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Non-Fragile H_∞ State Estimation for Recurrent Neural Networks with Time-Varying Delays: On Proportional-Integral Observer Design

Di Zhao, Zidong Wang, Guoliang Wei and Xiaohui Liu

***Abstract*—In this paper, a novel proportional-integral observer (PIO) design approach is proposed for the non-fragile H_∞ state estimation problem for a class of discrete-time recurrent neural networks with time-varying delays. The developed PIO is equipped with more design freedom leading to better steady-state accuracy as compared with the conventional Luenberger observer. The phenomena of randomly occurring gain variations, which are characterized by Bernoulli distributed random variables with certain probabilities, are taken into consideration in the implementation of the addressed PIO. Attention is focused on the design of a non-fragile PIO such that the error dynamics of the state estimation is exponentially stable in mean-square sense and the prescribed H_∞ performance index is also achieved. Sufficient conditions for the existence of the desired PIO are established by virtue of the Lyapunov-Krasovskii functional approach and the matrix inequality technique. Finally, a simulation example is provided to demonstrate the effectiveness of the proposed PIO design scheme.**

***Index Terms*—Recurrent neural networks, proportional-integral observer, non-fragile state estimation, H_∞ performance, randomly occurring gain variations, time-varying delays.**

I. Introduction

In the past few years, recurrent neural networks (RNNs), which are composed of a large number of interconnected neurons, have been received successful applications in a variety of fields including artificial intelligence, optimization, control and signal processing [3], [7], [10], [14], [16], [18], [19]. RNNs are typically implemented by simulating the information processing mechanism of human brain or biological nervous systems, and the success of RNNs is largely credited to their significant superiorities in parallelism, nonlinear mapping, self-learning adaptability, fault tolerance and associative memory. An increasingly attractive research topic along the line of RNN research is the dynamical analysis problem that has led to a rich body of remarkable results appeared in the literature. For example, the analysis problems for stability, adaptability, robustness and fault tolerance have been extensively investigated in [31], [35], [42], [45]. On the other hand, it is well recognized that the acquisition of the state information of certain primary neurons is critically important in enabling RNNs to perform specific tasks such as optimization, classification and approximation. Unfortunately, it is often the case that the neuron state information is not readily available due mainly to the inherent characteristics of RNNs such as huge dimensions, tight couplings, strong nonlinearities and resource constraints. As such, a natural yet efficient way is to estimate the states of the RNNs through available information of the network measurements, which gives rise to the state estimation issue of RNNs [28], [43].

Time-delays are well known to be inevitable in RNNs for three reasons identified as follows: 1) the speed of information transmission between neurons is limited due to physical constraints; 2) the switching speed of electronic components (e.g. the amplifiers) among large-scale integrated circuits when implementing RNNs is inherently bounded; and 3) the time-delays might be purposely introduced into RNNs in order to reflect the problem-specific nature in certain applications such as mobile image processing [23], [26], [45]. In the context of dynamic analysis, time-delays in RNNs contribute much to the system complexities that are likely to cause performance deterioration and undesirable behaviors such as oscillation, divergence, chaos or even instability. Accordingly, time-delays add substantial difficulties/challenges to the state estimation problems of RNNs. As a result, in the past two decades or so, much effort has been devoted to the state estimation problems for RNNs suffering from various kinds of time-delays that include, but are not limited to, constant time-delays, time-varying delays (TVDs), discrete time-delays, distributed time-delays as well as mixed time-delays, see [1], [2], [20], [38], [40], [46]–[48] and the references therein.

As is well known, the H_∞ state estimation (HSE) method has proven to be a powerful tool for evaluating the disturbance attenuation/resistance capacity of the estimation error dynamics, and this warrants the promising application prospect of the HSE algorithm in aerospace, aviation, power system, measuring equipment, robotics and other fields [13], [33], [37]. The main idea of the HSE algorithm is to construct an estimator, on the premise that the disturbance input is energy-bounded, such that the H_∞ norm of the transfer function from the disturbance input to the estimation error is no more than a deterministic value (also called disturbance attenuation level). In comparison with the Kalman filter method that assumes the noises to be strictly Gaussian, the HSE method works well

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under less stringent assumption that the process/measurement noises are arbitrary but energy-bounded. Up to date, the HSE problem has stirred considerable research interest as evidenced by a large number of reported results that can be categorized by employed methodologies such as the linear matrix inequality (LMI) approach and the Riccati matrix equation approach [8], [11], [21], [32]. For example, in [9], a theoretical framework has been established by utilizing the recursive Riccati equation method in order to cope with the distributed HSE problem for time-varying stochastic parameter system. The event-based HSE problem has been investigated in [12] for a class of nonlinear time-varying systems by means of the LMI technique.

During the past few decades, the so-called proportional-integral observer (PIO) has received an ever-increasing interest from a variety of research communities such as manufacturing process, network communication systems, power supply systems and economic systems [4], [6], [41]. To be more specific, the structure of a typical PIO consists of two terms, namely, the proportional term (proportional to the output estimation error) and the integral term (integral to the output estimation error), by which both the current and the historical information can be ideally exploited. In comparison with the conventional Luenberger observer, the PIO possesses certain distinguishing merits such as better steady-state accuracy, stronger robustness, more insensitive to exogenous noises and more freedom to observer design. Thanks to the extra integral term in its structure, the PIO has long been an attractive research topic leading to fruitful results in the literature [5], [17], [36]. Nevertheless, to the best of authors' knowledge, very few results have been available on the PIO design problem for RNNs, not to mention the case where the H_∞ performance index is a major concern as well, and this leaves a gap that will be narrowed through our endeavors in this paper.

An implicit assumption with almost all available PIO design schemes is that the designed PIO can be precisely implemented in practice. Such an assumption, however, is not always reasonable in reality because the imprecision in implementing the PIO parameters is a frequently occurred phenomenon for various reasons such as 1) the finite precision of measuring equipment; 2) the round-off error in numerical calculation; 3) the random failures/repairs of system components; and 4) the requirement of safe-tuning margin reserved for practicing engineers [22], [44]. In other words, the gains of the designed PIOs might encounter undesired fluctuations during the execution process, which could jeopardize the estimation performance to a great extent. In this sense, a natural idea is to design a PIO that is insensitive/invulnerable to the gain variations, and this gives rise to the so-called *non-fragile* PIO design problem. On the other hand, the gain variations might take place on a random basis owing mainly to the network-induced complexities (e.g. quantizations, saturations, disorders or channel fadings) and changes of network conditions (e.g. network load, network congestion and network transmission rate) whose occurrences are typically random [15], [24], [29]. Consequently, it is of both theoretical importance and practical significance to design a non-fragile PIO in case of the randomly occurring gain variations (ROGVs) in order to maintain the satisfactory estimation performance, and this leads to another motivation for the current investigation.

Motivated by the above discussions, in this paper, we strive to challenge the non-fragile H_∞ PIO design problem for the discrete-time recurrent neural network (DRNN) in the presence of TVDs. The main difficulties stem from 1) the design of the non-fragile PIO for the DRNN that ensures the exponentially mean-square (EM-S) stability and the H_∞ performance of the estimation error dynamics; and 2) the establishment of a unified framework to quantify the joint impact from the ROGVs and TVDs on estimation performance. The novelties of this paper are summarized as follows: *1) the first attempt is made to investigate the new PIO design problem for DRNN; 2) a non-fragile PIO is proposed that maintains a satisfactory estimation performance subject to simultaneous presence of ROGVs and TVDs; and 3) sufficient conditions are derived to ensure the EM-S stability and H_∞ performance of the estimation error dynamics.*

The outline of this paper is as follows. Section II formulates the PIO design problem for DRNN in the presence of TVDs. In Section III, both the EM-S stability and the H_∞ performance of the estimation error dynamics are analyzed. Sufficient conditions are characterized for the existence of the desired PIO by virtue of LMI technique. Section IV illustrates the validity of the designed PIO via numerical simulation. Section V concludes this paper.

Notation. For stochastic variables μ and ν, $\mathbb{E}\{\mu\}$ (respectively, $\mathbb{E}\{\mu|\nu\}$) denotes the expectation of μ (respectively, the expectation of μ conditional on ν). I_s refers to the identity matrix of dimension $s \times s$ and the symbol $*$ stands for the ellipsis for symmetric terms. Moreover, for a symmetric matrix Ξ, $\lambda_{\max}(\Xi)$ and $\lambda_{\min}(\Xi)$ are the maximum and minimum eigenvalues, respectively.

II. Problem Formulation and Preliminaries

Consider the following DRNN consisting of m neurons with TVDs:

$$\begin{cases} x(k+1) = Ax(k) + F\ell\big(x(k)\big) + H\jmath\big(x(k-\hbar(k))\big) \\ \qquad\qquad + \vartheta\big(k, x(k), x(k-\hbar(k))\big)v(k) \\ \qquad\qquad + M\varpi(k) \\ y(k) = Cx(k) \\ z(k) = Dx(k) \\ x(\imath) = \varphi(\imath), \quad \imath \in \mathfrak{H} \triangleq \{-\bar{\hbar}, \ldots, -1, 0\} \end{cases} \tag{1}$$

where $x(k) = \big[x_1(k)\ x_2(k)\ \cdots\ x_m(k)\big]^T \in \mathbb{R}^m$ is the neural state vector, $A = \operatorname{diag}\{a_1, a_2, \ldots, a_m\}$ is a diagonal matrix with positive entries $a_i > 0$ ($i \in \mathfrak{M} \triangleq \{1, 2, \ldots, m\}$), $y(k) = \big[y_1(k)\ y_2(k)\ \cdots\ y_p(k)\big]^T \in \mathbb{R}^p$ is the measurement output, and $z(k) = [z_1(k)\ z_2(k)\ \cdots\ z_r(k)]^T \in \mathbb{R}^r$ is the linear combination of the states to be estimated. C and D are deterministic matrices with appropriate dimensions. $\ell\big(x(k)\big) = \big[\ell_1\big(x_1(k)\big)\ \ell_2\big(x_2(k)\big)\ \cdots\ \ell_m\big(x_m(k)\big)\big]^T$ and $\jmath\big(x(k-\hbar(k))\big) = \big[\jmath_1\big(x_1(k-\hbar(k))\big)\ \jmath_2\big(x_2(k-\hbar(k))\big)\ \cdots\ \jmath_m\big(x_m(k-\hbar(k))\big)\big]^T$ are the neuron activation functions and $\hbar(k)$ denotes the TVD. $F = [f_{ij}]_{m\times m}$ and $H = [h_{ij}]_{m\times m}$ represent the connection weight matrix and

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the delayed connection weigh matrix, respectively. $\varpi(k) \in \mathbb{R}^q$ is the exogenous disturbance input belonging to $l_2[0, +\infty)$. $v(k) \in \mathbb{R}$ is a scalar Wiener process on the probability space $(\Omega, \mathscr{F}, \text{Prob})$ with

$$\mathbb{E}\{v(k)\} = 0, \quad \mathbb{E}\{v(s)v(t)\} = \begin{cases} 1, & \text{if } s = t \\ 0, & \text{if } s \neq t \end{cases}$$

and $\vartheta(\cdot, \cdot, \cdot) : \mathbb{R} \times \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m$ stands for the noise intensity vector function. $\varphi(\imath)$ is a given initial condition sequence.

Assumption 1: The neuron activation functions $\ell_i(\cdot) : \mathbb{R} \to \mathbb{R}$ and $\jmath_i(\cdot) : \mathbb{R} \to \mathbb{R}$ $(i \in \mathfrak{M})$ satisfy the following conditions:

$$\begin{aligned} l_i^- \leqslant & \frac{\ell_i(p) - \ell_i(q)}{p - q} \leqslant l_i^+ \\ j_i^- \leqslant & \frac{\jmath_i(p) - \jmath_i(q)}{p - q} \leqslant j_i^+, \ \forall p, q \in \mathbb{R} \end{aligned} \tag{2}$$

where l_i^-, l_i^+, j_i^- and j_i^+ are deterministic constants.

