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Solution of three-dimensional free boundary problems by variable interchange

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The three-dimensional problem of the seepage of fluid through a circular earth dam is solved to illustrate a new approach to more general free boundary problems. The method is based on Boadway's transformation in which the dependent variable, representing velocity potential, is interchanged with *one* of the independent space variables, which then becomes the new dependent variable to be computed. The need to determine the position of the whole of the free surface in the three-dimensional physical space is reduced to locating the position of the separation line on a fixed plane boundary in the transformed domain.

An iterative algorithm approximates within each single loop both a finite-difference solution of the partial differential equation and the position of the separation line.

1. Introduction

A free boundary problem may be defined as a steady-state boundary value problem, typically an elliptic partial differential equation with given boundary conditions, which has to be solved in a domain in which the positions of parts of the boundary are unknown. These are the free boundaries and they must be determined as part of the solution.

The model problem discussed in this paper refers to the seepage of water through an earth dam which is constructed of isotropic permeable material and rests on an impermeable base. The walls of the dam form the inner and outer surfaces of a quadrant of a hollow circular cylinder (Figs. 1 and 2). The water from the upstream reservoir seeps through the circular earth dam and emerges at an initially unknown height on the downstream face and then drains freely down the wall into a lower reservoir. The heights of the water in the higher and lower reservoirs are known. The upper surface of the water within the dam forms the "free surface" whose shape and position are to be determined.

Successive authors have approached the free boundary problem by solving a sequence of fixed boundary problems corresponding to the successive iteratively-computed positions of the free boundary.

Various methods have been used to carry out such interacting iterative procedures for the two-dimensional rectangular dam. Key references are to be found in Cryer (1976), Furzeland (1977,1979), Crank and Ozis (1980) and Ozis (1981).

Recently, Aitchison (1977) and others referred to in Furzeland (1977, 1979) and Ozis (1981) have avoided the iterations by using the Baiocchi transformation (1972) to reformulate the problem as a variational inequality over a fixed domain. Also the present authors (Crank and Ozis (1980)) have transformed the physical domain with a free boundary into a domain with fixed known boundaries by interchanging the dependent variable with one of the independent, space variables.

Relatively few solutions for three-dimensional free surface flows have been computed to date. Jeppson (1972) presented an inverse solution for three-dimensional potential flows with a free surface for seepage through an earth dam. Mogal & Street (1974) suggested the use of a three-dimensional hodograph space with dependent variables (x,y,z) and independent variables which are the velocity components (u,v,w), but did not apply the- transformation to any model problem. Ozis (1981) solved the model problem (Fig. 1) as a genuine three-dimensional problem by extending the approach used in Crank and Ozis (1980) to three-space dimensions.

The present paper, like Ozis (1981), transforms the physical space within the dam contained partly by the free surface into a domain with fixed, known boundaries but the present transformed equations are different. Here we use an extension of Boadway's transformation (1976). The transformation used by Crank & Ozis (1980) and Ozis (1981) is referred to as a "partial transformation" compared with Boadway's "complete transformation".

This transformation first introduced by Thom and Apelt (1961) has been modified by Boadway (1976). More recently it has been applied to the clasical problem of seepage of fluid in the two dimensional problem of a rectangular dam by Crank & Sabouri (1980).

In the transformed, three—dimensional space it is possible to solve the model problem by an iteractive algorithm which approximates within each single iterative loop both the solution of the partial differential equation and the position of the free boundary including the separation line. As a test of our new algorithm we are treating this problem as a genuine three-dimensional one in (x,y,z) coordinates and making no use of the circular symmetry which we know to exist. Afterwards, and quite independently, we solved the problem reformulated in the r-z plane and compared the two sets of results.

2. Three-dimensional seepage problem

The mathematical formulation of the problem illustrated in Fig. 1 and 2 in terms of velocity potential φ is

