(\left(\frac{1}{|\Omega|\tau\lambda_\epsilon}+\frac{1}{2\pi D_1}\log\frac{1}{\epsilon|y-z|}-H_{D_1}(\epsilon y,\epsilon z)\right)\nabla_z(u^2_{1,\epsilon}(\epsilon z))-\\ &\quad\nabla_y\left(\frac{1}{2\pi D_1}\log\frac{1}{\epsilon|y-z|}-H_{D_1}(\epsilon y,\epsilon z)\right)u^2_{1,\epsilon}(\epsilon z)\Bigg)dz\left(1+\mathcal{O}\left(\frac{1}{|\log\epsilon|}+\tau|\lambda_\epsilon|\right)\right)\\ &= -\frac{a_1 S_\epsilon(0)}{|\log\epsilon|}\int_{\mathbb{R}^2}(\nabla_z H_{D_1}(\epsilon y,\epsilon z)+\nabla_y H_{D_1}(\epsilon y,\epsilon z))u^2_{1,\epsilon}(\epsilon z)dz\left(1+\mathcal{O}\left(\frac{1}{|\log\epsilon|}+\tau|\lambda_\epsilon|\right)\right)\\ &= -\frac{a_1 S_\epsilon(0)}{|\log\epsilon|}\int_{\mathbb{R}^2}\epsilon^2 y(\nabla_{z'}\nabla_x H_{D_1}(0,0)+\nabla^2_x H_{D_1}(0,0))u^2_{1,\epsilon}(\epsilon z)dz\left(1+\mathcal{O}\left(\frac{1}{|\log\epsilon|}+\tau|\lambda_\epsilon|\right)\right)\\ &= -\frac{a_1\epsilon^2 y S_\epsilon(0)}{2|\log\epsilon|}\nabla^2 H_{D_1}(0,0)\int_{\mathbb{R}^2}u^2_{1,\epsilon}(\epsilon z)dz\left(1+\mathcal{O}\left(\frac{1}{|\log\epsilon|}+\tau|\lambda_\epsilon|\right)\right)\\ &= -\frac{a_1\epsilon^2 y\xi_\epsilon}{2|\log\epsilon|}\nabla^2 H_{D_1}(0,0)\int_{\mathbb{R}^2}w^2dy\left(1+\mathcal{O}\left(\frac{1}{|\log\epsilon|}+\tau|\lambda_\epsilon|\right)\right),\end{aligned}$$

here ∇_x and $\nabla_{z'}$ are the derivatives of the first component and the second component, respectively. Thus

$$\begin{aligned} I_1 &:= a^\epsilon u_{1,\epsilon}^2(\Psi_{1,\epsilon}^0 - \epsilon\nabla_x S_\epsilon) \\ &= -\frac{a^\epsilon a_1 \epsilon^2 y \xi_\epsilon^3}{2|\log\epsilon|}\nabla^2 H_{D_1}(0,0) w^2 \int_{\mathbb{R}^2} w^2 dy \left(1 + \mathcal{O}\left(\frac{1}{|\log\epsilon|} + \tau|\lambda_\epsilon|\right)\right). \end{aligned}$$

Similarly,

$$\begin{aligned} \epsilon\nabla_x u_{2,\epsilon}(\epsilon y) &= \frac{1}{\epsilon^2|\log\epsilon|}\int_\Omega \nabla_y G_{D_2}(\epsilon y, z) u_{1,\epsilon}(z) u_{2,\epsilon}^2(z) dz \\ &= \frac{u_{2,\epsilon}^2(0)}{|\log\epsilon|}\int_{\mathbb{R}^2} \nabla_y \left(\frac{1}{2\pi D_2}\log\frac{1}{\epsilon|y-z|} - H_{D_2}(\epsilon y, \epsilon z)\right) u_{1,\epsilon}(\epsilon z) dz \left(1 + \mathcal{O}\left(\frac{1}{|\log\epsilon|}\right)\right), \end{aligned}$$

$$\begin{aligned} \Psi_{2,\epsilon}^0(\epsilon y) &= \frac{1}{\epsilon^2|\log\epsilon|}\int_\Omega G_{D_2,\tau_1\lambda_\epsilon}(\epsilon y, z) u_{2,\epsilon}^2(z) \nabla_{\tilde{z}} u_{1,\epsilon}(z) dz \\ &= \frac{u_{2,\epsilon}^2(0)}{|\log\epsilon|}\int_{\Omega_\epsilon} G_{D_2,\tau_1\lambda_\epsilon}(\epsilon y, \epsilon z)\nabla_z u_{1,\epsilon}(\epsilon z) dz \left(1 + \mathcal{O}\left(\frac{1}{|\log\epsilon|}\right)\right) \\ &= \frac{u_{2,\epsilon}^2(0)}{|\log\epsilon|}\int_{\mathbb{R}^2}\left(\frac{1}{|\Omega|(1+\tau_1\lambda_\epsilon)} + \frac{1}{2\pi D_2}\log\frac{1}{\epsilon|y-z|} - H_{D_2}(\epsilon y, \epsilon z)\right) \\ &\quad \times \nabla_z u_{1,\epsilon}(\epsilon z) dz \left(1 + \mathcal{O}\left(\frac{1}{|\log\epsilon|} + \tau_1|\lambda_\epsilon|\right)\right), \end{aligned}$$