Remark 1: As discussed in [27], the constants l_i^-, l_i^+, j_i^- and j_i^+ in Assumption 1 could be positive, negative, or zero. Consequently, the activation functions $\ell_i(\cdot)$ and $\jmath_i(\cdot)$ $(i \in \mathfrak{M})$ could be nonmonotonic and more general than the usual sigmoid functions and Lipschitz-type conditions.

Assumption 2: The noise intensity vector function $\vartheta(\cdot, \cdot, \cdot) : \mathbb{R} \times \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m$ with $\vartheta(k, 0, 0) = 0$ satisfies the following condition:

$$\vartheta^T(k, \epsilon, \varepsilon)\vartheta(k, \epsilon, \varepsilon) \leqslant \iota_1 \epsilon^T \epsilon + \iota_2 \varepsilon^T \varepsilon, \ \forall \epsilon, \varepsilon \in \mathbb{R}^m \tag{3}$$

where ι_1 and ι_2 are deterministic constants.

Assumption 3: The positive integer $\hbar(k)$ in (1), which stands for the TVD, satisfies

$$\underline{\hbar} \leqslant \hbar(k) \leqslant \bar{\hbar}, \quad k \in \mathbb{N}$$

where $\underline{\hbar}$ and $\bar{\hbar}$ are known positive integers.

In this paper, the phenomenon of ROGVs is taken into consideration in order to accommodate the engineering practice. To estimate the neuron states of (1), a non-fragile PIO is constructed as follows:

$$\begin{cases} \hat{x}(k+1) = A\hat{x}(k) + F\ell\big(\hat{x}(k)\big) + H\jmath\big(\hat{x}(k - \hbar(k))\big) \\ \qquad\qquad + \big(G_P + \varrho(k)\Delta G_P(k)\big)\big(y(k) - C\hat{x}(k)\big) \\ \qquad\qquad + \big(G_I + \sigma(k)\Delta G_I(k)\big)\xi(k) \\ \xi(k+1) = \xi(k) + y(k) - C\hat{x}(k) \\ \hat{z}(k) = D\hat{x}(k) \\ \xi(0) = 0 \\ \hat{x}(\imath) = 0, \quad \imath \in \mathfrak{H} \end{cases} \tag{4}$$

where $\hat{x}(k) \in \mathbb{R}^m$ is the estimate of $x(k)$, $\hat{z}(k) \in \mathbb{R}^r$ is the estimate of $z(k)$ and $\xi(k) \in \mathbb{R}^p$ is a vector representing the integral of the output estimation error. G_P and G_I are the observer gain matrices to be designed.

The mutually *uncorrelated* stochastic variables $\varrho(k)$ and $\sigma(k)$, which govern the phenomenon of ROGVs, are two independent-identical-distribution Bernoulli sequences with the following probabilities:

$$\text{Prob}\{\varrho(k) = 1\} = \bar{\varrho}, \quad \text{Prob}\{\varrho(k) = 0\} = 1 - \bar{\varrho}$$

$$\text{Prob}\{\sigma(k) = 1\} = \bar{\sigma}, \quad \text{Prob}\{\sigma(k) = 0\} = 1 - \bar{\sigma}$$

where $\bar{\varrho} \in [0, 1)$ and $\bar{\sigma} \in [0, 1)$ are two known constants. Here, $\varrho(k)$ and $\sigma(k)$ are uncorrelated with $v(k)$.

The real matrices $\Delta G_P(k)$ and $\Delta G_I(k)$, which denote the observer gain variations, are presented as follows:

$$\Delta G_P(k) = S_P P(k) T_P \tag{5}$$

$$\Delta G_I(k) = S_I I(k) T_I \tag{6}$$

where S_P, S_I, T_P and T_I are deterministic constant matrices with appropriate dimensions, $P(k)$ and $I(k) \in \mathbb{R}^{t \times t}$ are unknown matrix functions satisfying the following norm-bounded conditions

$$P^T(k)P(k) \leqslant I \tag{7}$$

$$I^T(k)I(k) \leqslant I. \tag{8}$$

Remark 2: Compared with the conventional Luenberger observer, the PIO proposed in (4) is equipped with an extra integral term, which renders more design freedom for achieving better steady-state accuracy. On the other hand, the gains of the PIO are subject to undesirable fluctuations that should be adequately tackled in order to mitigate the possible deterioration of the estimation performance. For this purpose, the non-fragile PIO is put forward in (4) with hope to maintain the satisfactory estimation performance in the case of the gain variations on the PIO parameter implementation.

Remark 3: It should be emphasized that, in practical engineering, the gain variations are often inevitable due to a variety of reasons such as finite precision, rounding errors, analog-to-digital conversion and finite word length in computation. Moreover, the phenomenon of gain variation may appear in a random manner owing mainly to the random fluctuations of the network environments (e.g. network load, transmission rate and network bandwidth). Under such circumstances, the occurrence mechanism of the gain variations can be mathematically modeled by Bernoulli processes with certain statistical properties.

Denoting $\tilde{x}(k) \triangleq x(k) - \hat{x}(k)$ and $\tilde{z}(k) \triangleq z(k) - \hat{z}(k)$, we obtain the estimation error dynamics from (1) and (4) as follows:

$$\begin{cases} \tilde{x}(k+1) = A\tilde{x}(k) + F\tilde{\ell}(k) + H\tilde{\jmath}\big(k - \hbar(k)\big) \\ \qquad\qquad - \big(G_P + \varrho(k)\Delta G_P(k)\big)C\tilde{x}(k) \\ \qquad\qquad - \big(G_I + \sigma(k)\Delta G_I(k)\big)\xi(k) \\ \qquad\qquad + \vartheta\big(k, x(k), x(k - \hbar(k))\big)v(k) \\ \qquad\qquad + M\varpi(k) \\ \xi(k+1) = \xi(k) + C\tilde{x}(k) \\ \tilde{z}(k) = D\tilde{x}(k) \\ \tilde{x}(\imath) = \varphi(\imath), \quad \imath \in \mathfrak{H} \end{cases} \tag{9}$$

where

$$\begin{aligned} \tilde{\ell}(k) \triangleq & \ell\big(x(k)\big) - \ell\big(\hat{x}(k)\big) \\ \tilde{\jmath}\big(k - \hbar(k)\big) \triangleq & \jmath\big(x(k - \hbar(k))\big) - \jmath\big(\hat{x}(k - \hbar(k))\big). \end{aligned}$$

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Furthermore, setting $\chi(k) \triangleq \begin{bmatrix} x^T(k) & \tilde{x}^T(k) & \xi^T(k) \end{bmatrix}^T$, we have the following augmented system:

$$\begin{cases} \chi(k+1) = \big(\mathcal{A} + \Delta\mathcal{G}(k)\big)\chi(k) + \tilde{\varrho}(k)\Delta\mathcal{G}_P(k)\chi(k) \\ \qquad + \tilde{\sigma}(k)\Delta\mathcal{G}_I(k)\chi(k) + \mathcal{H}\jmath\big(k-\hbar(k)\big) \\ \qquad + \mathcal{V}\big(k,\chi(k),\chi(k-\hbar(k))\big)v(k) \\ \qquad + \mathcal{F}\ell(k) + \mathcal{M}\varpi(k) \\ \tilde{z}(k) = \mathcal{D}\chi(k) \\ \chi(\imath) = \phi(\imath), \quad \imath \in \mathfrak{H} \end{cases} \tag{10}$$

where $\phi(\imath) \triangleq \begin{bmatrix} \varphi^T(\imath) & \varphi^T(\imath) & 0_{1\times p} \end{bmatrix}^T$ and

$$\mathcal{A} \triangleq \begin{bmatrix} A & 0_{m\times m} & 0_{m\times p} \\ 0_{m\times m} & A - G_P C & -G_I \\ 0_{p\times m} & C & I_p \end{bmatrix}$$

$$\mathcal{F} \triangleq \begin{bmatrix} F & 0_{m\times m} \\ 0_{m\times m} & F \\ 0_{p\times m} & 0_{p\times m} \end{bmatrix}, \quad \mathcal{H} \triangleq \begin{bmatrix} H & 0_{m\times m} \\ 0_{m\times m} & H \\ 0_{p\times m} & 0_{p\times m} \end{bmatrix}$$

$$\mathcal{D} \triangleq \begin{bmatrix} 0_{r\times m} & D & 0_{r\times p} \end{bmatrix}, \quad \mathcal{I} \triangleq \begin{bmatrix} I_m & 0_{m\times m} & 0_{m\times p} \end{bmatrix}$$

$$\mathcal{M} \triangleq \begin{bmatrix} M \\ M \\ 0_{p\times q} \end{bmatrix}, \ \ell(k) \triangleq \begin{bmatrix} \ell(x(k)) \\ \tilde{\ell}(k) \end{bmatrix}, \ \tilde{\varrho}(k) \triangleq \varrho(k) - \bar{\varrho}$$

$$\jmath\big(k-\hbar(k)\big) \triangleq \begin{bmatrix} \jmath\big(x(k-\hbar(k))\big) \\ \tilde{\jmath}\big(k-\hbar(k)\big) \end{bmatrix}, \ \tilde{\sigma}(k) \triangleq \sigma(k) - \bar{\sigma}$$

$$\Delta\mathcal{G}(k) \triangleq \begin{bmatrix} 0_{m\times m} & 0_{m\times m} & 0_{m\times p} \\ 0_{m\times m} & -\bar{\varrho}\Delta G_P(k)C & -\bar{\sigma}\Delta G_I(k) \\ 0_{p\times m} & 0_{p\times m} & 0_{p\times p} \end{bmatrix}$$

$$\Delta\mathcal{G}_P(k) \triangleq \begin{bmatrix} 0_{m\times m} & 0_{m\times m} & 0_{m\times p} \\ 0_{m\times m} & -\Delta G_P(k)C & 0_{m\times p} \\ 0_{p\times m} & 0_{p\times m} & 0_{p\times p} \end{bmatrix}$$

$$\Delta\mathcal{G}_I(k) \triangleq \begin{bmatrix} 0_{m\times m} & 0_{m\times m} & 0_{m\times p} \\ 0_{m\times m} & 0_{m\times m} & -\Delta G_I(k) \\ 0_{p\times m} & 0_{p\times m} & 0_{p\times p} \end{bmatrix}$$

$$\mathcal{V}\big(k,\chi(k),\chi(k-\hbar(k))\big) \triangleq \begin{bmatrix} \vartheta\big(k,\mathcal{I}\chi(k),\mathcal{I}\chi(k-\hbar(k))\big) \\ \vartheta\big(k,\mathcal{I}\chi(k),\mathcal{I}\chi(k-\hbar(k))\big) \\ 0_{p\times 1} \end{bmatrix}.$$

To facilitate later discussions, the definition of EM-S stability is given as follows.