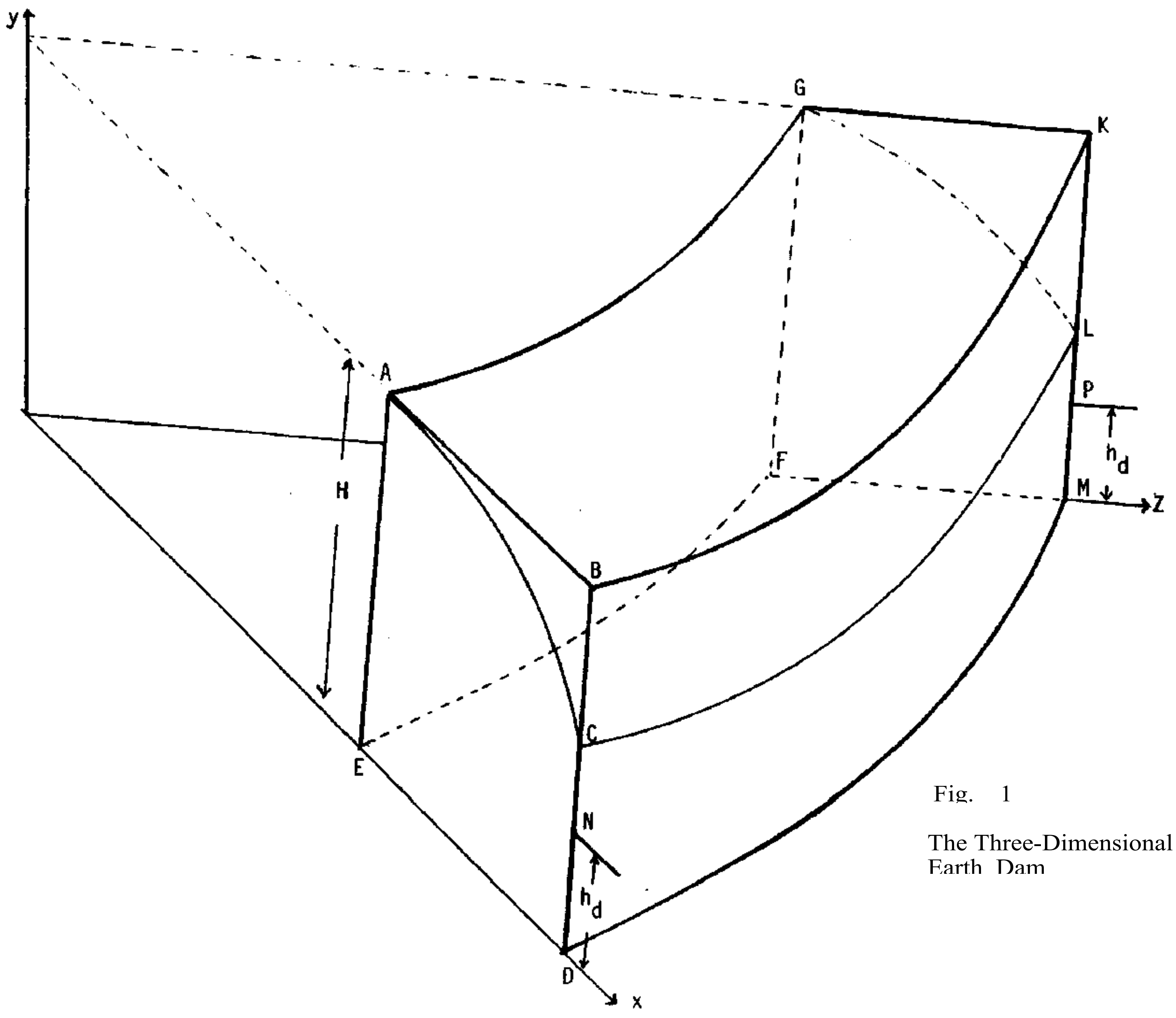

Fig. 1

The Three-Dimensional Circular Earth Dam

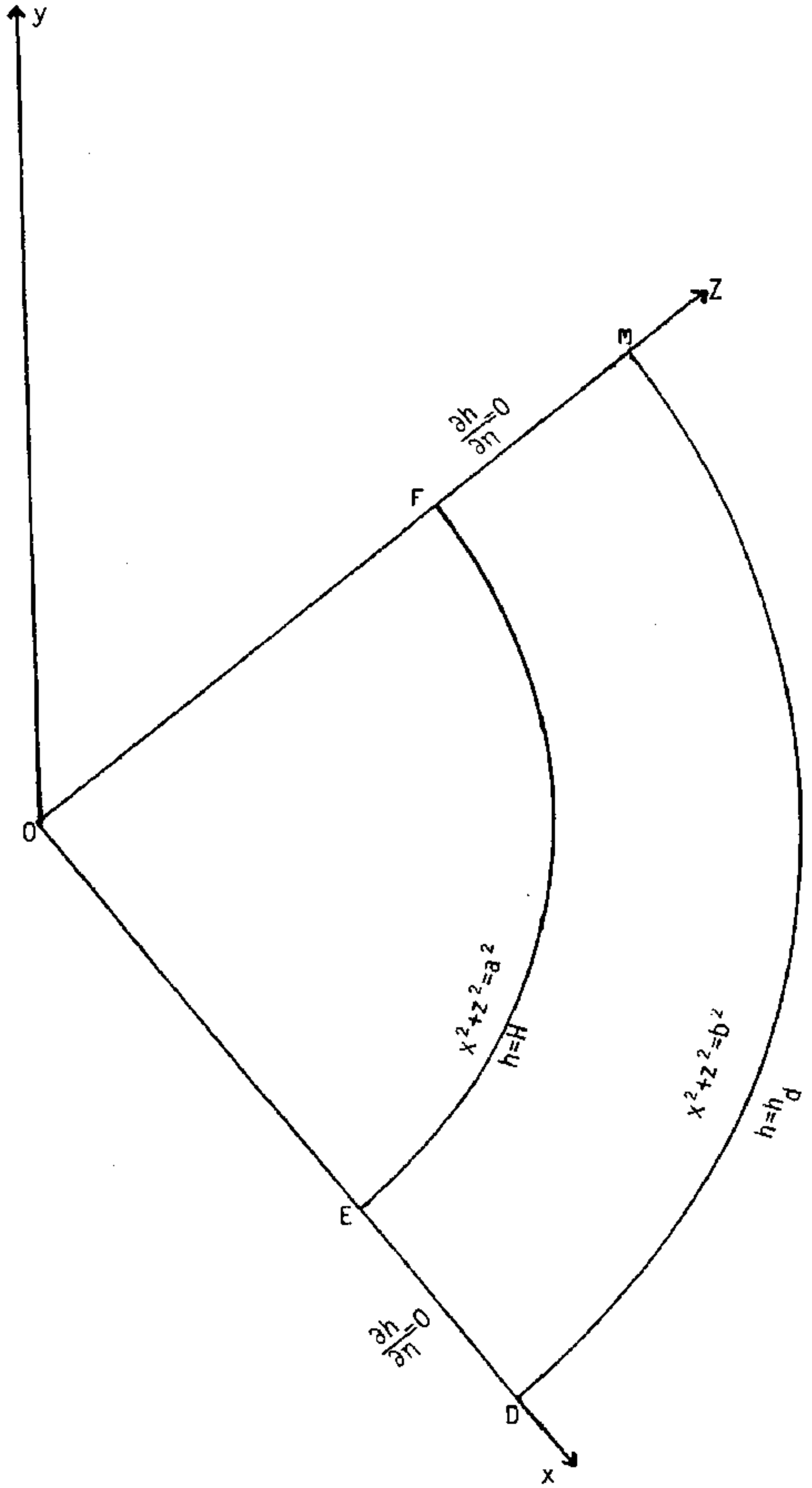

Fig. 2

The Cross-Section of the Dam for y=0

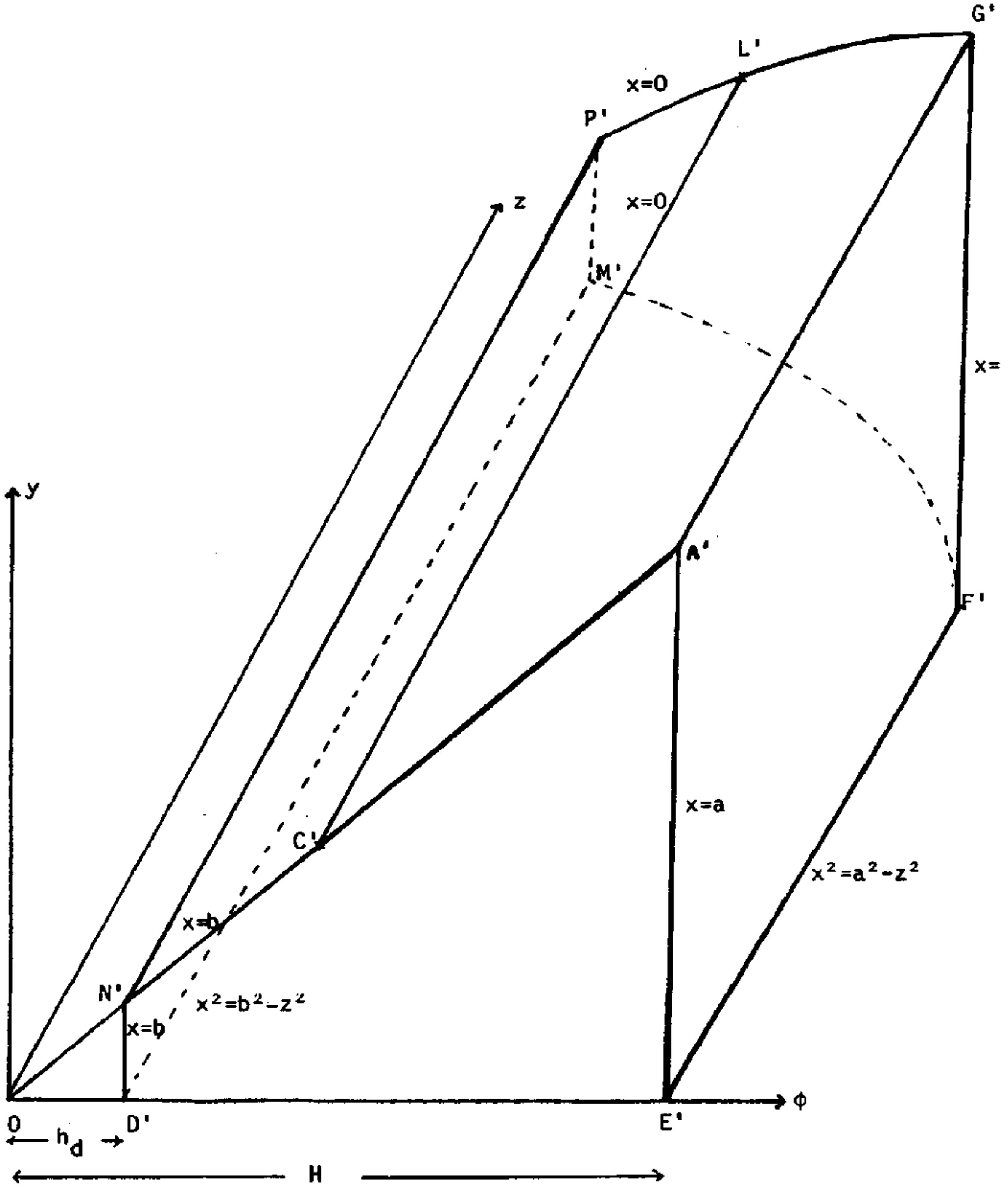

Fig. 3

Transformed Space

$$\frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^2\varphi}{\partial y^2}+\frac{\partial^2\varphi}{\partial z^2}=0 \quad \text{in AEDNCLPMFG} \tag{1}$$

$\varphi = H$ on AEFC (Interface with water at rest) (2)

$\frac{\partial\varphi}{\partial n}=0$ on EDMF (impervious boundary) (3)

$\varphi = h_d$ on DMPN (Interface with water at rest) (4)

$\varphi = y$ on CNPL (Interface with air, seepage face) (5)

$\frac{\partial\varphi}{\partial n}=0$on ABDE (Side face of the dam) (6)

$\frac{\partial\varphi}{\partial n}=0$on GFMK (Side face of the dam) (7)

$\frac{\partial\varphi}{\partial n}=0, \varphi = y$ on ACLG (Stream surface i.e. free surface) (8)

where n is the outwards normal.

The double condition on ACLG is needed here as in all free boundary problems in order to determine the position of ACLG as well as to solve the differential equation (1). In the two-dimensional case Boadway (1976) interchanges one of the independent variables with the dependent variable φ and introduces a dummy variable $\psi = \psi,(x,y)$. In the three-dimensional problem we do a similar interchange and introduce two dummy variables $x = \psi(x,y,x)$ and $\theta = \theta(x,y,z)$. The equations are re-written with $x = x(\psi,\theta,\varphi)$, $y = y(\psi,\theta<,\varphi)$ and $z - z(\psi,\theta,\varphi)$ as the new dependent variables. After considerable manipulation and putting $\Psi = y$ and $\theta = z$ the equation (1) becomes