$$\begin{aligned} \Psi_{2,\epsilon}^0(\epsilon y) - \epsilon\nabla_x u_{2,\epsilon}(\epsilon y) &= \frac{u_{2,\epsilon}^2(0)}{|\log\epsilon|}\int_{\mathbb{R}_2}\left(\left(\frac{1}{|\Omega|(1+\tau_1\lambda_\epsilon)} + \frac{1}{2\pi D_2}\log\frac{1}{\epsilon|y-z|} - H_{D_2}(\epsilon y, \epsilon z)\right)\right. \\ &\quad \times \nabla_z u_{1,\epsilon}(\epsilon z) \\ &\quad \left. -\nabla_y\left(\frac{1}{2\pi D_2}\log\frac{1}{\epsilon|y-z|} - H_{D_2}(\epsilon y, \epsilon z)\right) u_{1,\epsilon}(\epsilon z)\right) dz \\ &\quad \times \left(1 + \mathcal{O}\left(\frac{1}{|\log\epsilon|} + \tau_1|\lambda_\epsilon|\right)\right) \\ &= \frac{\epsilon^2 y u_{2,\epsilon}^2(0)\xi_\epsilon}{2(1+\alpha_\epsilon)|\log\epsilon|}\nabla^2 H_{D_2}(0,0)\int_{\mathbb{R}^2} w dy \left(1 + \mathcal{O}\left(\frac{1}{|\log\epsilon|} + \tau_1|\lambda_\epsilon|\right)\right). \end{aligned}$$

Thus

$$\begin{aligned} I_2 &:= -2a^\epsilon a_2 u_{1,\epsilon} u_{2,\epsilon}(\Psi_{2,\epsilon}^0 - \epsilon\nabla_x u_{2,\epsilon}) \\ &= -\epsilon^2 y a^\epsilon a_2 \frac{u_{2,\epsilon}^3(0)\xi_\epsilon^2}{(1+\alpha_\epsilon)|\log\epsilon|}\nabla^2 H_{D_2}(0,0) w \int_{\mathbb{R}^2} w dy \left(1 + \mathcal{O}\left(\frac{1}{|\log\epsilon|} + \tau_1|\lambda_\epsilon|\right)\right). \end{aligned}$$

We now estimate the orthogonal part of the eigenfunction by using the equation (6.4). Since $\Phi_\epsilon^\perp \perp \mathcal{K}_\epsilon$, then similar to the proof of Proposition 4.1, we conclude that

$$\|\Phi_\epsilon^\perp\|_{H^2(\Omega_\epsilon)} \le C\|I_1 + I_2\|_{L^2(\Omega_\epsilon)} = \mathcal{O}\left(\frac{\epsilon^2}{|\log\epsilon|}\right). \tag{6.5}$$

This implies

$$\|T'_{1,\tau\lambda_\epsilon}\Phi_\epsilon^\perp\|_{H^2(\Omega_\epsilon)} = \mathcal{O}\left(\frac{\epsilon^2}{|\log\epsilon|}\right), \tag{6.6}$$

and

$$\|T'_{2,\tau_1\lambda_\epsilon}\Phi_\epsilon^\perp\|_{H^2(\Omega_\epsilon)} = \mathcal{O}\left(\frac{\epsilon^2}{|\log\epsilon|}\right). \tag{6.7}$$

We calculate

$$\begin{aligned}\int_{\Omega_\epsilon} I_3\nabla_y u_{1,\epsilon}dy &= 2a_2\int_{\Omega_\epsilon} u_{1,\epsilon}u_{2,\epsilon}\epsilon\nabla_x u_{2,\epsilon}\Phi_\epsilon^\perp dy - \int_\Omega \epsilon\nabla_x S_\epsilon u_{1,\epsilon}^2\Phi_\epsilon^\perp dy \\ &\quad - 2a_2\int_{\Omega_\epsilon} u_{1,\epsilon}u_{2,\epsilon}\nabla_y u_{1,\epsilon}\Psi_{2,\epsilon}^\perp dy + \int_\Omega u_{1,\epsilon}^2\nabla_y u_{1,\epsilon}\Psi_{1,\epsilon}^\perp dy \\ &= \mathcal{O}\left(\frac{\epsilon^2}{|\log\epsilon|^2}\right),\end{aligned}$$

by using (6.2), (6.3) and the estimate

$$\epsilon\nabla_x u_{2,\epsilon} = \mathcal{O}\left(\frac{1}{|\log\epsilon|}\right), \qquad \epsilon\nabla_x S_\epsilon = \mathcal{O}\left(\frac{1}{|\log\epsilon|}\right).$$