Definition 1: The augmented system (10) with $\varpi(k) = 0$ is said to be EM-S stable if there exist constants $\lambda > 0$ and $\pi \in (0,1)$ such that

$$\mathbb{E}\{\|\chi(k)\|^2\} \leqslant \lambda\pi^k \sup_{\imath\in\mathfrak{H}} \mathbb{E}\{\|\phi(\imath)\|^2\}, \quad \forall k \in \mathbb{N}. \tag{11}$$

The main purpose of this paper is to design a non-fragile PIO in the form of (4) for the DRNN (1) with TVDs. Specifically, we aim to determine the observer gain matrices G_P and G_I such that the augmented system (10) satisfies the following two requirements simultaneously:

1) the augmented system (10) with $\varpi(k) = 0$ is EM-S stable;

2) for a given disturbance attenuation level $\delta > 0$ and all non-zero $\varpi(k)$, under the zero-initial condition, the output $\tilde{z}(k)$ satisfies

$$\sum_{k=0}^{\infty} \mathbb{E}\big\{\|\tilde{z}(k)\|^2\big\} \leqslant \delta^2 \sum_{k=0}^{\infty} \|\varpi(k)\|^2. \tag{12}$$

III. MAIN RESULTS

Lemma 1: [39] Let $U = U^T$, X and Z be real matrices of appropriate dimensions, and $Y(k)$ satisfy $Y^T(k)Y(k) \leqslant I$. Then

$$U + XY(k)Z + Z^TY^T(k)X^T < 0 \tag{13}$$

if and only if there exists a positive scalar κ such that

$$U + \kappa XX^T + \frac{1}{\kappa}Z^TZ < 0 \tag{14}$$

or

$$\begin{bmatrix} U & \kappa X & Z^T \\ \kappa X^T & -\kappa I & 0 \\ Z & 0 & -\kappa I \end{bmatrix} < 0. \tag{15}$$

Lemma 2: Based on condition (2), we have

$$\begin{aligned} &\Big(\ell(k) - \mathcal{L}_+\chi(k)\Big)^T\Big(\ell(k) - \mathcal{L}_-\chi(k)\Big) \leqslant 0 \\ &\Big(\jmath\big(k-\hbar(k)\big) - \mathcal{J}_+\chi\big(k-\hbar(k)\big)\Big)^T \\ &\times \Big(\jmath\big(k-\hbar(k)\big) - \mathcal{J}_-\chi\big(k-\hbar(k)\big)\Big) \leqslant 0 \end{aligned} \tag{16}$$

where

$$\mathcal{L}_+ \triangleq \begin{bmatrix} L^+ & 0 & 0 \\ 0 & L^+ & 0 \end{bmatrix}, \quad \mathcal{L}_- \triangleq \begin{bmatrix} L^- & 0 & 0 \\ 0 & L^- & 0 \end{bmatrix}$$

$$\mathcal{J}_+ \triangleq \begin{bmatrix} J^+ & 0 & 0 \\ 0 & J^+ & 0 \end{bmatrix}, \quad \mathcal{J}_- \triangleq \begin{bmatrix} J^- & 0 & 0 \\ 0 & J^- & 0 \end{bmatrix}$$

$$L^+ \triangleq \mathrm{diag}\{l_1^+, l_2^+, \ldots, l_m^+\}, \quad L^- \triangleq \mathrm{diag}\{l_1^-, l_2^-, \ldots, l_m^-\}$$

$$J^+ \triangleq \mathrm{diag}\{j_1^+, j_2^+, \ldots, j_m^+\}, \quad J^- \triangleq \mathrm{diag}\{j_1^-, j_2^-, \ldots, j_m^-\}.$$

Proof: From the definition of $\tilde{x}(k)$, $\chi(k)$ and combining with the notations in (1), (9) and (10), we have

$$\begin{aligned} &\Big(\ell(k) - \mathcal{L}_+\chi(k)\Big)^T\Big(\ell(k) - \mathcal{L}_-\chi(k)\Big) \\ =&\Big(\ell(x(k)) - L^+x(k)\Big)^T\Big(\ell(x(k)) - L^-x(k)\Big) \\ &+ \Big(\tilde{\ell}(k) - L^+\tilde{x}(k)\Big)^T\Big(\tilde{\ell}(k) - L^-\tilde{x}(k)\Big) \\ =&\sum_{i=1}^{m}\Big(\ell_i\big(x_i(k)\big) - l_i^+x_i(k)\Big)^T\Big(\ell_i\big(x_i(k)\big) - l_i^-x_i(k)\Big) \\ &+ \sum_{i=1}^{m}\Big(\ell_i\big(x_i(k)\big) - \ell_i\big(\hat{x}_i(k)\big) - l_i^+\big(x_i(k) - \hat{x}_i(k)\big)\Big)^T \\ &\times \Big(\ell_i\big(x_i(k)\big) - \ell_i\big(\hat{x}_i(k)\big) - l_i^-\big(x_i(k) - \hat{x}_i(k)\big)\Big) \end{aligned} \tag{17}$$

and

$$\begin{aligned} &\Big(\jmath\big(k-\hbar(k)\big) - \mathcal{J}_+\chi\big(k-\hbar(k)\big)\Big)^T \\ &\times \Big(\jmath\big(k-\hbar(k)\big) - \mathcal{J}_-\chi\big(k-\hbar(k)\big)\Big) \\ =&\Big(\jmath\big(x(k-\hbar(k))\big) - J^+x(k-\hbar(k))\Big)^T \\ &\times \Big(\jmath\big(x(k-\hbar(k))\big) - J^-x(k-\hbar(k))\Big) \\ &+ \Big(\tilde{\jmath}(k-\hbar(k)) - J^+\tilde{x}(k-\hbar(k))\Big)^T \\ &\times \Big(\tilde{\jmath}(k-\hbar(k)) - J^-\tilde{x}(k-\hbar(k))\Big) \end{aligned}$$

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$$
\begin{aligned}
&= \sum_{i=1}^{m} \Big(\jmath_i\big(x_i(k-\hbar(k))\big) - j_i^{+} x_i(k-\hbar(k)) \Big)^T \\
&\quad \times \Big(\jmath_i\big(x_i(k-\hbar(k))\big) - j_i^{-} x_i(k-\hbar(k)) \Big) \\
&\quad + \sum_{i=1}^{m} \Big(\jmath_i\big(x_i(k-\hbar(k))\big) - \jmath_i\big(\hat{x}_i(k-\hbar(k))\big) \\
&\quad - j_i^{+}\big(x_i(k-\hbar(k)) - \hat{x}_i(k-\hbar(k))\big) \Big)^T \\
&\quad \times \Big(\jmath_i\big(x_i(k-\hbar(k))\big) - \jmath_i\big(\hat{x}_i(k-\hbar(k))\big) \\
&\quad - j_i^{-}\big(x_i(k-\hbar(k)) - \hat{x}_i(k-\hbar(k))\big) \Big). \qquad (18)
\end{aligned}
$$

Furthermore, in light of (2), we obtain that

$$
\begin{aligned}
&\Big(\ell_i\big(x_i(k)\big) - l_i^{+} x_i(k)\Big)^T \Big(\ell_i\big(x_i(k)\big) - l_i^{-} x_i(k)\Big) \leqslant 0, \\
&\Big(\ell_i\big(x_i(k)\big) - \ell_i\big(\hat{x}_i(k)\big) - l_i^{+}\big(x_i(k) - \hat{x}_i(k)\big)\Big)^T \\
&\times \Big(\ell_i\big(x_i(k)\big) - \ell_i\big(\hat{x}_i(k)\big) - l_i^{-}\big(x_i(k) - \hat{x}_i(k)\big)\Big) \leqslant 0, \\
&\Big(\jmath_i\big(x_i(k-\hbar(k))\big) - j_i^{+} x_i(k-\hbar(k))\Big)^T \\
&\times \Big(\jmath_i\big(x_i(k-\hbar(k))\big) - j_i^{-} x_i(k-\hbar(k))\Big) \leqslant 0, \\
&\Big(\jmath_i\big(x_i(k-\hbar(k))\big) - \jmath_i\big(\hat{x}_i(k-\hbar(k))\big) \\
&- j_i^{+}\big(x_i(k-\hbar(k)) - \hat{x}_i(k-\hbar(k))\big)\Big)^T \\
&\times \Big(\jmath_i\big(x_i(k-\hbar(k))\big) - \jmath_i\big(\hat{x}_i(k-\hbar(k))\big) \\
&- j_i^{-}\big(x_i(k-\hbar(k)) - \hat{x}_i(k-\hbar(k))\big)\Big) \leqslant 0.
\end{aligned}
$$

Therefore, (16) is true and the proof is complete. ■

A. Exponentially Mean-Square Stability Analysis

In this subsection, we shall give a sufficient condition for examining the EM-S stability of the augmented system (10).

Theorem 1: Let the PIO gain matrices G_P and G_I be given. The system (10) is EM-S stable with $\varpi(k) = 0$ if there exist positive-definite matrices P, Q and positive scalars ς_0, ς_1, ς_2 satisfying the following inequalities:

$$
\begin{cases}
\Omega = \begin{bmatrix} \Omega_{11} & * \\ \Omega_{21} & \Omega_{22} \end{bmatrix} < 0 & \text{(19a)} \\
P < \dfrac{\varsigma_0}{2} I & \text{(19b)}
\end{cases}
$$

where

$$
\Omega_{11} \triangleq \begin{bmatrix} \Omega_{11}^1 & * & * & * \\ 0 & \Omega_{11}^2 & * & * \\ \Omega_{11}^3 & 0 & -\varsigma_1 I & * \\ 0 & \Omega_{11}^4 & 0 & -\varsigma_2 I \end{bmatrix}
$$

$$
\Omega_{21} \triangleq \begin{bmatrix} \lambda_1 \Delta\mathcal{G}_P(k) & 0 & 0 & 0 \\ \lambda_2 \Delta\mathcal{G}_I(k) & 0 & 0 & 0 \\ \mathcal{A} + \Delta\mathcal{G}(k) & 0 & \mathcal{F} & \mathcal{H} \end{bmatrix}
$$

$$
\begin{aligned}
&\Omega_{11}^1 \triangleq -P + (\bar{\hbar} - \underline{\hbar} + 1)Q + \varsigma_0 \iota_1 \mathcal{I}^T\mathcal{I} - \varsigma_1 \mathcal{L}_1 \\
&\Omega_{11}^2 \triangleq -Q + \varsigma_0 \iota_2 \mathcal{I}^T\mathcal{I} - \varsigma_2 \mathcal{J}_1, \quad \Omega_{22} \triangleq -I_3 \otimes P^{-1} \\
&\Omega_{11}^3 \triangleq -\varsigma_1 \mathcal{L}_2, \quad \Omega_{11}^4 \triangleq -\varsigma_2 \mathcal{J}_2, \quad \breve{\varrho} \triangleq \bar{\varrho}(1-\bar{\varrho}) \\
&\breve{\sigma} \triangleq \bar{\sigma}(1-\bar{\sigma}), \quad \lambda_1 \triangleq \sqrt{\breve{\varrho}}, \quad \lambda_2 \triangleq \sqrt{\breve{\sigma}} \\
&\mathcal{L}_1 \triangleq \frac{1}{2}\left(\mathcal{L}_+^T\mathcal{L}_- + \mathcal{L}_-^T\mathcal{L}_+\right), \ \mathcal{L}_2 \triangleq -\frac{1}{2}\left(\mathcal{L}_+ + \mathcal{L}_-\right) \\
&\mathcal{J}_1 \triangleq \frac{1}{2}\left(\mathcal{J}_+^T\mathcal{J}_- + \mathcal{J}_-^T\mathcal{J}_+\right), \ \mathcal{J}_2 \triangleq -\frac{1}{2}\left(\mathcal{J}_+ + \mathcal{J}_-\right).
\end{aligned}
$$

Proof: To derive the criterion for the EM-S stability of the augmented system (10), we construct the following Lyapunov-Krasovskii functional:

$$
V\big(\chi(k)\big) = V_1\big(\chi(k)\big) + V_2\big(\chi(k)\big) + V_3\big(\chi(k)\big) \qquad (20)
$$

where

$$
\begin{aligned}
V_1\big(\chi(k)\big) &\triangleq \chi^T(k)P\chi(k) \\
V_2\big(\chi(k)\big) &\triangleq \sum_{\mu=k-\hbar(k)}^{k-1} \chi^T(\mu)Q\chi(\mu) \\
V_3\big(\chi(k)\big) &\triangleq \sum_{\nu=k-\bar{\hbar}+1}^{k-\underline{\hbar}} \sum_{\mu=\nu}^{k-1} \chi^T(\mu)Q\chi(\mu).
\end{aligned}
$$