$$\frac{\partial^2 x}{\partial\phi^2}\left[1+\left(\frac{\partial x}{\partial y}\right)^2+\left(\frac{\partial x}{\partial z}\right)^2\right]+\left[\frac{\partial^2 x}{\partial\gamma^2}-\frac{\partial^2 x}{\partial z^2}\right]\left(\frac{\partial x}{\partial\phi}\right)^2+$$

$$2\frac{\partial^2 x}{\partial\phi\partial z}\frac{\partial x}{\partial\phi}\frac{\partial x}{\partial z}-2\frac{\partial^2 x}{\partial\phi\partial y}\frac{\partial x}{\partial y}\frac{\partial x}{\partial\phi}=0\,. \tag{9}$$

This equation is to be solved for $x(\varphi,y,z)$ in the (φ,y,z) domain shown in Fig. 3

In general, on a surface y = y(x,z), we have

$$\phi_n = \frac{\left.\frac{\partial\phi}{\partial x}\right)_{y,z}\left.\frac{\partial y}{\partial x}\right)_z - \left.\frac{\partial\phi}{\partial y}\right)_{x,z} + \left.\frac{\partial\phi}{\partial z}\right)_{x,y}\left.\frac{\partial y}{\partial z}\right)_x}{\left[\left(\frac{\partial y}{\partial x}\right)^2 + 1 + \left(\frac{\partial y}{\partial z}\right)^2\right]^{\frac{1}{2}}} \tag{10}$$

Where $\phi_n = \frac{\partial\phi}{\partial n}$ and n is the outward normal to y = y(x,z).

Provided $\partial y/\partial x \neq \infty$ and $\partial y/\partial z \neq \infty$, we have on the free surface ACLG.

$$\left.\frac{\partial\phi}{\partial x}\right)_{y,z}\left.\frac{\partial y}{\partial x}\right)_z - \left.\frac{\partial\phi}{\partial y}\right)_{x,z} + \left.\frac{\partial\phi}{\partial z}\right)_{x,y}\left.\frac{\partial y}{\partial z}\right)_x = 0\,,\ y = \phi \tag{11}$$

which can readily be re-written

$$1 + \left.\frac{\partial x}{\partial y}\right)_{\phi,z}\left.\frac{\partial x}{\partial y}\right)_z - \left.\frac{\partial x}{\partial z}\right)_{\phi,y}\left.\frac{\partial x}{\partial z}\right)_y = 0\,,\ y = \phi \tag{12}$$

It is true that $\partial y/\partial x = \infty$ and $\partial y/\partial z = \infty$ at the separation line CL, but condition (12) is not applied at that line.