Note that

$$\int_{\mathbb{R}^2} w^2\nabla wy = -\frac{1}{3}\int_{\mathbb{R}^2} w^3, \quad \int_{\mathbb{R}^2} w\nabla wy = -\frac{1}{2}\int_{\mathbb{R}^2} w^2, \quad \int_{\mathbb{R}^2} w^3dy = \frac{3}{2}\int_{\mathbb{R}^2} w^2dy. \tag{6.8}$$

Thus we multiply the eigenvalue problem (5.2) by ∇w and integrating, we get

$$\begin{aligned}\text{l.h.s.} &= \int_{\mathbb{R}^2}(I_1+I_2+I_3)\nabla wdy \\ &= -\frac{a^\epsilon a_1\epsilon^2\xi_\epsilon^3}{2|\log\epsilon|}\nabla^2 H_{D_1}(0,0)\int_{\mathbb{R}^2} w^2\nabla wydy\int_{\mathbb{R}^2} w^2dy\left(1+\mathcal{O}\left(\frac{1}{|\log\epsilon|}+\tau|\lambda_\epsilon|\right)\right) \\ &\quad - \epsilon^2 a^\epsilon a_2\frac{u_{2,\epsilon}^3(0)\xi_\epsilon^2}{(1+\alpha_\epsilon)|\log\epsilon|}\nabla^2 H_{D_2}(0,0)\int_{\mathbb{R}^2} w\nabla wydy\int_{\mathbb{R}^2} wdy\left(1+\mathcal{O}\left(\frac{1}{|\log\epsilon|}+\tau_1|\lambda_\epsilon|\right)\right) \\ &\quad + \mathcal{O}\left(\frac{\epsilon^2}{|\log\epsilon|^2}\right) \\ &= \frac{a^\epsilon\epsilon^2\xi_\epsilon^2}{|\log\epsilon|}\left(\frac{a_1\xi_\epsilon}{4}\nabla^2 H_{D_1}(0,0)+\frac{a_2u_{2,\epsilon}^3(0)}{2(1+\alpha_\epsilon)}\nabla^2 H_{D_2}(0,0)\right)(\int_{\mathbb{R}^2} w^2dy)^2+\mathcal{O}\left(\frac{\epsilon^2}{|\log\epsilon|^2}\right).\end{aligned}$$

Further, we compute

$$r.h.s. = \lambda_\epsilon a^\epsilon\xi_\epsilon\int_{\mathbb{R}^2}(\nabla w)^2dy\left(1+\mathcal{O}\left(\frac{1}{|\log\epsilon|}\right)\right).$$

Combining the l.h.s. and r.h.s., we have

$$a^\epsilon\left(\lambda_\epsilon I - \frac{\epsilon^2\xi_\epsilon}{|\log\epsilon|}(\frac{a_1\xi_\epsilon}{4}\nabla^2 H_{D_1}(0,0)+\frac{a_2u_{2,\epsilon}^3(0)}{2(1+\alpha_\epsilon)}\nabla^2 H_{D_2}(0,0))\frac{(\int_{\mathbb{R}^2} w^2dy)^2}{\int_{\mathbb{R}^2}(\nabla w)^2dy}\right) = \mathcal{O}(\frac{\epsilon^2}{|\log\epsilon|^2}).$$

Denote $M = \frac{a_1\xi_\epsilon}{4}\nabla^2 H_{D_1}(0,0)+\frac{a_2u_{2,\epsilon}^3(0)}{2(1+\alpha_\epsilon)}\nabla^2 H_{D_2}(0,0)$, then we have

$$\frac{\lambda_\epsilon}{\frac{\epsilon^2}{|\log\epsilon|}} \to \xi_\epsilon\frac{(\int_{\mathbb{R}^2} w^2dy)^2}{\int_{\mathbb{R}^2}(\nabla w)^2dy}\sigma,$$

where σ is an eigenvalue of M, and the vector a^ϵ approaches the eigenvector of M corresponding to σ.

Remark 6.1. *From the study of the regular part of the Green's function in* [1], *we know that both* $\nabla^2 H_{D_1}$ *and* $\nabla^2 H_{D_2}$ *are negative definite matrices at the origin.*

We summarize our result on the small eigenvalues in the following theorem.

Theorem 6.2. *The eigenvalues of (5.1) with* $\lambda_\epsilon \to 0$ *satisfy* $|\lambda_\epsilon| \sim \frac{\epsilon^2}{|\log \epsilon|}$. *Furthermore,*

$$\frac{\lambda_\epsilon}{\frac{\epsilon^2}{|\log \epsilon|}} \to \xi_\epsilon \frac{(\int_{\mathbb{R}^2} w^2 dy)^2}{\int_{\mathbb{R}^2} (\nabla w)^2 dy} \sigma. \tag{6.9}$$

In particular these eigenvalues are stable.