Then, the difference of the Lyapunov-Krasovskii functional $V\big(\chi(k)\big)$ is given as follows:

$$
\Im V\big(\chi(k)\big) = \Im V_1\big(\chi(k)\big) + \Im V_2\big(\chi(k)\big) + \Im V_3\big(\chi(k)\big) \qquad (21)
$$

where

$$
\begin{aligned}
\Im V_1\big(\chi(k)\big) &\triangleq \mathbb{E}\{V_1\big(\chi(k+1)\big)|\chi(k)\} - V_1\big(\chi(k)\big) \\
\Im V_2\big(\chi(k)\big) &\triangleq \mathbb{E}\{V_2\big(\chi(k+1)\big)|\chi(k)\} - V_2\big(\chi(k)\big) \\
\Im V_3\big(\chi(k)\big) &\triangleq \mathbb{E}\{V_3\big(\chi(k+1)\big)|\chi(k)\} - V_3\big(\chi(k)\big).
\end{aligned}
$$

In the case of $\varpi(k) = 0$, calculating the difference of $V_1\big(\chi(k)\big)$ along the trajectory of system (10) and taking the mathematical expectation, one has

$$
\begin{aligned}
&\mathbb{E}\left\{\Im V_1\big(\chi(k)\big)\right\} \\
=&\mathbb{E}\left\{V_1\big(\chi(k+1)\big) - V_1\big(\chi(k)\big)\right\} \\
=&\mathbb{E}\Big\{\Big(\big(\mathcal{A} + \Delta\mathcal{G}(k)\big)\chi(k) + \tilde{\varrho}(k)\Delta\mathcal{G}_P(k)\chi(k) \\
&+ \tilde{\sigma}(k)\Delta\mathcal{G}_I(k)\chi(k) + \mathcal{H}\jmath\big(k-\hbar(k)\big) \\
&+ \mathcal{F}\ell(k) + \mathcal{V}\big(k, \chi(k), \chi(k-\hbar(k))\big)v(k)\Big)^T \\
&\times P\Big(\big(\mathcal{A} + \Delta\mathcal{G}(k)\big)\chi(k) + \tilde{\varrho}(k)\Delta\mathcal{G}_P(k)\chi(k) \\
&+ \tilde{\sigma}(k)\Delta\mathcal{G}_I(k)\chi(k) + \mathcal{H}\jmath\big(k-\hbar(k)\big) \\
&+ \mathcal{F}\ell(k) + \mathcal{V}\big(k, \chi(k), \chi(k-\hbar(k))\big)v(k)\Big) \\
&- \chi^T(k)P\chi(k)\Big\} \\
=&\mathbb{E}\Big\{\chi^T(k)\big(\mathcal{A}^T + \Delta\mathcal{G}^T(k)\big)P\big(\mathcal{A} + \Delta\mathcal{G}(k)\big)\chi(k) - \chi^T(k) \\
&\times P\chi(k) + \ell^T(k)\mathcal{F}^T P\mathcal{F}\ell(k) + \tilde{\varrho}^2(k)\chi^T(k)\Delta\mathcal{G}_P{}^T(k) \\
&\times P\Delta\mathcal{G}_P(k)\chi(k) + \tilde{\sigma}^2(k)\chi^T(k)\Delta\mathcal{G}_I{}^T(k)P\Delta\mathcal{G}_I(k) \\
&\times \chi(k) + \jmath^T\big(k-\hbar(k)\big)\mathcal{H}^T P\mathcal{H}\jmath\big(k-\hbar(k)\big) + v^T(k) \\
&\times \mathcal{V}\big(k, \chi(k), \chi(k-\hbar(k))\big)^T P\mathcal{V}\big(k, \chi(k), \chi(k-\hbar(k))\big) \\
&\times v(k) + 2\tilde{\varrho}(k)\chi^T(k)\Delta\mathcal{G}_P{}^T(k)P\big(\mathcal{A} + \Delta\mathcal{G}(k)\big)\chi(k) \\
&+ 2\tilde{\sigma}(k)\chi^T(k)\Delta\mathcal{G}_I{}^T(k)P\big(\mathcal{A} + \Delta\mathcal{G}(k)\big)\chi(k) \\
&+ 2\jmath^T\big(k-\hbar(k)\big)\mathcal{H}^T P\big(\mathcal{A} + \Delta\mathcal{G}(k)\big)\chi(k) + 2\ell^T(k)\mathcal{F}^T P
\end{aligned}
$$

$$
\begin{aligned}
&\times \big(\mathcal{A}+\Delta\mathcal{G}(k)\big)\chi(k)+2v^T(k)\mathcal{V}\big(k,\chi(k),\chi(k-\hbar(k))\big)^T \\
&\times P\big(\mathcal{A}+\Delta\mathcal{G}(k)\big)\chi(k)+2\tilde{\varrho}(k)\tilde{\sigma}(k)\chi^T(k)\Delta\mathcal{G}_I{}^T(k)P \\
&\times \Delta\mathcal{G}_P(k)\chi(k)+2\tilde{\varrho}(k)\jmath^T\big(k-\hbar(k)\big)\mathcal{H}^T P\Delta\mathcal{G}_P(k)\chi(k) \\
&+2\tilde{\varrho}(k)\ell^T(k)\mathcal{F}^T P\Delta\mathcal{G}_P(k)\chi(k)+2\tilde{\varrho}(k)v^T(k) \\
&\times \mathcal{V}\big(k,\chi(k),\chi(k-\hbar(k))\big)^T P\Delta\mathcal{G}_P(k)\chi(k) \\
&+2\tilde{\sigma}(k)\jmath^T\big(k-\hbar(k)\big)\mathcal{H}^T P\Delta\mathcal{G}_I(k)\chi(k)+2\tilde{\sigma}(k)\ell^T(k)\mathcal{F}^T \\
&\times P\Delta\mathcal{G}_I(k)\chi(k)+2\tilde{\sigma}(k)v^T(k)\mathcal{V}\big(k,\chi(k),\chi(k-\hbar(k))\big)^T \\
&\times P\Delta\mathcal{G}_I(k)\chi(k)+2\jmath^T\big(k-\hbar(k)\big)\mathcal{H}^T P\mathcal{F}\ell(k) \\
&+2v^T(k)\mathcal{V}\big(k,\chi(k),\chi(k-\hbar(k))\big)^T P\mathcal{H}\jmath\big(k-\hbar(k)\big) \\
&+2v^T(k)\mathcal{V}\big(k,\chi(k),\chi(k-\hbar(k))\big)^T P\mathcal{F}\ell(k)\Big\} \\
=&\mathbb{E}\Big\{\chi^T(k)\big(\mathcal{A}^T+\Delta\mathcal{G}^T(k)\big)P\big(\mathcal{A}+\Delta\mathcal{G}(k)\big)\chi(k) \\
&-\chi^T(k)P\chi(k)+\ell^T(k)\mathcal{F}^T P\mathcal{F}\ell(k) \\
&+\tilde{\varrho}^2(k)\chi^T(k)\Delta\mathcal{G}_P{}^T(k)P\Delta\mathcal{G}_P(k)\chi(k) \\
&+\tilde{\sigma}^2(k)\chi^T(k)\Delta\mathcal{G}_I{}^T(k)P\Delta\mathcal{G}_I(k)\chi(k) \\
&+\jmath^T\big(k-\hbar(k)\big)\mathcal{H}^T P\mathcal{H}\jmath\big(k-\hbar(k)\big) \\
&+v^T(k)\mathcal{V}\big(k,\chi(k),\chi(k-\hbar(k))\big)^T P \\
&\times \mathcal{V}\big(k,\chi(k),\chi(k-\hbar(k))\big)v(k) \\
&+2\jmath^T\big(k-\hbar(k)\big)\mathcal{H}^T P\big(\mathcal{A}+\Delta\mathcal{G}(k)\big)\chi(k) \\
&+2\ell^T(k)\mathcal{F}^T P\big(\mathcal{A}+\Delta\mathcal{G}(k)\big)\chi(k) \\
&+2\jmath^T\big(k-\hbar(k)\big)\mathcal{H}^T P\mathcal{F}\ell(k)\Big\}.
\end{aligned}
\tag{22}
$$

Furthermore, the differences for $V_2\big(\chi(k)\big)$ and $V_3\big(\chi(k)\big)$ can be computed as follows:

$$
\begin{aligned}
&\mathbb{E}\left\{\Im V_2\big(\chi(k)\big)\right\} \\
=&\mathbb{E}\left\{V_2\big(\chi(k+1)\big)-V_2\big(\chi(k)\big)\right\} \\
=&\mathbb{E}\left\{\sum_{\mu=k-\hbar(k+1)+1}^{k}\chi^T(\mu)Q\chi(\mu)-\sum_{\mu=k-\hbar(k)}^{k-1}\chi^T(\mu)Q\chi(\mu)\right\} \\
=&\mathbb{E}\Bigg\{\chi^T(k)Q\chi(k)-\chi^T\big(k-\hbar(k)\big)Q\chi\big(k-\hbar(k)\big) \\
&+\sum_{\mu=k-\hbar(k+1)+1}^{k-1}\chi^T(\mu)Q\chi(\mu)-\sum_{\mu=k-\hbar(k)+1}^{k-1}\chi^T(\mu)Q\chi(\mu)\Bigg\} \\
=&\mathbb{E}\Bigg\{\chi^T(k)Q\chi(k)-\chi^T\big(k-\hbar(k)\big)Q\chi\big(k-\hbar(k)\big) \\
&+\sum_{\mu=k-\underline{\hbar}+1}^{k-1}\chi^T(\mu)Q\chi(\mu)+\sum_{\mu=k-\hbar(k+1)+1}^{k-\underline{\hbar}}\chi^T(\mu)Q\chi(\mu) \\
&-\sum_{\mu=k-\hbar(k)+1}^{k-1}\chi^T(\mu)Q\chi(\mu)\Bigg\} \\
\leqslant&\mathbb{E}\Bigg\{\chi^T(k)Q\chi(k)-\chi^T\big(k-\hbar(k)\big)Q\chi\big(k-\hbar(k)\big) \\
&+\sum_{\mu=k-\bar{\hbar}+1}^{k-\underline{\hbar}}\chi^T(\mu)Q\chi(\mu)\Bigg\}
\end{aligned}
\tag{23}
$$

and

$$
\begin{aligned}
&\mathbb{E}\left\{\Im V_3\big(\chi(k)\big)\right\} \\
=&\mathbb{E}\left\{V_3\big(\chi(k+1)\big)-V_3\big(\chi(k)\big)\right\} \\
=&\mathbb{E}\left\{\sum_{\nu=k-\bar{\hbar}+2}^{k-\underline{\hbar}+1}\sum_{\mu=\nu}^{k}\chi^T(\mu)Q\chi(\mu)-\sum_{\nu=k-\bar{\hbar}+1}^{k-\underline{\hbar}}\sum_{\mu=\nu}^{k}\chi^T(\mu)Q\chi(\mu)\right\} \\
=&\mathbb{E}\left\{\sum_{\nu=k-\bar{\hbar}+1}^{k-\underline{\hbar}}\Big(\chi^T(k)Q\chi(k)-\chi^T(\nu)Q\chi(\nu)\Big)\right\} \\
=&\mathbb{E}\left\{(\bar{\hbar}-\underline{\hbar})\chi^T(k)Q\chi(k)-\sum_{\mu=k-\bar{\hbar}+1}^{k-\underline{\hbar}}\chi^T(\mu)Q\chi(\mu)\right\}.
\end{aligned}
\tag{24}
$$