On the impervious foundation EDMF, we have

$$\left.\frac{\partial x}{\partial y}\right)_{\phi,z} = 0\,,.\ y = 0\,. \tag{13}$$

On the side face ACDE, we have

$$\left.\frac{\partial x}{\partial z}\right)_{\phi,y} = 0\ ,\ z = 0\,. \tag{14}$$

On the side face GLMF in Fig. 1, we know that x = 0 everywhere, but what we do not know is the shape of the surface F'M'P'G' in Fig. 3, corresponding to the face GLMF. However, we can determine the shape of F'M'P'G' by using the second boundary condition on GLMF which is

$$\left.\frac{\partial\phi}{\partial x}\right)_z = 0\,,\ \ x = 0\ . \tag{15}$$

But

$$\left.\frac{\partial\phi}{\partial x}\right)_z = \frac{\left.\frac{\partial z}{\partial x}\right)_\phi}{\left.\frac{\partial z}{\partial \phi}\right)_x} = 0 \tag{16}$$

which for $\left.\frac{\partial z}{\partial \varphi}\right)_x \neq 0$ gives the condition

$$\left.\frac{\partial z}{\partial \varphi}\right)\phi = 0 \ , \ x = 0 \tag{17}$$

Condition (17) suggests a quadratic relationship for small x on a constant φ Line of the form on any constant y — plane (Fig- 4),

$$z = ax^2 + b \ \bullet \tag{18}$$

Extrapolating along AB in Fig. 4 using (18) where a,b are determined by fitting the (x,y) values at A,B we find the values of z at C

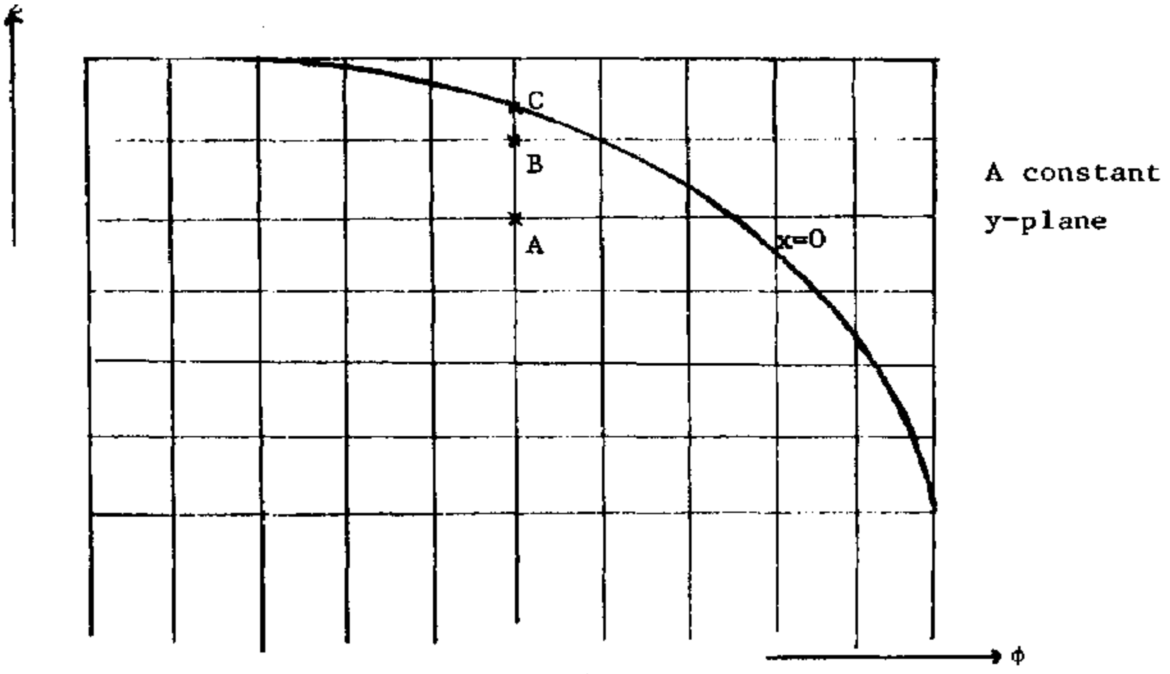

Fig. 4

The other boundary conditions in Fig. 3 are

$$x = \sqrt{a^2 - z^2} \quad , \quad \phi = H \tag{19}$$

$$x = \sqrt{b^2 - z^2} \quad , \quad \phi = h_d \qquad o \leq y \leq d \tag{20}$$

$$x = \sqrt{a^2 - z^2} \quad , \quad \phi = y \qquad \text{on } C'N'P'L \tag{21}$$

Thus we wish to solve equation (9) subject to conditions (12) to (14) ; (17) and (19),(20),(21) in the region D'E'A'C'N'P'L'G'F'M' *in* Fig. 3 in the (φ,y,z) space. All the boundaries are fixed. The original free surface ACLG has become the known plane A'C'L'G', $y = \phi$. What we do not know, however, is the position of the line C'L' on the plane A'N'P'G' corresponding to the separation line CL in Fig. 1, on which the boundary condition on the plane A'N'P'G' changes from $x = \sqrt{b^2 - z^2}$ to the condition (12). The shape of face F'M'P'G' on which $x = 0$ everywhere is determined by the extrapolation given above in (18).

We cover the region with a mesh of spacing $\delta\phi$, δy and δz and denote $x(i\delta\varphi, j\delta y, k\delta z)$ by $X_{i,j,k}$ . . Then equation (9) can be approximated by replacing all the derivatives by central finite-differences for the typical internal point $(i\delta\phi, j\delta y, k\delta z)$. If we collect together terms in $X_{.i\,j\,k}$. , , we can write the difference equation in the iterative form

$$\left\{2\left(\frac{1}{\delta y^2}-\frac{1}{\delta z^2}\right)\left(\frac{X^n_{i+1,j,k}-X^n_{i-1,j,k}}{2\delta\phi}\right)+\frac{2}{\delta\phi^2}\left[1+\left(\frac{X^n_{i,j+1,k}-X^n_{i,j-1,k}}{2\delta y}\right)^2+\left(\frac{X^n_{i,j,k+1}-X^n_{i,j,k-1}}{2\delta z}\right)^2\right]\right\}X^{n+1}_{i,j,k}=$$

$$\left(\frac{X^n_{i+1,j,k}-X^n_{i-1,j,k}}{2\delta\phi}\right)\left[\frac{X^n_{i,j+1,k}-X^n_{i,j-1,k}}{2\delta y}-\frac{X^n_{i,j,k+1}-X^n_{i,j,k-1}}{2\delta z}\right]+$$

$$\left(\frac{X^n_{i+1,j,k}-X^n_{i-1,j,k}}{\delta\phi^2}\right)\left[1+\left(\frac{X^n_{i,j+1,k}-X^n_{i,j-1,k}}{\delta y}\right)^2+\left(\frac{X^n_{i,j,k+1}-X^n_{i,j,k-1}}{\delta z}\right)^2\right]+$$

$$2\left(\frac{X^n_{i+1,j,k}-X^n_{i-1,j,k}-X^n_{i,j+1,k}-X^n_{i,j-1,k}+X^n_{i,j-1,k}}{4\delta y\delta\phi}\right)\left(\frac{X^n_{i,j+1,k}-X^n_{i,j-1,k}}{2\delta y}\right)\left(\frac{X^n_{i+1,j,k}-X^n_{i-1,j,k}}{2\delta\phi}\right)-$$

$$2\left(\frac{X^n_{i+1,j,k}-X^n_{i-1,j,k}-X^n_{i,j+1,k}-X^n_{i,j-1,k}+X^n_{i,j-1,k}}{4\delta z\delta\phi}\right)\left(\frac{X^n_{i,j+1,k}-X^n_{i,j-1,k}}{2\delta z}\right)\left(\frac{X^n_{i+1,j,k}-X^n_{i-1,j,k}}{2\delta\phi}\right),\quad (22)$$

where $X^{n}_{i,j,k}$ . . is the nth iterative of $X._{i,j,k}$. , .