This completes the proof of Theorem 2.2.

7. Appendix

7.1. **Proof for Proposition 4.1.** We will divide the proof into two parts:
Part I : There exist positive constants ϵ' and λ such that for all $\epsilon \in (0, \epsilon')$,

$$\|\mathcal{L}_\epsilon \Phi\|_{L^2(\Omega_\epsilon)} \geq \lambda \|\Phi\|_{H^2_N(\Omega_\epsilon)} \quad \text{for all } \Phi \in \mathcal{K}_\epsilon^\perp. \tag{7.1}$$

Suppose that (7.1) is false. Then there exist sequences $\{\epsilon_k\}$, $\{\Phi^k\}$ with $\epsilon_k \to 0$, $\Phi^k = \Phi_{\epsilon_k}$, $k = 1, 2, ...$ such that

$$\|\mathcal{L}_{\epsilon_k} \Phi^k\|_{L^2(\Omega_\epsilon)} \to 0, \quad k \to \infty, \tag{7.2}$$

$$\|\Phi^k\|_{H^2(\Omega_\epsilon)} = 1, \qquad k = 1, 2, ... \tag{7.3}$$

By using the cut-off function χ defined in (3.2), we define the following functions:

$$\begin{aligned} \Phi_{1,\epsilon}(y) &= \Phi_\epsilon(y)\chi(|x|), \qquad y \in \Omega_\epsilon. \\ \Phi_{2,\epsilon}(y) &= \Phi_\epsilon(y)(1 - \chi(|x|)), \quad y \in \Omega_\epsilon. \end{aligned} \tag{7.4}$$

Let $\Phi_{1,\epsilon} = 0$ and $\Phi_{2,\epsilon} = 0$ in $\mathbb{R}^2 \setminus \Omega_\epsilon$, then by a standard procedure, we extend $\Phi_{1,\epsilon}$ and $\Phi_{2,\epsilon}$ to a function defined on $\mathbb{R}^2$, respectively, such that

$$\|\Phi_{1,\epsilon}\|_{H^2(\mathbb{R}^2)} \leq C \|\Phi_{1,\epsilon}\|_{H^2(\Omega_\epsilon)}, \qquad \|\Phi_{2,\epsilon}\|_{H^2(\mathbb{R}^2)} \leq C \|\Phi_{2,\epsilon}\|_{H^2(\Omega_\epsilon)},$$

for some positive constant C.
Then from $\|\Phi_\epsilon\|_{H^2(\Omega_\epsilon)} = 1$ we have

$$\|\Phi_{1,\epsilon}\|_{H^2(\mathbb{R}^2)} \leq C, \quad \|\Phi_{2,\epsilon}\|_{H^2(\mathbb{R}^2)} \leq C.$$

By taking a subsequence of ϵ we may also assume that for $i = 1, 2$,

$$\Phi_{i,\epsilon} \to \Phi_i \quad \text{as} \quad \epsilon \to 0 \quad \text{in } H^2(\mathbb{R}^2).$$

Taking the limit $\epsilon \to 0$ in (4.4), then $\Phi = \begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix}$ satisfies

$$\int_{\mathbb{R}^2} \Phi_1 \nabla w dy = 0, \tag{7.5}$$

and it solves the system

$$\mathcal{L}\Phi_1 = \Delta_y \Phi_1 - (1+\alpha)\Phi_1 + 2(1+\alpha) w \Phi_1 - 2(1+\alpha) \frac{\int_{\mathbb{R}^2} w \Phi_1 dy}{\int_{\mathbb{R}^2} w^2 dy} w^2 + 2\alpha \frac{\int_{\mathbb{R}^2} \Phi_1 dy}{\int_{\mathbb{R}^2} w dy} w = 0. \tag{7.6}$$

In Lemma 5.6 we have show that the system (7.5) and (7.6) has only the solution $\Phi_1 = 0$ in $\mathbb{R}^2$.
Further, trivially, $\Phi_2 = 0$ in $\mathbb{R}^2$.

By standard elliptic estimates we get $\|\Phi_{i,\epsilon_k}\|_{H^2(\Omega_\epsilon)} \to 0$ for $i = 1, 2$ as $k \to \infty$. This contradicts the assumption that $\|\Phi^k\|_{H^2(\Omega_\epsilon)} = 1$.

Part II : The linear operator $\mathcal{L}_\epsilon$ is surjective.

Consider the adjoint operator $\mathcal{L}_\epsilon^*$ to the linear operator $\mathcal{L}_\epsilon$. To show the linear operator $\mathcal{L}_\epsilon$ is surjective, we just need to show that $\mathcal{L}_\epsilon^*$ is injective from $\mathcal{K}_\epsilon^\perp$ to $\mathcal{C}_\epsilon^\perp$. We first pass to the limit $\epsilon \to 0$ for the adjoint operator $\mathcal{L}_\epsilon^*$. Then we have to show that the limiting adjoint operator $\mathcal{L}^*$ has only the trivial kernel.