Noting the statistical characteristics of $v(k)$ and combining with (3) and (19b), we calculate the term $v^T(k)\mathcal{V}\big(k,\chi(k),\chi(k-\hbar(k))\big)^T P\mathcal{V}\big(k,\chi(k),\chi(k-\hbar(k))\big)v(k)$ (contained in (22)) as follows:

$$
\begin{aligned}
&\mathbb{E}\Big\{v^T(k)\mathcal{V}\big(k,\chi(k),\chi(k-\hbar(k))\big)^T P \\
&\quad\times \mathcal{V}\big(k,\chi(k),\chi(k-\hbar(k))\big)v(k)\Big\} \\
=&\mathcal{V}\big(k,\chi(k),\chi(k-\hbar(k))\big)^T P \\
&\quad\times \mathcal{V}\big(k,\chi(k),\chi(k-\hbar(k))\big) \\
\leqslant&\lambda_{\max}(P)\mathcal{V}\big(k,\chi(k),\chi(k-\hbar(k))\big)^T \\
&\quad\times \mathcal{V}\big(k,\chi(k),\chi(k-\hbar(k))\big) \\
\leqslant&\varsigma_0\vartheta\big(k,\mathcal{I}\chi(k),\mathcal{I}\chi(k-\hbar(k))\big)^T \\
&\quad\times \vartheta\big(k,\mathcal{I}\chi(k),\mathcal{I}\chi(k-\hbar(k))\big) \\
\leqslant&\varsigma_0\Big(\iota_1\chi^T(k)\mathcal{I}^T\mathcal{I}\chi(k) \\
&\quad+\iota_2\chi^T(k-\hbar(k))\mathcal{I}^T\mathcal{I}\chi(k-\hbar(k))\Big).
\end{aligned}
\tag{25}
$$

Substituting (22)-(25) into (21), we have

$$
\begin{aligned}
&\mathbb{E}\big\{\Im V\big(\chi(k)\big)\big\} \\
=&\mathbb{E}\Big\{\Im V_1\big(\chi(k)\big)+\Im V_2\big(\chi(k)\big)+\Im V_3\big(\chi(k)\big)\Big\} \\
\leqslant&\mathbb{E}\Big\{\chi^T(k)\Big(\big(\mathcal{A}^T+\Delta\mathcal{G}^T(k)\big)P\big(\mathcal{A}+\Delta\mathcal{G}(k)\big) \\
&+(\bar{\hbar}-\underline{\hbar}+1)Q-P+\tilde{\varrho}^2(k)\Delta\mathcal{G}_P{}^T(k)P\Delta\mathcal{G}_P(k) \\
&+\tilde{\sigma}^2(k)\Delta\mathcal{G}_I{}^T(k)P\Delta\mathcal{G}_I(k)+\varsigma_0\iota_1\mathcal{I}^T\mathcal{I}\Big)\chi(k) \\
&+\chi^T\big(k-\hbar(k)\big)\Big(\varsigma_0\iota_2\mathcal{I}^T\mathcal{I}-Q\Big)\chi\big(k-\hbar(k)\big) \\
&+\jmath^T\big(k-\hbar(k)\big)\mathcal{H}^T P\mathcal{H}\jmath\big(k-\hbar(k)\big) \\
&+\ell^T(k)\mathcal{F}^T P\mathcal{F}\ell(k)+2\jmath^T\big(k-\hbar(k)\big)\mathcal{H}^T P\mathcal{F}\ell(k) \\
&+2\jmath^T\big(k-\hbar(k)\big)\mathcal{H}^T P\big(\mathcal{A}+\Delta\mathcal{G}(k)\big)\chi(k) \\
&+2\ell^T(k)\mathcal{F}^T P\big(\mathcal{A}+\Delta\mathcal{G}(k)\big)\chi(k)\Big\} \\
=&\mathbb{E}\Big\{\chi^T(k)\Big(\big(\mathcal{A}^T+\Delta\mathcal{G}^T(k)\big)P\big(\mathcal{A}+\Delta\mathcal{G}(k)\big) \\
&+(\bar{\hbar}-\underline{\hbar}+1)Q-P+\check{\varrho}\Delta\mathcal{G}_P{}^T(k)P\Delta\mathcal{G}_P(k) \\
&+\check{\sigma}\Delta\mathcal{G}_I{}^T(k)P\Delta\mathcal{G}_I(k)+\varsigma_0\iota_1\mathcal{I}^T\mathcal{I}\Big)\chi(k) \\
&+\chi^T\big(k-\hbar(k)\big)\Big(\varsigma_0\iota_2\mathcal{I}^T\mathcal{I}-Q\Big)\chi\big(k-\hbar(k)\big) \\
&+\jmath^T\big(k-\hbar(k)\big)\mathcal{H}^T P\mathcal{H}\jmath\big(k-\hbar(k)\big)
\end{aligned}
$$

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$$
\begin{aligned}
&+\ell^T(k)\mathcal{F}^T P\mathcal{F}\ell(k)+2\jmath^T\big(k-\hbar(k)\big)\mathcal{H}^T P\mathcal{F}\ell(k)\\
&+2\jmath^T\big(k-\hbar(k)\big)\mathcal{H}^T P\big(\mathcal{A}+\Delta\mathcal{G}(k)\big)\chi(k)\\
&+2\ell^T(k)\mathcal{F}^T P\big(\mathcal{A}+\Delta\mathcal{G}(k)\big)\chi(k)\Big\}\\
=&\mathbb{E}\Big\{\aleph_1^T(k)\Pi_1\aleph_1(k)\Big\} \qquad (26)
\end{aligned}
$$

where

$$
\begin{aligned}
\aleph_1(k)\triangleq&\big[\chi^T(k)\quad \chi^T\big(k-\hbar(k)\big)\quad \ell^T(k)\quad \jmath^T\big(k-\hbar(k)\big)\big]^T\\
\Pi_1\triangleq&\begin{bmatrix}\Pi_1^{11} & * & * & *\\ 0 & \Pi_1^{22} & * & *\\ \Pi_1^{31} & 0 & \Pi_1^{33} & *\\ \Pi_1^{41} & 0 & \Pi_1^{43} & \Pi_1^{44}\end{bmatrix}\\
\Pi_1^{11}\triangleq&-P+\big(\mathcal{A}^T+\Delta\mathcal{G}^T(k)\big)P\big(\mathcal{A}+\Delta\mathcal{G}(k)\big)\\
&+(\bar{\hbar}-\underline{\hbar}+1)Q+\check{\varrho}\Delta\mathcal{G}_P{}^T(k)P\Delta\mathcal{G}_P(k)\\
&+\check{\sigma}\Delta\mathcal{G}_I{}^T(k)P\Delta\mathcal{G}_I(k)+\varsigma_0\iota_1\mathcal{I}^T\mathcal{I}\\
\Pi_1^{22}\triangleq&-Q+\varsigma_0\iota_2\mathcal{I}^T\mathcal{I},\quad \Pi_1^{33}\triangleq\mathcal{F}^TP\mathcal{F}\\
\Pi_1^{44}\triangleq&\mathcal{H}^TP\mathcal{H},\quad \Pi_1^{31}\triangleq\mathcal{F}^TP\big(\mathcal{A}+\Delta\mathcal{G}(k)\big)\\
\Pi_1^{41}\triangleq&\mathcal{H}^TP\big(\mathcal{A}+\Delta\mathcal{G}(k)\big),\quad \Pi_1^{43}\triangleq\mathcal{H}^TP\mathcal{F}.
\end{aligned}
$$

From (16), it can be readily verified that

$$
\begin{aligned}
&\begin{bmatrix}\chi(k)\\ \ell(k)\end{bmatrix}^T\begin{bmatrix}\mathcal{L}_1 & *\\ \mathcal{L}_2 & I\end{bmatrix}\begin{bmatrix}\chi(k)\\ \ell(k)\end{bmatrix}\leqslant 0,\\
&\begin{bmatrix}\chi\big(k-\hbar(k)\big)\\ \jmath\big(k-\hbar(k)\big)\end{bmatrix}^T\begin{bmatrix}\mathcal{J}_1 & *\\ \mathcal{J}_2 & I\end{bmatrix}\begin{bmatrix}\chi\big(k-\hbar(k)\big)\\ \jmath\big(k-\hbar(k)\big)\end{bmatrix}\leqslant 0. \qquad (27)
\end{aligned}
$$

Furthermore, it results from (26) that

$$
\begin{aligned}
&\mathbb{E}\big\{\Im V\big(\chi(k)\big)\big\}\\
\leqslant&\mathbb{E}\big\{\aleph_1^T(k)\Pi_1\aleph_1(k)\big\}-\varsigma_1\mathbb{E}\left\{\begin{bmatrix}\chi(k)\\ \ell(k)\end{bmatrix}^T\begin{bmatrix}\mathcal{L}_1 & *\\ \mathcal{L}_2 & I\end{bmatrix}\begin{bmatrix}\chi(k)\\ \ell(k)\end{bmatrix}\right\}\\
&-\varsigma_2\mathbb{E}\left\{\begin{bmatrix}\chi\big(k-\hbar(k)\big)\\ \jmath\big(k-\hbar(k)\big)\end{bmatrix}^T\begin{bmatrix}\mathcal{J}_1 & *\\ \mathcal{J}_2 & I\end{bmatrix}\begin{bmatrix}\chi\big(k-\hbar(k)\big)\\ \jmath\big(k-\hbar(k)\big)\end{bmatrix}\right\}\\
=&\mathbb{E}\Big\{\aleph_1^T(k)\Pi_2\aleph_1(k)\Big\} \qquad (28)
\end{aligned}
$$

where

$$
\begin{aligned}
\Pi_2\triangleq&\Pi_1+\Pi_3\\
\Pi_3\triangleq&\begin{bmatrix}\Pi_3^{11} & * & * & *\\ 0 & \Pi_3^{22} & * & *\\ \Pi_3^{31} & 0 & \Pi_3^{33} & *\\ 0 & \Pi_3^{42} & 0 & \Pi_3^{44}\end{bmatrix}\\
\Pi_3^{11}\triangleq&-\varsigma_1\mathcal{L}_1,\quad \Pi_3^{22}\triangleq-\varsigma_2\mathcal{J}_1\\
\Pi_3^{33}\triangleq&-\varsigma_1 I,\quad \Pi_3^{44}\triangleq-\varsigma_2 I\\
\Pi_3^{31}\triangleq&-\varsigma_1\mathcal{L}_2,\quad \Pi_3^{42}\triangleq-\varsigma_2\mathcal{J}_2.
\end{aligned}
$$

By virtue of the Schur Complement Lemma, we conclude from (19a) that $\Pi_2<0$, which further indicates

$$
\mathbb{E}\big\{\Im V\big(\chi(k)\big)\big\}\leqslant-\lambda_{\min}(-\Pi_2)\mathbb{E}\{\|\chi(k)\|^2\}. \qquad (29)
$$

In what follows, we shall proceed to analyze the EM-S stability of the augmented system (10). According to the definition of $V(\chi(k))$, we know that

$$
\mathbb{E}\{V(\chi(k))\}\leqslant\eta_1\mathbb{E}\{\|\chi(k)\|^2\}+\eta_2\sum_{\nu=k-\bar{\hbar}}^{k-1}\mathbb{E}\{\|\chi(\nu)\|^2\} \quad (30)
$$

where

$$
\eta_1\triangleq\lambda_{\max}(P),\quad \eta_2\triangleq(\bar{\hbar}-\underline{\hbar}+1)\lambda_{\max}(Q).
$$