On the free surface A'NP'G' we either have $X = \sqrt{b^2 - Z^2}$ or

$$X^{n+1}_{i,j,k} = X^{n}_{i,j-1,k} - \frac{2(\delta y)^2}{X^{n}_{i+1,j+1,k} - X^{n}_{i-1,j-1,k}}$$
$$\frac{(\delta y)^2 \left(X^{n}_{i,j,k+1} - (\delta y)^2 X^{n}_{i,j,k-1} \right)^2}{2(\delta y)^2 \left(X^{n}_{i+1,j+1,k} - X^{n}_{i-1,j-1,k} \right)} \tag{23}$$

from equation (12) where a one-sided difference replacement of $\left.\partial x/\partial \gamma\right)\phi, z$ is used to avoid fictitious points outside the region. Similarly, a modified form of equation (22) is applied to all the mesh points which are one step δy below the plane A' N' P' G' in order to avoid the use of fictitious points outside the domain. On the lower boundary y = 0, i.e. j = 0 and on the side-face A'E'D'N', z = 0, i.e. k = 0 fictitious points outside the domain arise. These can be eliminated from (22) since condition (13) implies $X._{i,-l,k}$ . . = $X_{i,l,k}$. for all values of i,k and (14) implies $X_{i,j,-1}$. = $X.i_{j,l}$. , for all values of i,j, respectively. Application of the extrapolation (18) to each constant ϕ line on each constant y-plane, forces us to unequal mesh sizes in the equation (22) for the points one step inside the side face F'M'P'G'.

Also application of the equation (23) to the points along the line ∧'N', z = 0, i.e. k = 0, on the free surface introduces fictitious points outside the region, and these are eliminated, similarly, using the condition (14) which implies $X._{i,i,-1}$ . . = $X._{i,i,l}$. for all i's. In addition to this, the extrapolation which is applied to each constant φ line introduces unequal mesh sizes in equation (23) one step inside the curve P'G' on which x = 0.

3. The iterative cycle

We start by assuming the separation line C'L' to be at one of the mesh lines on the plane A'N'P'G' in Fig. 3. Then we know that $X = \sqrt{b^2 - Z^2}$ on the face D'N'P'M' and on C'N'P'L' and also that $X = \sqrt{a^2 - Z^2}$ on the face E'A'G'F'. For every other mesh point within the region and on the remaining parts of the boundary, we have derived an equation.

We carry out one iterative cycle by sweeping along successive j-planes, in the order j = 0,1,2,... where y = j = 0 is the lower boundary. In each consecutive plane, we sweep each mesh point in the order i = 0 and k = 0,1,2,...; i = 1 and k = 0,1,2,... so on. The new values of $X_{i,j,k}$. . , are retained for use in the next cycle subject to the proviso that on the boundary plane A'N'P'G' we take the new value $X^{n+1}_{i,j,k}$ to be

$$X^{n+1}_{i,j,k} = \min\left(\sqrt{b^2 - z^2}, X^{n+1}_{i,i,k}\right) \tag{24}$$

since we know that $X_{i,j,k} < \sqrt{b^2 - Z^2}$ Also on F'G'P'M' new values of z are retained for use in the next cycle. We proceed with successive loops, iterating values of the solution and the position of the separation line C'L' on the plane A'N'P'G' in the same loop by using (2 4). The iteration proceeds until the difference between successive iterations at each point of the mesh is less than some precribed amount. The highest mesh point on each z-line on the boundary plane A'N'P'G' (Fig, 3) at which $Z = \sqrt{b^2 - Z^2}$ provides the best approximation to t he separation line that can be obtained directly from the set of finite difference equations (22) and (23). In the absence of any knowledge of tliis approximate position of the line C'L' , we start the iterative process by assuming C'L' to be at N'p', its lowest possible position. We thus ensure that the line C'L' is always approached from below *in* the iterative process.

The initial values of x in the free surface were taken as linear interpolates between $X=\sqrt{b^2-Z^2}$ on the line C'L' and $X=\sqrt{a^2-Z^2}$ on the line A'G' along the constant z lines in Fig. 3.

On each constant y plane, the initial value of x at each internal point was obtained by parobolic interpolation along each constant z-line between the known values of x at each mesh point on D'N'C'A'G'L'P' M' and the known value of x at the corresponding mesh point on E'A'G'F' . The parobolic interpolation is chosen by exploiting the property of the free surface in the physical domain that $\partial\phi/\partial r = 0$ at the edge AG in Fig. 1, where the radial co-ordinate r is normal to AG on a constant y-plane. In the transformed domain, we define the interpolation parabola, using the property given above and taking the geometry of the domain into consideration- To do this, a cross-section on any constant z-plane (Fig. 3) is depicted in Fig. 5. We are seeking a parabolic relation between x and ϕ on each constant y line in this constant z-plane.

In general, this relation can be given by the equation

$$\left(x - \sqrt{a^2 - z^2}\right)^2 = \gamma(H - \phi) \tag{25}$$

where γ' is a constant to be determined and H is the known constant upstream hydraulic head.