Expressing $\mathcal{L}_\epsilon^*$ explicitly, we can rewrite the adjoint eigenvalue problem as follows:

$$\begin{cases} D_1 \Delta \Psi_{1,\epsilon} - \frac{a_1}{\epsilon^2|\log\epsilon|} u_{1,\epsilon}^2 \Psi_{1,\epsilon} + u_{1,\epsilon}^2 \Phi_\epsilon = \tau \lambda_\epsilon \Psi_{1,\epsilon}, \\ \epsilon^2 \Delta \Phi_\epsilon - \Phi_\epsilon + 2 S_\epsilon u_{1,\epsilon} \Phi_\epsilon - a_2 u_{2,\epsilon}^2 \Phi_\epsilon - \frac{2a_1}{\epsilon^2|\log\epsilon|} S_\epsilon u_{1,\epsilon} \Psi_{1,\epsilon} + \frac{1}{\epsilon^2|\log\epsilon|} u_{2,\epsilon}^2 \Psi_{2,\epsilon} = \lambda_\epsilon \Phi_\epsilon, \\ D_2 \Delta \Psi_{2,\epsilon} - \Psi_{2,\epsilon} + \frac{2}{\epsilon^2|\log\epsilon|} u_{1,\epsilon} u_{2,\epsilon} \Psi_{2,\epsilon} - 2a_2 u_{1,\epsilon} u_{2,\epsilon} \Phi_\epsilon = \tau_1 \lambda_\epsilon \Psi_{2,\epsilon}. \end{cases} \tag{7.7}$$

Integrating the first equation, we have

$$\frac{a_1}{\epsilon^2|\log\epsilon|} \int_\Omega u_{1,\epsilon}^2 \Psi_{1,\epsilon} dx = \int_\Omega u_{1,\epsilon}^2 \Phi_\epsilon dx,$$

taking the limit $\epsilon \to 0$ as in the proof of part I, then we have

$$\Psi_{1,\epsilon}(0) = \frac{\epsilon^2|\log\epsilon| \int_{\mathbb{R}^2} w^2 \Phi dy}{a_1 \int_{\mathbb{R}^2} w^2 dy} \left(1 + \mathcal{O}\left(\frac{1}{|\log\epsilon|}\right)\right),$$

which implies that

$$-\frac{2a_1}{\epsilon^2|\log\epsilon|} S_\epsilon u_{1,\epsilon} \Psi_{1,\epsilon} = -2(1+\alpha) \frac{\int_{\mathbb{R}^2} w^2 \Phi}{\int_{\mathbb{R}^2} w^2} w \left(1 + \mathcal{O}\left(\frac{1}{|\log\epsilon|}\right)\right). \tag{7.8}$$

We also compute that

$$\Psi_{2,\epsilon}(0) = \int_{\Omega_\epsilon} G_{D_2}(0, \epsilon z) \left[\frac{2}{|\log\epsilon|} u_{1,\epsilon}(\epsilon z) u_{2,\epsilon}(\epsilon z) \Psi_{2,\epsilon}(\epsilon z) - 2\epsilon^2 a_2 u_{1,\epsilon}(\epsilon z) u_{2,\epsilon}(\epsilon z) \Phi_\epsilon(\epsilon z)\right] dz,$$

then by (3.7) we compute that

$$\Psi_{2,\epsilon}(0) = 2\epsilon^2|\log\epsilon| a_2 \frac{\int_{\mathbb{R}^2} w \Phi dy}{\int_{\mathbb{R}^2} w dy} \left(1 + \mathcal{O}\left(\frac{1}{|\log\epsilon|}\right)\right).$$

Therefore we have

$$\frac{1}{\epsilon^2|\log\epsilon|} u_{2,\epsilon}^2 \Psi_{2,\epsilon} = 2\alpha \frac{\int_{\mathbb{R}^2} w \Phi dy}{\int_{\mathbb{R}^2} w dy} \left(1 + \mathcal{O}\left(\frac{1}{|\log\epsilon|}\right)\right). \tag{7.9}$$

Putting the expressions (7.8) and (7.9) into (7.7), Then by (3.4) and (3.5) we derive the following nonlocal linear operator which is the adjoint operator of (5.7):

$$\mathcal{L}^* \Phi = \Delta_y \Phi - (1+\alpha)\Phi + 2(1+\alpha) w \Phi - 2(1+\alpha) \frac{\int_{\mathbb{R}^2} w^2 \Phi dy}{\int_{\mathbb{R}^2} w^2 dy} w + 2\alpha \frac{\int_{\mathbb{R}^2} w \Phi dy}{\int_{\mathbb{R}^2} w dy} = 0. \tag{7.10}$$

Then we have the following claim:

Claim. The kernel of the operator $\mathcal{L}^*$ defined in (7.10) is trivial.