Furthermore, for any $\rho>1$, it follows from (29) that

$$
\begin{aligned}
&\mathbb{E}\{\rho^{k+1}V(\chi(k+1))\}-\mathbb{E}\{\rho^kV(\chi(k))\}\\
=&\rho^{k+1}\mathbb{E}\{\Im V(\chi(k))\}+\rho^{k+1}\mathbb{E}\{V(\chi(k))\}\\
&-\rho^k\mathbb{E}\{V(\chi(k))\}\\
\leqslant&\rho^{k+1}\Big(-\lambda_{\min}(-\Pi_2)\mathbb{E}\{\|\chi(k)\|^2\}\Big)\\
&+\rho^k(\rho-1)\mathbb{E}\{V(\chi(k))\}\\
\leqslant&\alpha_1(\rho)\rho^k\mathbb{E}\{\|\chi(k)\|^2\}\\
&+\alpha_2(\rho)\sum_{\nu=k-\bar{\hbar}}^{k-1}\rho^k\mathbb{E}\{\|\chi(\nu)\|^2\} \qquad (31)
\end{aligned}
$$

where

$$
\begin{aligned}
\alpha_1(\rho)\triangleq&-\lambda_{\min}(-\Pi_2)\rho+(\rho-1)\eta_1\\
\alpha_2(\rho)\triangleq&(\rho-1)\eta_2.
\end{aligned}
$$

For any integer $\theta\geqslant 1$, taking summation on both sides of (31) from 0 to $\theta-1$ with respect to k yields

$$
\begin{aligned}
&\mathbb{E}\{\rho^\theta V(\chi(\theta))\}-\mathbb{E}\{V(\chi(0))\}\\
\leqslant&\alpha_1(\rho)\sum_{k=0}^{\theta-1}\rho^k\mathbb{E}\{\|\chi(k)\|^2\}\\
&+\alpha_2(\rho)\sum_{k=0}^{\theta-1}\sum_{\nu=k-\bar{\hbar}}^{k-1}\rho^k\mathbb{E}\{\|\chi(\nu)\|^2\}. \qquad (32)
\end{aligned}
$$

Additionally, the last item in (32) can be computed as

$$
\begin{aligned}
&\sum_{k=0}^{\theta-1}\sum_{\nu=k-\bar{\hbar}}^{k-1}\rho^k\mathbb{E}\{\|\chi(\nu)\|^2\}\\
\leqslant&\left(\sum_{\nu=-\bar{\hbar}}^{-1}\sum_{k=0}^{\nu+\bar{\hbar}}+\sum_{\nu=0}^{\theta-\bar{\hbar}-1}\sum_{k=\nu+1}^{\nu+\bar{\hbar}}+\sum_{\nu=\theta-\bar{\hbar}}^{\theta-1}\sum_{k=\nu+1}^{\theta-1}\right)\rho^k\mathbb{E}\{\|\chi(\nu)\|^2\}\\
\leqslant&\frac{\rho^{\bar{\hbar}}-1}{\rho-1}\sum_{\nu=-\bar{\hbar}}^{-1}\mathbb{E}\{\|\chi(\nu)\|^2\}+\frac{\rho(\rho^{\bar{\hbar}}-1)}{\rho-1}\sum_{\nu=0}^{\theta-1}\rho^\nu\mathbb{E}\{\|\chi(\nu)\|^2\}\\
&+\frac{\rho(\rho^{\bar{\hbar}}-1)}{\rho-1}\sum_{\nu=0}^{\theta-1}\rho^\nu\mathbb{E}\{\|\chi(\nu)\|^2\}. \qquad (33)
\end{aligned}
$$

Then, it follows from (32) and (33) that

$$
\begin{aligned}
&\mathbb{E}\{\rho^\theta V(\chi(\theta))\}-\mathbb{E}\{V(\chi(0))\}\\
\leqslant&\alpha_1(\rho)\sum_{k=0}^{\theta-1}\rho^k\mathbb{E}\{\|\chi(k)\|^2\}\\
&+\alpha_2(\rho)\Bigg(\frac{\rho^{\bar{\hbar}}-1}{\rho-1}\sum_{\nu=-\bar{\hbar}}^{-1}\mathbb{E}\{\|\chi(\nu)\|^2\}\\
&+\frac{\rho(\rho^{\bar{\hbar}}-1)}{\rho-1}\sum_{\nu=0}^{\theta-1}\rho^\nu\mathbb{E}\{\|\chi(\nu)\|^2\}\\
&+\frac{\rho(\rho^{\bar{\hbar}}-1)}{\rho-1}\sum_{\nu=0}^{\theta-1}\rho^\nu\mathbb{E}\{\|\chi(\nu)\|^2\}\Bigg)
\end{aligned}
$$

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$$\begin{aligned}&\leqslant \beta_1(\rho)\sum_{k=0}^{\theta-1}\rho^k\mathbb{E}\{\|\chi(k)\|^2\}\\&\quad+\beta_2(\rho)\sup_{\imath\in\mathfrak{H}}\mathbb{E}\{\|\phi(\imath)\|^2\}\end{aligned}\tag{34}$$

where

$$\begin{aligned}\beta_1(\rho)&\triangleq\alpha_1(\rho)+\alpha_2(\rho)\frac{2\rho^{\bar{\hbar}+1}-2\rho}{\rho-1}\\\beta_2(\rho)&\triangleq\alpha_2(\rho)\bar{\hbar}\frac{\rho^{\bar{\hbar}}-1}{\rho-1}.\end{aligned}$$

Since $\beta_1(1)=-\lambda_{\min}(-\Pi_2)<0$ and $\lim_{\rho\to\infty}\beta_1(\rho)=+\infty$, we can infer that there exists a scalar $\gamma>1$ such that $\beta_1(\gamma)=0$, which implies that

$$\begin{aligned}&\mathbb{E}\{\gamma^\theta V(\chi(\theta))\}-\mathbb{E}\{V(\chi(0))\}\\&\leqslant\beta_2(\gamma)\sup_{\imath\in\mathfrak{H}}\mathbb{E}\{\|\phi(\imath)\|^2\}.\end{aligned}\tag{35}$$

Noting

$$\mathbb{E}\{V(\chi(0))\}\leqslant\bar{\eta}\sup_{\imath\in\mathfrak{H}}\mathbb{E}\{\|\phi(\imath)\|^2\}\tag{36}$$

and

$$\mathbb{E}\{\gamma^\theta V(\chi(\theta))\}\geqslant\lambda_{\min}(P)\gamma^\theta\mathbb{E}\{\|\chi(\theta)\|^2\}\tag{37}$$

where

$$\bar{\eta}\triangleq(\bar{\hbar}+1)\max\{\eta_1,\eta_2\},$$

we obtain

$$\begin{aligned}\mathbb{E}\{\|\chi(\theta)\|^2\}&\leqslant\frac{\bar{\eta}+\beta_2(\gamma)}{\lambda_{\min}(P)\gamma^\theta}\sup_{\imath\in\mathfrak{H}}\mathbb{E}\{\|\phi(\imath)\|^2\}\\&=\lambda\pi^\theta\sup_{\imath\in\mathfrak{H}}\mathbb{E}\{\|\phi(\imath)\|^2\}\end{aligned}\tag{38}$$

with

$$\lambda\triangleq\frac{\bar{\eta}+\beta_2(\gamma)}{\lambda_{\min}(P)},\quad\pi\triangleq\frac{1}{\gamma}.$$

Consequently, according to Definition 1, it is easy to conclude that the augmented system (10) with $\varpi(k)=0$ is EM-S stable, which completes the proof. ∎

B. H_∞ Performance Analysis

In this subsection, the analysis on the H_∞ performance constraint (12) will be conducted for the augmented system (10) with non-zero $\varpi(k)$ under the zero-initial condition.

Theorem 2: Let the PIO gain matrices G_P, G_I and the disturbance attenuation level $\delta>0$ be given. The system (10) is EM-S stable and also satisfies the H_∞ performance constraint (12) for all non-zero $\varpi(k)$ under the zero-initial condition if there exist positive-definite matrices P, Q and positive scalars ς_0, ς_1, ς_2 satisfying the following inequalities:

$$\begin{cases}\Theta=\begin{bmatrix}\Theta_{11}&*\\\Theta_{21}&\Theta_{22}\end{bmatrix}<0 & \text{(39a)}\\ P<\dfrac{\varsigma_0}{2}I & \text{(39b)}\end{cases}$$

where

$$\Theta_{11}\triangleq\begin{bmatrix}\Omega_{11}^1&*&*&*&*\\0&\Omega_{11}^2&*&*&*\\\Omega_{11}^3&0&-\varsigma_1 I&*&*\\0&\Omega_{11}^4&0&-\varsigma_2 I&*\\0&0&0&0&-\delta^2 I\end{bmatrix}$$

$$\Theta_{21}\triangleq\begin{bmatrix}\lambda_1\Delta\mathcal{G}_P(k)&0&0&0&0\\\lambda_2\Delta\mathcal{G}_I(k)&0&0&0&0\\\mathcal{A}+\Delta\mathcal{G}(k)&0&\mathcal{F}&\mathcal{H}&\mathcal{M}\\\mathcal{D}&0&0&0&0\end{bmatrix}$$

$$\Theta_{22}\triangleq\text{diag}\{-P^{-1},\ -P^{-1},\ -P^{-1},\ -I\}.$$

Proof: Note that $\Omega<0$ is implied by $\Theta<0$, hence the EM-S stability of the augmented system (10) in the case of $\varpi(k)=0$ can be inferred immediately from Theorem 1.

In what follows, in order to conduct the H_∞ performance analysis for the augmented system (10) under non-zero $\varpi(k)$, we define the following index functional:

$$J(\mu)\triangleq\sum_{k=0}^{\mu}\mathbb{E}\{\tilde{z}^T(k)\tilde{z}(k)-\delta^2\varpi^T(k)\varpi(k)\}\tag{40}$$

where μ is a non-negative integer.

According to the initial condition $\chi(0)=0$, we know that $V(\chi(0))=0$, and therefore

$$\begin{aligned}J(\mu)&=\sum_{k=0}^{\mu}\mathbb{E}\{\tilde{z}^T(k)\tilde{z}(k)-\delta^2\varpi^T(k)\varpi(k)\}\\&=\sum_{k=0}^{\mu}\mathbb{E}\{\Im V(\chi(k))+\tilde{z}^T(k)\tilde{z}(k)-\delta^2\varpi^T(k)\varpi(k)\}\\&\quad+\mathbb{E}\{V(\chi(0))\}-\mathbb{E}\{V(\chi(\mu+1))\}\\&=\sum_{k=0}^{\mu}\mathbb{E}\{\Im V(\chi(k))+\tilde{z}^T(k)\tilde{z}(k)-\delta^2\varpi^T(k)\varpi(k)\}\\&\quad-\mathbb{E}\{V(\chi(\mu+1))\}\\&\leqslant\sum_{k=0}^{\mu}\mathbb{E}\Big\{\aleph_1^T(k)\Pi_2\aleph_1(k)+2\varpi^T(k)\mathcal{M}^TP\big(\mathcal{A}+\Delta\mathcal{G}(k)\big)\\&\quad\times\chi(k)+2\tilde{\varrho}(k)\varpi^T(k)\mathcal{M}^TP\Delta\mathcal{G}_P(k)\chi(k)\\&\quad+2\tilde{\sigma}(k)\varpi^T(k)\mathcal{M}^TP\Delta\mathcal{G}_I(k)\chi(k)+2\varpi^T(k)\\&\quad\times\mathcal{M}^TP\mathcal{H}\jmath\big(k-\hbar(k)\big)+2\varpi^T(k)\mathcal{M}^TP\mathcal{F}\ell(k)\\&\quad+2\varpi^T(k)\mathcal{M}^TP\mathcal{V}\big(k,\chi(k),\chi(k-\hbar(k))\big)v(k)\\&\quad+\varpi^T(k)\mathcal{M}^TP\mathcal{M}\varpi(k)+\chi(k)\mathcal{D}^T\mathcal{D}\chi(k)\\&\quad-\delta^2\varpi^T(k)\varpi(k)\Big\}-\mathbb{E}\{V(\chi(\mu+1))\}\\&\leqslant\sum_{k=0}^{\mu}\mathbb{E}\Big\{\aleph_1^T(k)\Pi_2\aleph_1(k)+2\varpi^T(k)\mathcal{M}^TP\big(\mathcal{A}+\Delta\mathcal{G}(k)\big)\\&\quad\times\chi(k)+2\varpi^T(k)\mathcal{M}^TP\mathcal{H}\jmath\big(k-\hbar(k)\big)\\&\quad+2\varpi^T(k)\mathcal{M}^TP\mathcal{F}\ell(k)+\varpi^T(k)\mathcal{M}^TP\mathcal{M}\varpi(k)\\&\quad+\chi(k)\mathcal{D}^T\mathcal{D}\chi(k)-\delta^2\varpi^T(k)\varpi(k)\Big\}\\&\quad-\mathbb{E}\{V(\chi(\mu+1))\}\\&=\sum_{k=0}^{\mu}\mathbb{E}\{\aleph_2^T(k)\Pi_4\aleph_2(k)\}-\mathbb{E}\{V(\chi(\mu+1))\}\end{aligned}\tag{41}$$