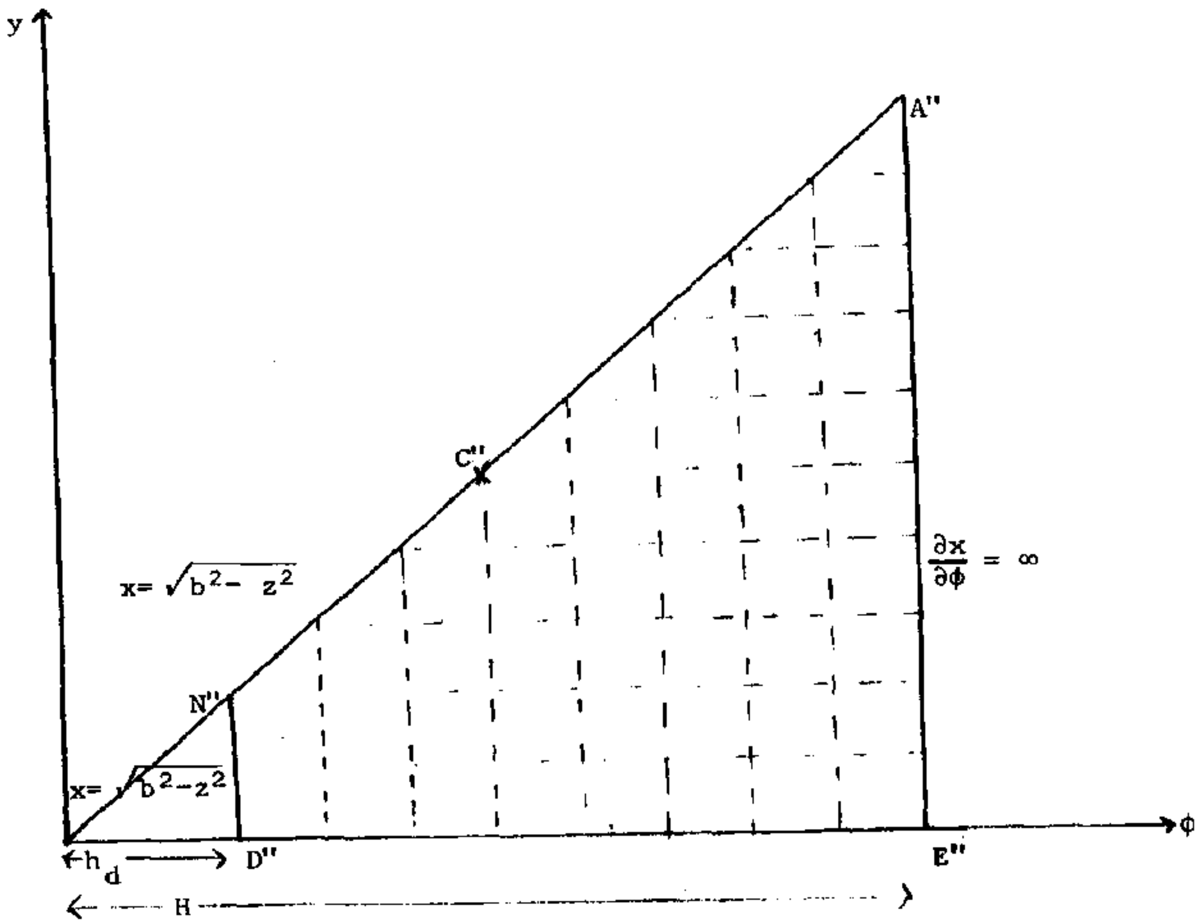

Fig. 5

To evaluate the necessary values of Υ we substitute in to (25) appropriate pairs of values of x and φ along D"N" N"C" and C'A" in Fig. 5 ; i.e.

i) along D"N", $\phi = h_d$, $x = x_{i,j,k} = \sqrt{b^2 - z_k^2}$, where $z_k = k\delta z$

ii) along N"C", $\phi = \phi_i$, $x = x_{i,j,k} = \sqrt{b^2 - z_k^2}$,

iii) along C"A", $\phi = \phi_i$. and the $X_{i,j,k}$. . are linear interpolates between

$X_{i,j,k} = \sqrt{b^2 - z_k^2}$ at C" and $X_{i,j,k} = \sqrt{a^2 - z_k^2}$ at A" .

Then these parabolas are used to estimate the initial $X_{i,j,k}$. . , values for each mesh point between D"N"C"A" and E"A" on each constant y-line on each constant z-plane.

Also, on the surface x = 0 (face M'P'G'F' in Fig. 3), we need to estimate the value of z, for each constant φ line to start with. Again, for the initial estimation of values of z, we used parabolas between the lines M'P' and F'G' in Fig. 3 for each constant y-plane. Thus, to define a parabola in the $\varphi = y$ plane, from the physical condition at G' corresponding to G (see Fig, 6),

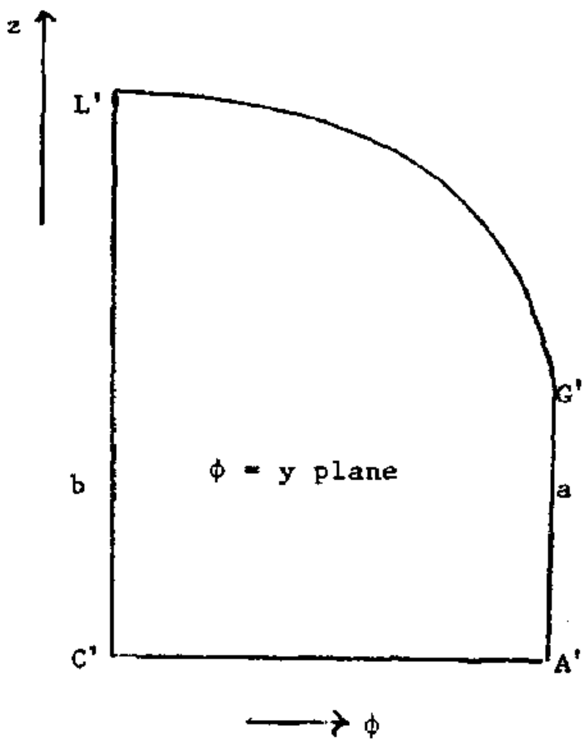

Fig. 6.

we have

$$\partial\phi|\partial z = 0 \ , \ \varphi = H, \ z = a.$$

These two conditions define the parabola in the form

$$(z-a)^2 = \Upsilon^1 \ (H - \phi),$$

where Υ' is a constant to be determined; H is the known constant upstream hydraulic head, where $z = a$, $x = 0$. To determine the value of Υ' we have

$$(b-a)^2 - \Upsilon'(H " \varphi_{L,})$$

on the ϕ - y plane; ϕ_L , is the value of ϕ at L'.

Also the same parabola is used to estimate the z values between the lines M'P' and F'G' but taking the value of $\Upsilon 1$ to be given by

$$(b-a)^2 - Y'(H - h_d)$$

for each constant y-plane-

4. Numerical Results

In this section the results from the solution to the seepage flow through the circular earth dam are given. Throughout the calculations we take a = 1, b - 5/3, d - 1/6, H = 1 and h_d = 1/6.

Calculations have been carried out with cubic meshes of size $\delta\varphi = \delta y = 1/12$ and $\delta z = 1/15$, In order to start the iteration process the separation line is first assumed to be at the line N'P' in Fig. 3, corresponding to NP in Fig. 1.