Proof. Integrating (7.10), we derive $\int_{\mathbb{R}^2} w\Phi dy = 0$ since otherwise there is an unbounded term. Furthur, we get the relation

$$\int_{\mathbb{R}^2} \Phi dy + 2\frac{\int_{\mathbb{R}^2} wdy}{\int_{\mathbb{R}^2} w^2 dy} \int_{\mathbb{R}^2} w^2 \Phi dy = 0. \tag{7.11}$$

Multiplying (7.10) by w and integrating, we derive

$$\int_{\mathbb{R}^2} w^2 \Phi dy = 0. \tag{7.12}$$

Then from (7.11) we get $\int_{\mathbb{R}^2} \Phi dy = 0$. Then all the nonlocal terms of (7.10) vanish and by Theorem 5.4 in the special case $\gamma = 0$ we derive $\Phi \in \text{span}\{\frac{\partial w}{\partial y_1}, \frac{\partial w}{\partial y_2}\}$. Thus the kernel of $\mathcal{L}^*$ is trivial. □

7.2. **The Green's Functions.** Let $G_{D_1}(x, \xi)$ be the Green's function of the Laplace operator with Neumann boundary conditions:

$$\begin{cases} D_1 \Delta G_{D_1}(x, \xi) - \frac{1}{|\Omega|} + \delta_\xi(x) = 0 \quad \text{in } \Omega \\ \int_\Omega G_{D_1}(x, \xi) dx = 0 \\ \frac{\partial}{\partial \nu_x} G_{D_1}(x, \xi)\Big|_{\partial\Omega} = 0. \end{cases} \tag{7.13}$$

Here $\delta_\xi(x)$ denotes the Dirac delta distribution concentrated at the point ξ.

We can decompose $G_{D_1}(x, \xi)$ as follows:

$$G_{D_1}(x, \xi) = \frac{1}{2\pi D_1} \log \frac{1}{|x - \xi|} - H_{D_1}(x, \xi), \tag{7.14}$$

where $H_{D_1}(x, \xi)$ is the regular part of $G_{D_1}(x, \xi)$.

Next we define

$$\begin{cases} D_1 \Delta G_{D_1,\tau\lambda}(x, \xi) - \tau\lambda G_{D_1,\tau\lambda}(x, \xi) + \delta_\xi(x) = 0 \quad \text{in } \Omega \\ \frac{\partial}{\partial \nu_x} G_{D_1,\tau\lambda}(x, \xi)\Big|_{\partial\Omega} = 0. \end{cases} \tag{7.15}$$

By simple calculation we can get that

$$\int_\Omega G_{D_1,\tau\lambda}(x, \xi) dx = \frac{1}{\tau\lambda}.$$

Let

$$G_{D_1,\tau\lambda}(x, \xi) = \frac{1}{|\Omega|\tau\lambda} + \hat{G}_{D_1,\tau\lambda}(x, \xi),$$

where

$$\begin{cases} D_1 \Delta \hat{G}_{D_1,\tau\lambda}(x, \xi) - \tau\lambda \hat{G}_{D_1,\tau\lambda}(x, \xi) - \frac{1}{|\Omega|} + \delta_\xi(x) = 0 \quad \text{in } \Omega \\ \int_\Omega \hat{G}_{D_1,\tau\lambda}(x, \xi) dx = 0 \\ \frac{\partial}{\partial \nu_x} \hat{G}_{D_1,\tau\lambda}(x, \xi)\Big|_{\partial\Omega} = 0. \end{cases}$$

From calculation we can get $\hat{G}_{D_1,\tau\lambda}(x, \xi)$ as follows:

$$\hat{G}_{D_1,\tau\lambda}(x, \xi) = G_{D_1}(x, \xi) + \mathcal{O}(\tau\lambda),$$

which means that

$$
\begin{aligned}
G_{D_1,\tau\lambda}(x,\xi) &= \frac{1}{|\Omega|\tau\lambda} + G_{D_1}(x,\xi) + \mathcal{O}(\tau\lambda) \\
&= \frac{1}{|\Omega|\tau\lambda} + \frac{1}{2\pi D_1}\log\frac{1}{|x-\xi|} - H_{D_1}(x,\xi) + \mathcal{O}(\tau\lambda) \\
&= \frac{1}{|\Omega|\tau\lambda} + \frac{1}{2\pi D_1}\log\frac{1}{|x-\xi|} - H_{D_1,\tau\lambda}(x,\xi), \qquad (7.16)
\end{aligned}
$$

where $H_{D_1,\tau\lambda}(x,\xi)$ is the regular part of $G_{D_1,\tau\lambda}(x,\xi)$.
Then an elementary computation shows that:

$$
\left| H_{D_1}(x,\xi) - H_{D_1,\tau\lambda}(x,\xi) - \frac{1}{|\Omega|\tau\lambda} \right| \le C|\tau\lambda|
$$

uniformly for all $(x,\xi)\in\Omega\times\Omega$. For the first two derivatives we have

$$
\left|\nabla[H_{D_1}(x,\xi) - H_{D_1,\tau\lambda}(x,\xi)]\right| \le C|\tau\lambda|
$$

uniformly for all $(x,\xi)\in\Omega\times\Omega$ and

$$
\left|\nabla^2[H_{D_1}(x,\xi) - H_{D_1,\tau\lambda}(x,\xi)]\right| \le C|\tau\lambda|
$$

uniformly for all $(x,\xi)\in\Omega\times\Omega$, where ∇ above can mean derivative w.r.t x or ξ.

Further, let $G_{D_2}(x,\xi)$ be the following Green's functions:

$$
\begin{cases} D_2\Delta G_{D_2}(x,\xi) - G_{D_2}(x,\xi) + \delta_\xi(x) = 0 \quad \text{in } \Omega \\ \frac{\partial}{\partial\nu_x} G_{D_2}(x,\xi)\Big|_{\partial\Omega} = 0 \end{cases} \qquad (7.17)
$$

then

$$
\int_\Omega G_{D_2}(x,\xi)dx = 1.
$$

Let

$$
G_{D_2}(x,\xi) = \frac{1}{|\Omega|} + \hat{G}_{D_2}(x,\xi),
$$

where

$$
\begin{cases} D_2\Delta \hat{G}_{D_2}(x,\xi) - \hat{G}_{D_2}(x,\xi) - \frac{1}{|\Omega|} + \delta_\xi(x) = 0 \quad \text{in } \Omega \\ \int_\Omega \hat{G}_{D_2}(x,\xi)dx = 0 \\ \frac{\partial}{\partial\nu_x} \hat{G}_{D_2}(x,\xi)\Big|_{\partial\Omega} = 0. \end{cases}
$$

By calculation we can get that

$$
\hat{G}_{D_2}(x,\xi) = \frac{1}{2\pi D_2}\log\frac{1}{|x-\xi|} - \hat{H}_{D_2}(x,\xi) + \mathcal{O}(1),
$$

then

$$
\begin{aligned}
G_{D_2}(x,\xi) &= \frac{1}{|\Omega|} + \hat{G}_{D_2}(x,\xi) \\
&= \frac{1}{|\Omega|} + \frac{1}{2\pi D_2}\log\frac{1}{|x-\xi|} - \hat{H}_{D_2}(x,\xi) + \mathcal{O}(1) \\
&= \frac{1}{|\Omega|} + \frac{1}{2\pi D_2}\log\frac{1}{|x-\xi|} - H_{D_2}(x,\xi), \qquad (7.18)
\end{aligned}
$$

where $H_{D_2}(x,\xi)$ is the regular part of $G_{D_2}(x,\xi)$.

Similarly, define

$$\begin{cases} D_2\Delta G_{D_2,\tau_1\lambda}(x,\xi) - (1+\tau_1\lambda)G_{D_2,\tau_1\lambda}(x,\xi) + \delta_\xi(x) = 0 & \text{in } \Omega \\ \frac{\partial}{\partial \nu_x} G_{D_2,\tau_1\lambda}(x,\xi)\Big|_{\partial\Omega} = 0. \end{cases} \tag{7.19}$$

By simple calculation we can get that

$$\int_\Omega G_{D_2,\tau_1\lambda}(x,\xi)dx = \frac{1}{1+\tau_1\lambda}.$$

Let

$$G_{D_2,\tau_1\lambda}(x,\xi) = \frac{1}{|\Omega|(1+\tau_1\lambda)} + \hat{G}_{D_2,\tau_1\lambda}(x,\xi),$$

where

$$\begin{cases} D_2\Delta \hat{G}_{D_2,\tau_1\lambda}(x,\xi) - (1+\tau_1\lambda)\hat{G}_{D_2,\tau_1\lambda}(x,\xi) - \frac{1}{|\Omega|} + \delta_\xi(x) = 0 & \text{in } \Omega \\ \int_\Omega \hat{G}_{D_2,\tau_1\lambda}(x,\xi)dx = 0 \\ \frac{\partial \hat{G}_{D_2,\tau_1\lambda}(x,\xi)}{\partial \nu_x}\Big|_{\partial\Omega} = 0. \end{cases}$$