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where

$$\aleph_2(k) \triangleq \begin{bmatrix}\aleph_1^T(k) & \varpi^T(k)\end{bmatrix}^T$$

$$\Pi_4 \triangleq \begin{bmatrix}\Pi_4^{11} & * & * & * & * \\ 0 & \Pi_4^{22} & * & * & * \\ \Pi_4^{31} & 0 & \Pi_4^{33} & * & * \\ \Pi_1^{41} & \Pi_3^{42} & \Pi_1^{43} & \Pi_4^{44} & * \\ \Pi_4^{51} & 0 & \Pi_4^{53} & \Pi_4^{54} & \Pi_4^{55}\end{bmatrix}$$

$$\begin{aligned}
\Pi_4^{11} &\triangleq \Pi_1^{11} + \Pi_3^{11} + \mathcal{D}^T\mathcal{D} \\
\Pi_4^{22} &\triangleq \Pi_1^{22} + \Pi_3^{22} \\
\Pi_4^{31} &\triangleq \Pi_1^{31} + \Pi_3^{31} \\
\Pi_4^{33} &\triangleq \Pi_1^{33} + \Pi_3^{33} \\
\Pi_4^{44} &\triangleq \Pi_1^{44} + \Pi_3^{44} \\
\Pi_4^{51} &\triangleq \mathcal{M}^T P\big(\mathcal{A} + \Delta\mathcal{G}(k)\big) \\
\Pi_4^{53} &\triangleq \mathcal{M}^T P\mathcal{F} \\
\Pi_4^{54} &\triangleq \mathcal{M}^T P\mathcal{H} \\
\Pi_4^{55} &\triangleq -\delta^2 I + \mathcal{M}^T P\mathcal{M}.
\end{aligned}$$

On the basis of the Schur Complement Lemma, we conclude from (39a) that $\mathbb{E}\{\aleph_2^T(k)\Pi_4\aleph_2(k)\} < 0$, which results in

$$\sum_{k=0}^{\mu}\mathbb{E}\big\{\tilde{z}^T(k)\tilde{z}(k) - \delta^2\varpi^T(k)\varpi(k)\big\} \leqslant -\mathbb{E}\big\{V(\chi(\mu+1))\big\}. \quad (42)$$

Letting $\mu \to \infty$ and considering $\mathbb{E}\{V(\chi(\infty))\} \geqslant 0$, it follows from (42) that

$$\begin{aligned}
&\sum_{k=0}^{\infty}\mathbb{E}\big\{\tilde{z}^T(k)\tilde{z}(k)\big\} - \delta^2\sum_{k=0}^{\infty}\varpi^T(k)\varpi(k) \\
&\leqslant -\mathbb{E}\big\{V(\chi(\infty))\big\} \leqslant 0 \qquad (43)
\end{aligned}$$

which further indicates

$$\sum_{k=0}^{\infty}\mathbb{E}\big\{\tilde{z}^T(k)\tilde{z}(k)\big\} \leqslant \delta^2\sum_{k=0}^{\infty}\varpi^T(k)\varpi(k). \quad (44)$$

The proof of Theorem 2 is complete. ■

C. Non-Fragile PIO Design

In this subsection, a non-fragile PIO is designed for the DRNN (1) by virtue of the LMI technique. Furthermore, based on the designed observer, both the EM-S stability and the H_∞ performance of the augmented system (10) can be simultaneously achieved.

Theorem 3: Let the disturbance attenuation level $\delta > 0$ be given. The system (10) is EM-S stable and also satisfies the H_∞ performance constraint (12) for all non-zero $\varpi(k)$ under the zero-initial condition if there exist positive-definite matrices $\hat{P}$, Q, matrices $\hat{G}_P$, $\hat{G}_I$ and positive scalars ς_0, ς_1, ς_2, κ_1, κ_2 satisfying the following inequalities:

$$\begin{cases} \Lambda = \begin{bmatrix}\Theta_{11} & * \\ \Lambda_{21} & \Lambda_{22}\end{bmatrix} < 0 & (45a) \\ P < \frac{\varsigma_0}{2} I & (45b)\end{cases}$$

where

$$\Lambda_{21} \triangleq \begin{bmatrix}\Lambda_{21}^1 \\ \Lambda_{21}^2\end{bmatrix}, \ \Lambda_{22} \triangleq \begin{bmatrix}\Lambda_{22}^1 & * \\ \Lambda_{22}^2 & \Lambda_{22}^3\end{bmatrix}, \ P \triangleq \text{diag}\{\hat{P}, \hat{P}, I\}$$

$$\Lambda_{21}^1 \triangleq \begin{bmatrix}0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \tilde{\mathcal{A}} & 0 & \tilde{\mathcal{F}} & \tilde{\mathcal{H}} & \tilde{\mathcal{M}} \\ \mathcal{D} & 0 & 0 & 0 & 0\end{bmatrix}, \ \Lambda_{21}^2 \triangleq \begin{bmatrix}0 & 0 & 0 & 0 & 0 \\ \mathcal{T}_P & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \mathcal{T}_I & 0 & 0 & 0 & 0\end{bmatrix}$$

$$\Lambda_{22}^1 \triangleq \text{diag}\{-P, -P, -P, -I\}$$

$$\Lambda_{22}^2 \triangleq \begin{bmatrix}\kappa_1\lambda_1\mathcal{S}_P^T & 0 & \kappa_1\bar{\varrho}\mathcal{S}_P^T & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \kappa_2\lambda_2\mathcal{S}_I^T & \kappa_2\bar{\sigma}\mathcal{S}_I^T & 0 \\ 0 & 0 & 0 & 0\end{bmatrix}$$

$$\Lambda_{22}^3 \triangleq \text{diag}\{-\kappa_1 I, -\kappa_1 I, -\kappa_2 I, -\kappa_2 I\}$$

$$\mathcal{S}_P \triangleq \begin{bmatrix}0_{m\times t} \\ S_P \\ 0_{p\times t}\end{bmatrix}, \quad \mathcal{T}_P \triangleq \begin{bmatrix}0_{t\times m} & -T_P C & 0_{t\times p}\end{bmatrix}$$

$$\mathcal{S}_I \triangleq \begin{bmatrix}0_{m\times t} \\ S_I \\ 0_{p\times t}\end{bmatrix}, \quad \mathcal{T}_I \triangleq \begin{bmatrix}0_{t\times m} & 0_{t\times m} & -T_I\end{bmatrix}$$

$$\tilde{\mathcal{A}} \triangleq \begin{bmatrix}\hat{P}A & 0_{m\times m} & 0_{m\times p} \\ 0_{m\times m} & \hat{P}A - \hat{G}_P C & -\hat{G}_I \\ 0_{p\times m} & C & I\end{bmatrix}, \ \tilde{\mathcal{M}} \triangleq \begin{bmatrix}\hat{P}M \\ \hat{P}M \\ 0_{p\times q}\end{bmatrix}$$

$$\tilde{\mathcal{F}} \triangleq \begin{bmatrix}\hat{P}F & 0_{m\times m} \\ 0_{m\times m} & \hat{P}F \\ 0_{p\times m} & 0_{p\times m}\end{bmatrix}, \quad \tilde{\mathcal{H}} \triangleq \begin{bmatrix}\hat{P}H & 0_{m\times m} \\ 0_{m\times m} & \hat{P}H \\ 0_{p\times m} & 0_{p\times m}\end{bmatrix}.$$

In addition, the desired PIO gains are calculated by

$$G_P = \hat{P}^{-1}\hat{G}_P, \quad G_I = \hat{P}^{-1}\hat{G}_I. \quad (46)$$

Proof: To begin with, in order to eliminate the parameter uncertainties occurred in the gain matrices, we rewrite (39a) in the form of (15). According to the notations in (10), we have

$$\Delta\mathcal{G}(k) = \bar{\varrho}\mathcal{S}_P P(k)\mathcal{T}_P + \bar{\sigma}\mathcal{S}_I I(k)\mathcal{T}_I \quad (47)$$

$$\Delta\mathcal{G}_P(k) = \mathcal{S}_P P(k)\mathcal{T}_P \quad (48)$$

$$\Delta\mathcal{G}_I(k) = \mathcal{S}_I I(k)\mathcal{T}_I. \quad (49)$$

Accordingly, we can rewrite (39a) as follows:

$$\begin{aligned}
\Theta =& \Xi_0 + \Xi_{1_S}P(k)\Xi_{1_T} + \Xi_{1_S}^T P(k)\Xi_{1_T}^T \\
&+ \Xi_{2_S}I(k)\Xi_{2_T} + \Xi_{2_S}^T I(k)\Xi_{2_T}^T < 0 \qquad (50)
\end{aligned}$$

where

$$\Xi_0 \triangleq \begin{bmatrix}\Theta_{11} & * \\ \Xi_0^{21} & \Theta_{22}\end{bmatrix}, \quad \Xi_0^{21} \triangleq \begin{bmatrix}0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \mathcal{A} & 0 & \mathcal{F} & \mathcal{H} & \mathcal{M} \\ \mathcal{D} & 0 & 0 & 0 & 0\end{bmatrix}$$

$$\Xi_{1_S} \triangleq \begin{bmatrix}0 & 0 & 0 & 0 & 0 & \lambda_1\mathcal{S}_P^T & 0 & \bar{\varrho}\mathcal{S}_P^T & 0\end{bmatrix}^T$$

$$\Xi_{1_T} \triangleq \begin{bmatrix}\mathcal{T}_P & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\end{bmatrix}$$

$$\Xi_{2_S} \triangleq \begin{bmatrix}0 & 0 & 0 & 0 & 0 & 0 & \lambda_2\mathcal{S}_I^T & \bar{\sigma}\mathcal{S}_I^T & 0\end{bmatrix}^T$$

$$\Xi_{2_T} \triangleq \begin{bmatrix}\mathcal{T}_I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\end{bmatrix}.$$

Applying Lemma 1 to (50), we know that (50) holds if and only if there exist positive scalars κ_1 and κ_2 such that the following inequality holds:

$$\begin{bmatrix} \Xi_0 & * & * & * & * \\ \kappa_1\Xi_{1_S}^T & -\kappa_1 I & * & * & * \\ \Xi_{1_T} & 0 & -\kappa_1 I & * & * \\ \kappa_2\Xi_{2_S}^T & 0 & 0 & -\kappa_2 I & * \\ \Xi_{2_T} & 0 & 0 & 0 & -\kappa_2 I \end{bmatrix} < 0. \tag{51}$$

Performing the congruence transformation to the inequality (51) by diag$\{I, I, I, I, I, P, P, P, I, I, I, I, I\}$ and utilizing the variable substitution

$$\hat{G}_P = \hat{P}G_P, \quad \hat{G}_I = \hat{P}G_I \tag{52}$$

we conclude that (51) holds if and only if (45a) holds.