It has not been possible to carry out a formal study of convergence of the iterative process for solving the non-linear equation (22). Instead Table 1 demonstrates the convergence of some of the numerical values obtained on the plane y = 1/3 at selected stages of the iteration so1ut ion. The iteration process was terminated when

$$\left| X^{n+1}_{i,j,k} - X^{n}_{i,j,;} \right| < 10^{-3}$$

at all points of the mesh. table Z shows convergence of the position of the free surface on selected points. Figs. 7, 8 and 9 show extracts from the results obtained by the present method and for the problem reformulated in the r - z plane i.e. the axially symmetric solution. Table 3 shows an extract from the free boundary results obtained by the present method and the solution of the problem reformulated in the r - z plane i.e. the axially symmetric soluLion.

Table I

Convergence of $10^4 x$ at Selected Internal Mesh Points on y = 1/3 : The Iteration Started with Parabolically Interpolated Initial Values in the Domain

z	Iteration ϕ	5/12	7/12	9/12	11/12
	0	14078	13117	11912	10157
2/3	50	14066	13199	11821	10129
	100	14061	13283	11655	9906
	1 11	14061	13284	11654	9906
	0	8053	6222	2127	-
4/3	50	9284	6929	2198	-
	100	9751	7213	3370	-
	1 1 1	9751	7213	3371	—

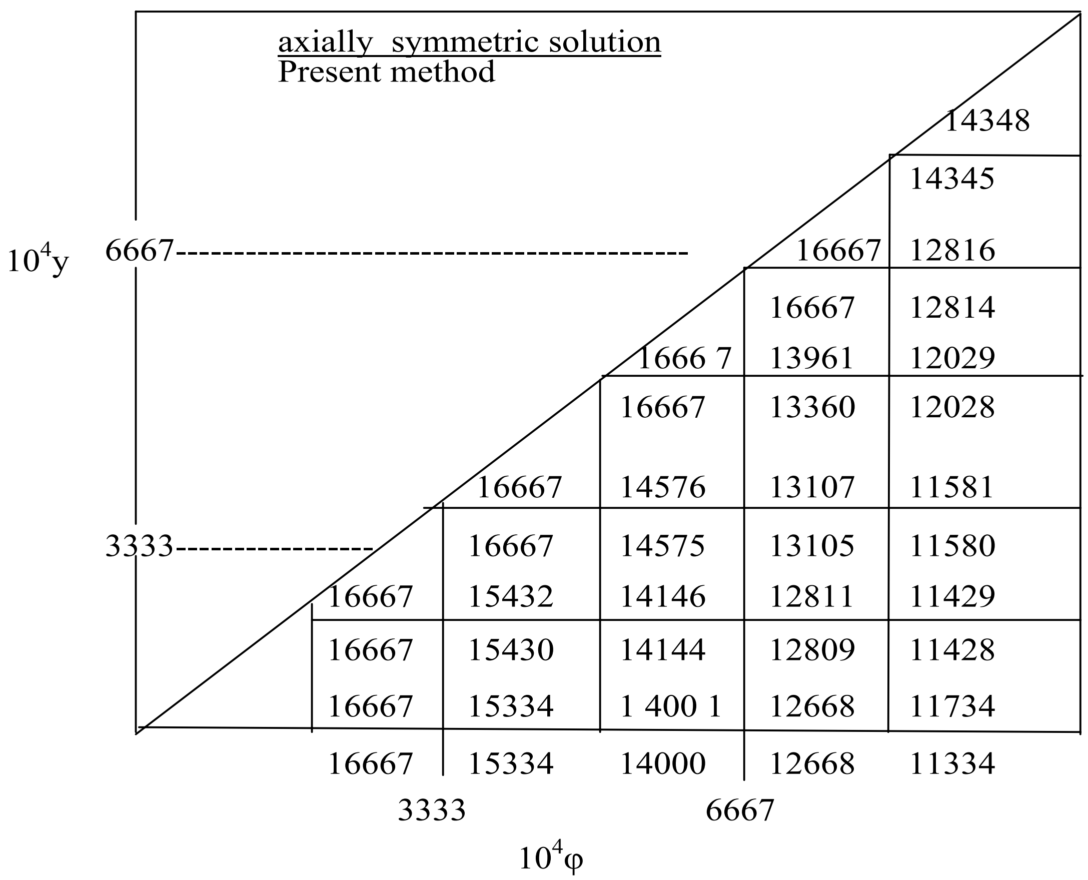

Fig. 7

Selected Values of $10^4 x$ from 12x12 Mesh for z = 0 i.e. k = 0, by the "Complete" Transformation