From calculation we can get $\hat{G}_{D_2,\tau_1\lambda}(x,\xi)$ as follows:

$$\hat{G}_{D_2,\tau_1\lambda}(x,\xi) = G_{D_2}(x,\xi) + \mathcal{O}(1+\tau_1\lambda),$$

which means that

$$\begin{aligned} G_{D_2,\tau_1\lambda}(x,\xi) &= \frac{1}{|\Omega|(1+\tau_1\lambda)} + G_{D_2}(x,\xi) + \mathcal{O}(1+\tau_1\lambda) \\ &= \frac{1}{|\Omega|(1+\tau_1\lambda)} + \frac{1}{2\pi D_2}\log\frac{1}{|x-\xi|} - H_{D_2}(x,\xi) + \mathcal{O}(1+\tau_1\lambda). \end{aligned} \tag{7.20}$$

REFERENCES

[1] Xinfu Chen and Michal Kowalczyk. Dynamics of an Interior Spike in the GiererMeinhardt System. *SIAM Journal on Mathematical Analysis*, 33:172–193, 2001.

[2] Edward N Dancer. On stability and hopf bifurcations for chemotaxis systems. *Methods and applications of analysis*, 8(2):245–256, 2001.

[3] Arjen Doelman, Robert A Gardner, and Tasso J Kaper. Stability analysis of singular patterns in the 1d gray-scott model: a matched asymptotics approach. *Physica D: Nonlinear Phenomena*, 122(1-4):1–36, 1998.

[4] Arjen Doelman, Robert A Gardner, and Tasso J Kaper. Large stable pulse solutions in reaction-diffusion equations. *Indiana University Mathematics Journal*, pages 443–507, 2001.

[5] Arjen Doelman, Robert A Gardner, and Tasso J Kaper. *A stability index analysis of 1-D patterns of the Gray-Scott model*. American Mathematical Soc., 2002.

[6] Arjen Doelman, Tasso J Kaper, and Paul A Zegeling. Pattern formation in the one-dimensional gray-scott model. *Nonlinearity*, 10(2):523, 1997.

[7] Alfred Gierer and Hans Meinhardt. A theory of biological pattern formation. *Kybernetik*, 12(1):30–39, 1972.

[8] David Iron, Juncheng Wei, and Matthias Winter. Stability analysis of turing patterns generated by the schnakenberg model. *Journal of mathematical biology*, 49(4):358–390, 2004.

[9] J Schnakenberg. Simple chemical reaction systems with limit cycle behaviour. *Journal of theoretical biology*, 81(3):389–400, 1979.

[10] Michael J Ward and Juncheng Wei. Asymmetric spike patterns for the one-dimensional gierer–meinhardt model: equilibria and stability. *European Journal of Applied Mathematics*, 13(3):283–320, 2002.

[11] Juncheng Wei. Existence, stability and metastability of point condensation patterns generated by the gray-scott system. *Nonlinearity*, 12(3):593, 1999.

[12] Juncheng Wei and Matthias Winter. On a two-dimensional reaction-diffusion system with hypercyclical structure. *Nonlinearity*, 13(6):2005, 2000.

[13] Juncheng Wei and Matthias Winter. Spikes for the two-dimensional gierer-meinhardt system: the weak coupling case. *Journal of Nonlinear Science*, 11(6):415–458, 2001.

[14] Juncheng Wei and Matthias Winter. Spikes for the two-dimensional gierer-meinhardt system: The strong coupling case. *J. Differential Equations*, 178:478–518, 2002.
[15] Juncheng Wei and Matthias Winter. Asymmetric spotty patterns for the gray–scott model in r2. *Studies in Applied Mathematics*, 110(1):63–102, 2003.
[16] Juncheng Wei and Matthias Winter. Existence and stability of multiple-spot solutions for the gray–scott model in r2. *Physica D: Nonlinear Phenomena*, 176(3-4):147–180, 2003.
[17] Juncheng Wei and Matthias Winter. Existence, classification and stability analysis of multiple-peaked solutions for the gierer-meinhardt system in r1. *Methods and Applications of Analysis*, 14(2):119–164, 2007.
[18] Juncheng Wei and Matthias Winter. Stationary multiple spots for reaction–diffusion systems. *Journal of mathematical biology*, 57(1):53–89, 2008.
[19] Juncheng Wei and Matthias Winter. Stationary multiple spots for reaction–diffusion systems. *Journal of mathematical biology*, 57(1):53–89, 2008.
[20] Juncheng Wei and Matthias Winter. Existence and stability of a spike in the central component for a consumer chain model. *Journal of Dynamics and Differential Equations*, 27(3):1141–1171, 2015.

WUHAN UNIVERSITY, DEPARTMENT OF MATHEMATICS, CHINA, 430072
E-mail address: wwao@whu.edu.cn

SCHOOL OF MATHEMATICS AND STATISTICS, WUHAN UNIVERSITY, WUHAN, CHINA, 430072
E-mail address: pyj_ahu@126.com

BRUNEL UNIVERSITY LONDON, DEPARTMENT OF MATHEMATICS, UXBRIDGE UB8 3PH, UNITED KINGDOM
E-mail address: matthias.winter@brunel.ac.uk