To this end, it follows immediately from Theorem 2 that, with the non-fragile PIO gain matrices G_P and G_I given in (46), the augmented system (10) is EM-S stable and the H_∞ performance constraint (12) is also met for any non-zero $\varpi(k)$ under the zero-initial condition. The proof is now complete. ■

Remark 4: In this paper, we aim to develop a non-fragile H_∞ PIO design scheme for the DRNN with TVDs. In Theorems 1-2, sufficient conditions are established such that the estimation error dynamics is EM-S stable and the H_∞ performance constraint is also satisfied. In Theorem 3, the gains of PIO are characterized in terms of the solutions to LMIs. It should be noted that the main results established in Theorems 1-3 can be extended to more general systems with network-induced phenomena such as packet dropouts, quantizations, disorders or saturations [25], [30], [34], [37].

Remark 5: For now, the non-fragile PIO design problem has been solved for the DRNN subject to ROGVs and TVDs. Comparing with the existing results, the distinctive characteristics of the main results in this paper are highlighted as follows: 1) the design problem of non-fragile PIO is, for the first time, investigated for DRNN with TVDs; and 2) a unified framework is established to account for the joint effects from the ROGVs and the TVDs on estimation performance.

IV. Numerical Simulation

In this section, we shall provide a simulation example in order to demonstrate the validity of the obtained theoretical results on the proposed PIO design problem for a class of DRNNs.

Consider a DRNN described by (1) with corresponding parameters as follows:

$$A = \begin{bmatrix} 0.681 & 0 & 0 \\ 0 & 0.637 & 0 \\ 0 & 0 & 0.806 \end{bmatrix}, \quad M = \begin{bmatrix} 1 \\ 1.5 \\ 2 \end{bmatrix}$$

$$F = \begin{bmatrix} 0.5 & 0.1 & 0.4 \\ 0.2 & 0.3 & 0.1 \\ 0.2 & 0.3 & 0.4 \end{bmatrix}, \quad C = \begin{bmatrix} 1.3 & 04 & 1.9 \\ 0.4 & 0.1 & 1.2 \end{bmatrix}$$

$$H = \begin{bmatrix} 0.3 & 0.1 & 0.2 \\ 0.2 & 0.3 & 0.4 \\ 0.1 & 0.2 & 0.3 \end{bmatrix}, \quad D = \begin{bmatrix} 0.3 & 0.4 & 0.1 \\ 0.2 & 0.1 & 0.3 \end{bmatrix}.$$

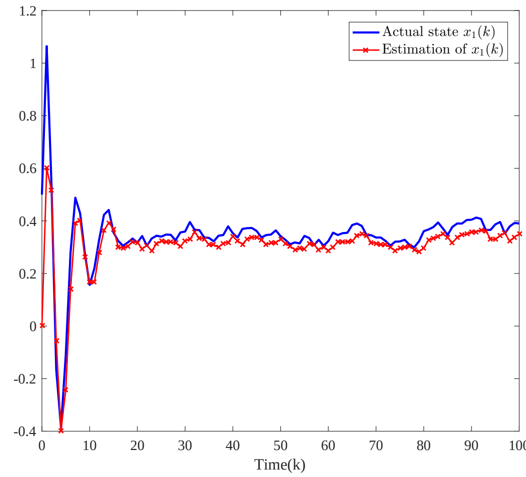

Fig. 1: Trajectories of state $x_1(k)$ and its estimate.

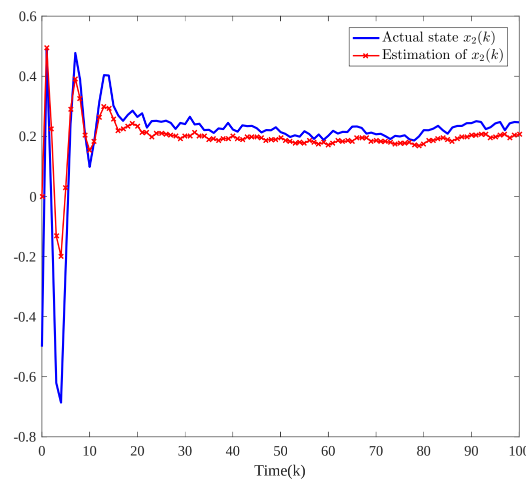

Fig. 2: Trajectories of state $x_2(k)$ and its estimate.

The activation functions are taken as

$$\ell(s) = \begin{bmatrix} \tanh(0.6s) - 0.2\sin s \\ \tanh(-0.4s) \\ \tanh(-0.2s) \end{bmatrix}$$

$$\jmath(s) = \begin{bmatrix} \tanh(-0.4s) - 0.2\cos s \\ \tanh(0.2s) \\ \tanh(0.4s) \end{bmatrix}, \quad \forall s \in \mathbb{R}$$

which meet (2) with

$$l_1^- = -0.2, \quad l_1^+ = 0.8$$
$$l_2^- = -0.4, \quad l_2^+ = 0$$
$$l_3^- = -0.2, \quad l_3^+ = 0$$
$$\jmath_1^- = -0.2, \quad \jmath_1^+ = 0.6$$
$$\jmath_2^- = 0, \quad \jmath_2^+ = 0.2$$

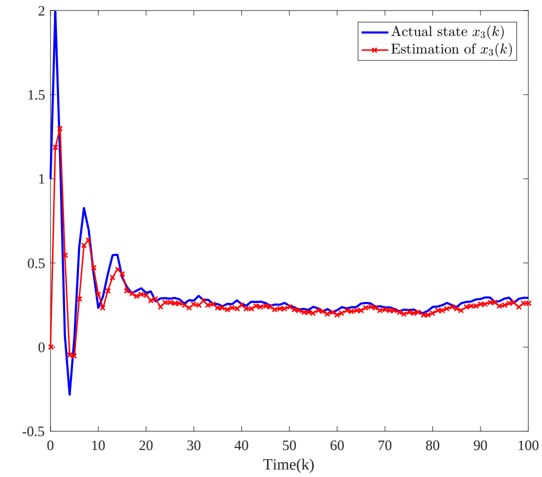

Fig. 3: Trajectories of state $x_3(k)$ and its estimate.

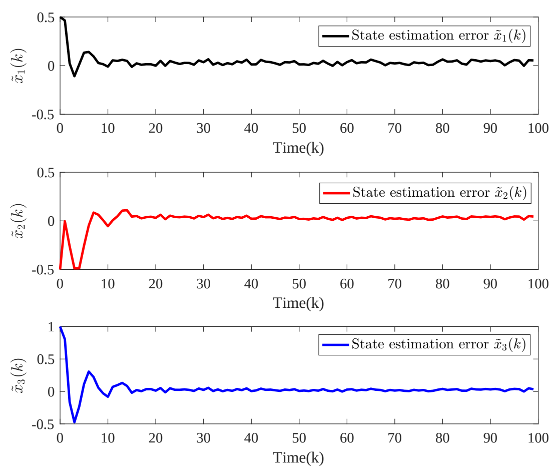

Fig. 4: Trajectories of estimation error $\tilde{x}(k)$ with PIO.

$$j_3^- = 0, \quad j_3^+ = 0.4.$$

The corresponding parameters of gain perturbation matrices in non-fragile PIO are given as follows:

$$\begin{aligned} S_P &= \begin{bmatrix} 0.6 & 0.5 & 0.4 \end{bmatrix}^T, \quad T_P = \begin{bmatrix} 0.1 & 0.2 \end{bmatrix} \\ S_I &= \begin{bmatrix} 0.5 & 0.4 & 0.3 \end{bmatrix}^T, \quad T_I = \begin{bmatrix} 0.2 & 0.3 \end{bmatrix} \\ P(k) &= 0.8\sin(k), \quad I(k) = 0.6\cos(k). \end{aligned}$$

In this example, the probabilities of the ROGVs are assumed to be $\bar{\varrho} = 0.65$ and $\bar{\sigma} = 0.7$. The external disturbance is taken as $\varpi(k) = 0.8\mathrm{e}^{-0.2k}\cos(k)$ and the disturbance attenuation level is given as $\delta = 0.28$. The TVD is chosen as $\hbar(k) = 3 - (\sin(k\pi))^2$, from which it is easy to verify that the upper bound and the lower bound of the TVD are $\bar{\hbar} = 3$ and $\underline{\hbar} = 1$, respectively. Moreover, the initial values of the states are set as $x_1(-3) = x_1(-2) = x_1(-1) = x_1(0) = 0.5$, $x_2(-3) = x_2(-2) = x_2(-1) = x_2(0) = -0.5$ and $x_3(-3) = x_3(-2) = x_3(-1) = x_3(0) = 1$.

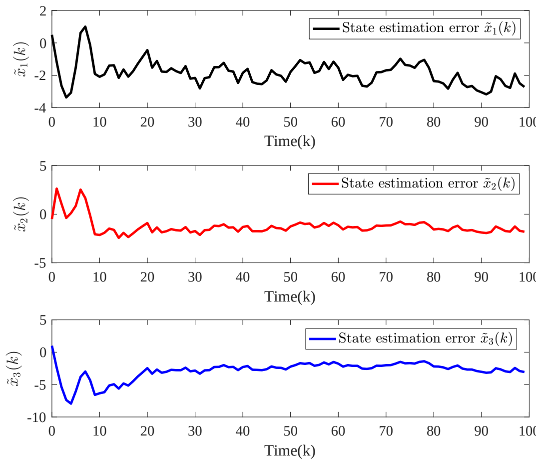

Fig. 5: Trajectories of estimation error $\tilde{x}(k)$ with Luenberger observer.

By means of the MATLAB software (with the YALMIP 3.0), the solutions to LMIs (45a)-(45b) and the desired PIO gains can be obtained immediately as follows (only the main parameters are listed):

$$\begin{aligned} \hat{P} &= \begin{bmatrix} 4.1777 & -1.2340 & -1.5194 \\ -1.2340 & 3.6153 & -2.1519 \\ -1.5194 & -2.1519 & 2.6332 \end{bmatrix} \\ G_P &= \begin{bmatrix} 1.8502 & -1.3418 \\ 0.3349 & -0.5081 \\ -1.3323 & 1.1808 \end{bmatrix}, \quad \kappa_1 = 0.2708 \\ G_I &= \begin{bmatrix} -1.6524 & 6.9657 \\ -1.0431 & 9.8022 \\ 1.2087 & 1.7936 \end{bmatrix}, \quad \kappa_2 = 0.3582 \\ \varsigma_0 &= 6.4878, \qquad \varsigma_1 = 7.2499 \\ \varsigma_2 &= 6.6950, \qquad \varsigma_3 = 6.7031. \end{aligned}$$

In order to manifest the validity of the proposed PIO design scheme, the simulation results are shown in Figs. 1-5. Figs. 1-3 depict the neuron states and their estimates and Fig. 4 plots the trajectories of estimation error $\tilde{x}(k)$, from which we can confirm that the desired PIO performs quite well as expected. Fig. 5 depicts the trajectories of estimation error $\tilde{x}(k)$ with Luenberger observer. It can be concluded from the above simulation results that the PIO proposed in this paper outperforms Luenberger observer adopted in most literature for estimation performance.

V. Conclusion

In this paper, we have dealt with the non-fragile H_∞ PIO design problem for a kind of DRNNs subject to ROGVs and TVDs. The phenomena of ROGVs concerned in this paper have been governed by Bernoulli distributed random variables with certain probabilities for the sake of accommodating the practical situations. A novel non-fragile PIO has been designed which can exhibit a satisfactory estimation performance in the simultaneous presence of ROGVs and TVDs. Furthermore, the EM-S stability and H_∞ performance of the estimation error dynamics have been analyzed respectively by means of Lyapunov theory, and the desired PIO gains have been obtained by resorting to LMI technique. A simulation example has been finally presented to reveal the effectiveness and superiority of the proposed design method. It is worth noting that the current method and results could be extended to the RNNs with constant time-delays, discrete time-delays, distributed time-delays or mixed time-delays [2], [46], [47].

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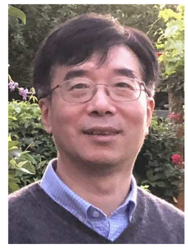

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