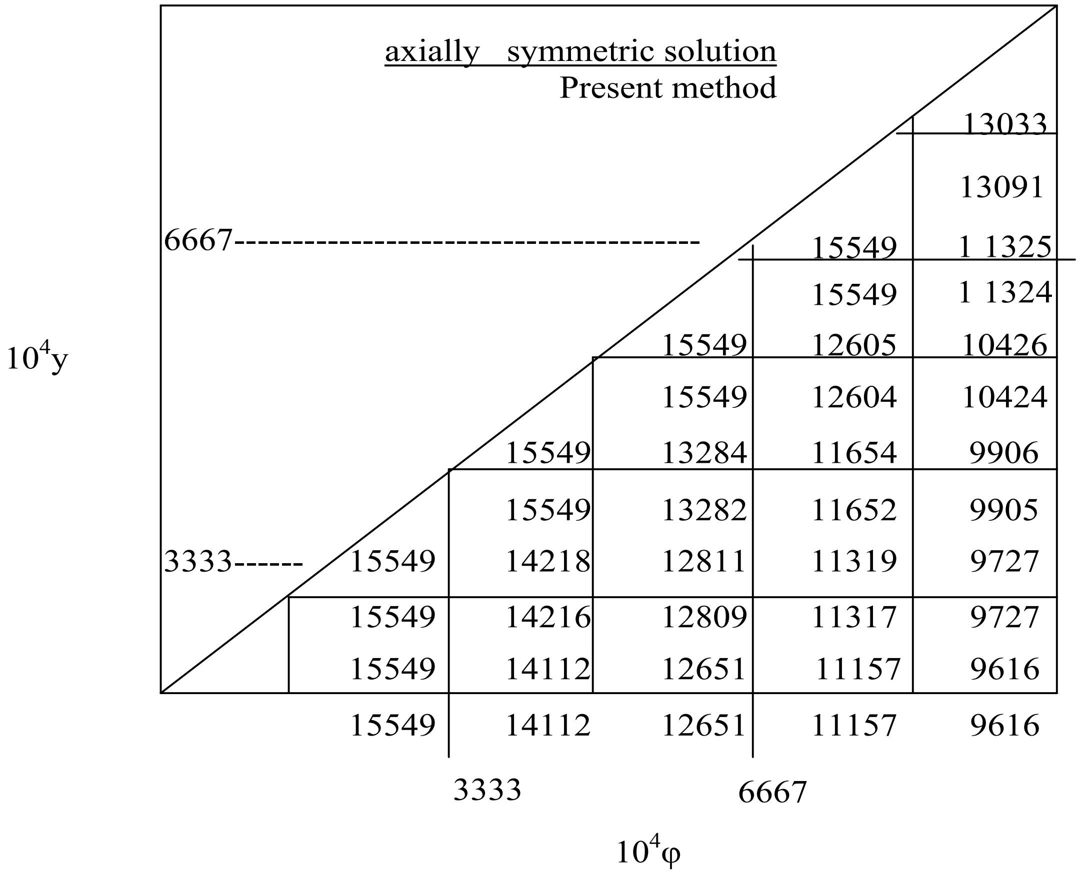

Fig. 8

Selected Values of 10^4x from 12x12 Mesh for z = 2_/3, i.e. k = 10, by the "complete" transformation

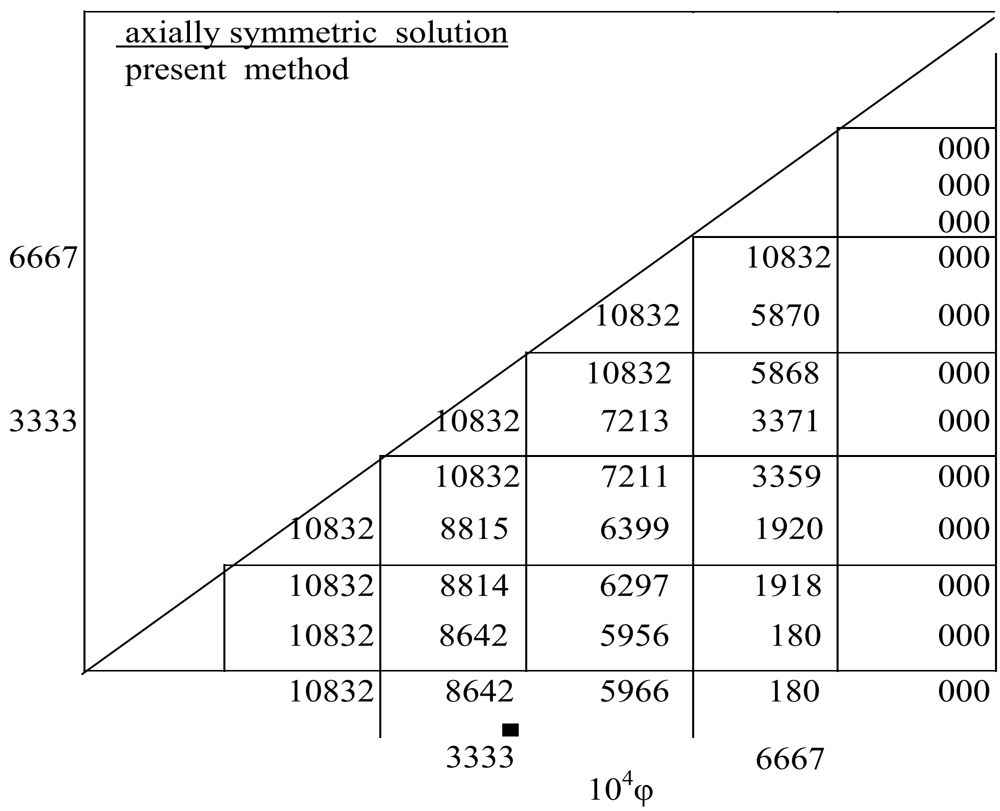

Fig. 9

Selected Values of . 10^4x from 12x12 Mesh for z = 4/3 i.e, k = 20, by "complete" Transformation

Table 2

Started with Parabolically Interpolated Initial Values in the Domain

z	Iteration	5/12	7/12	9/12	11/12
2/3	0	14078	13117	11912	10157
	50	15549	15549	13786	12567
	100	15549	15549	15548	13032
	111	15549	15549	15549	13033
4/3	0	8053	6222	2927	-
	50	10832	10832	7399	-
	100	10832	10832	10830	-
	111	10832	10832	10832	-

Table 3

Comparison of the positions of the free boundary calculated with

$\delta\varphi = \delta y = 1/12, \quad \delta z = 1/15.$

z	$10^4 y$	$10^4 x$	Radial Sol $10^4 x$
0	5000	16667	16667
	5833	16667	16667
	6667	16667	16667
	7500	15715	15715
	8333	14348	14348
	9167	12554	12554
2/3	5000	15549	15549
	5833	15549	15549
	6667	15549	15549
	7500	14525	14525
	8333	13033	13033
	9167	11027	11028
4/3	5000	10832	10832
	5833	10832	10832
	6667	10832	10832
	7500	9302	9303
	8333	6739	6739
	9167	0000	0000

5. Conclusions

The method described here, shows promise for obtaining finite- difference solutions to three-dimensional problems with free surfaces. By Boadway's transformation, a free surface, with an unknown position in the physical space, can become a plane of known position in a transformed space. The principal objectives of this study were to demonstrate the applicability of the method for solving a three-dimensional problem with free surfaces. Consequently, major emphasis was not given to obtaining as accurate a solution as would be possible. Nevertheless, the method yields reasonably accurate solutions.

6. References

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Captions and Titles

Table 1	Convergence of $10^4 x$ at Selected Internal Mesh Points on $y = 1/3$. The Iteration Started with Parabolically Interpolated Initial Values in the Domain.
Table 2	Convergence of Free Boundary Values of $10^4 x$ are Tabulated. The Iteration Started with Parabolically Interpolated Initial Values in the Domain.
Table 3	Comparison of the positions of the Free Boundary calculated with $\delta\varphi = \delta y = 1/12$, $\delta z = 1/15$.
Fig. 1	The Three-dimensional Circular Earth Dam.
Fig. 2	The Cross-Section of the Dam for $y = 0$.
Fig. 3	Transformed Space.
Fig. 4	The extrapolation on along the constant φ lines
Fig. 5	A cross-section on any constant z—plane.
Fig. 6	$\phi = y$ plane.
Fig. 7	Selected Values of $10^4 x$ from 12x12 Mesh for $z = 0$ i.e. $k = 0$, by the "Complete" Transformation.
Fig. 8	Selected Values of $10^4 x$ from 12x12 Mesh for $z = 2/3$, i.e. $k = 10$, by the "Complete" Transformation.
Fig. 9	Selected Values of $10^4 x$ from 12x12 Mesh for $z = 4/3$ i.e. $k = 20$, by "Complete" Transformation.