Multi-Channel Modulo Samplers Constructed from Gaussian Integers

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Abstract—Recently, there is an increased interest in the study of modulo analog to digital converters (ADCs). These new systems can reconstruct a signal whose amplitude is much higher than the conventional ADC's dynamic range. Modulo ADCs are characterized by their modulo threshold and in the current literature, all existing works are limited to real-valued moduli. In this paper, we propose multi-channel modulo samplers with complex-valued moduli to sample a band-limited complex signal. Specifically, we discuss the construction of complex divisors from Gaussian integers and propose their efficient implementations. A memory-efficient, closed-form recovery algorithm is also proposed. Simulation results demonstrate that the proposed systems can provide stable reconstruction of a high dynamic range complex-valued signal at low sampling rates.

Index Terms—Analog-to-digital conversion (ADC), Complex signal representations, Modulo samplers, Chinese reminder theorem (CRT), Gaussian integers

I. INTRODUCTION

Complex signal representations have found many applications in radar, control and communications. In this paper, we consider the sampling of a continuous-time, finite-energy, band-limited complex-valued signal $g(t) = g_I(t) + ig_Q(t)$ with a maximum angular frequency of Ω, where $g_I(t)$ and $g_Q(t)$ represent the real part (I-channel) and the imaginary part (Q-channel) of $g(t)$, respectively. We are interested in the scenario when the amplitudes of $g_I(t)$ or $g_Q(t)$ are much higher than the dynamic range of the analog-to-digital converter (ADC). Recently, unlimited sampling framework with modulo sampling hardware was developed in [1]–[4]. Specifically, for a given $x \in \mathbb{R}$ and $\Delta > 0$, the modulo operation is defined as [1], [5]

$$\langle x \rangle_\Delta = x \bmod \Delta \triangleq x - \Delta \left\lceil \frac{x}{\Delta} \right\rfloor, \qquad (1)$$

where $\lceil t \rfloor \triangleq \operatorname{argmin}_{b \in \mathbb{Z}} |t - b|$ is the "rounding" operation, which returns the closest integer to t [1], [5].

According to *Unlimited Sampling Theorem* in [1], [3], $g(t)$ can be recovered from its modulo samples through higher order differences (HOD) reconstruction if the sampling period T_{us} satisfies $T_{us} \leq 1/(2e\Omega)$. That is, for HOD, the required sampling rate F_s is at least $2e\pi f_{NYQ} \approx 17.1 f_{NYQ}$, in which f_{NYQ} is the Nyquist rate. For some wide-band applications, it

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TABLE I: Comparison of different modulo samplers for noise-free reconstruction

Methods	L	Minimum F_s	g_{max}	Same $|\Delta_l|$
[1], [3]	1	$2\pi e f_{NYQ}$	$< \infty$	Yes
[5]	1	$> f_{NYQ}$ [1]	$< \infty$	Yes
[13]	2	$2 f_{NYQ}$	$< \tau_0 \Delta / 2$	No
This work	2	$2 f_{NYQ}$	$< |\tau_0| \Delta / 2$	Yes

[1] Restriction: There exists a negative integer N so that $|g(n/Fs)| < \Delta/2$ when $n < N$ [5].

may be difficult to achieve such a high sampling rate. In [5], it was shown that if $F_s > f_{NYQ}$, reconstruction is possible by using a prediction filter constructed from Chebyshev polynomials (ChP). However, this method requires the knowledge of unfolded samples, i.e., there exists a negative integer N so that $|g(n/Fs)| < \Delta/2$ when $n < N$. For real-world applications, this condition is often impractical. Additionally, this method is sensitive to noises. For other works on single-channel modulo ADCs, we refer the readers to [2], [6]–[12] and the references therein.

In [13], multi-channel modulo samplers were developed based on Chinese remainder theorem (CRT). In an L-channel modulo ADC, each channel samples at f_{NYQ} with *different* dynamic ranges $\Delta_0 < \Delta_1 < \cdots < \Delta_{L-1}$

$$\Delta_l = \epsilon \tau_l \quad 0 \leq l \leq L-1, \qquad (2)$$

where ϵ is a positive real number and τ_l $(0 \leq l \leq L-1)$ are L jointly prime positive integers. In the noiseless case, only 2-channels are required with a total sampling rate of $F_s = 2 f_{NYQ}$. Besides, each signal sample can be recovered independently [13]. However, once Δ_l are fixed, the maximum signal amplitude g_{max} for reconstruction is bounded by $P\epsilon/2$, in which P is the least common multiple (lcm) of τ_l. In addition, as Δ_l are different, the calibrations of ADC dynamic ranges may be complicated.

In this paper, we propose multi-channel modulo samplers with *complex-valued* moduli Δ_l. Specifically, the positive integers τ_l in (2) are generalized to *Gaussian integers* of the form $\tau_l = p_l + iq_l$, in which p_l and q_l are non-negative integers. Our main contributions include: *i)* **Selection of moduli**: We propose to use pairwise co-prime Gaussian integers τ_l in (2), which allows for recovery of a signal with larger dynamic range than that of real-valued moduli (Section II). We prove that perfect recovery can be achieved at a total sampling rate of $2 f_{NYQ}$ (Proposition 1). *ii)* **Implementation**: We develop implementation diagram of complex modulo operations by applying analog phase-shifts before modulo ADCs (Section III-A). In the special case when $L = 2$, all modulo

$$-\frac{1}{2}+\frac{7}{2}i$$

$$\Delta = 3 + 4i$$

$$\frac{7}{2}+\frac{i}{2}$$

Im

Re

$$-\frac{7}{2}-\frac{i}{2}$$

$$\theta$$

$$\frac{1}{2}-\frac{7}{2}i$$

$$\theta = \text{atan}\left(\frac{4}{3}\right)$$
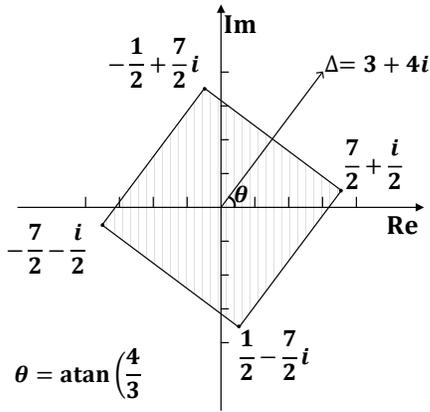
Fig. 1: Region of complex remainder sets $\mathbb{S}(3 + 4i)$.

ADCs have the *same* dynamic range in our proposed systems, which is impossible for multi-channel systems developed in [13]. *iii*) **Reconstruction**: For two co-prime Gaussian integers $x_0 = p + iq$ and $x_1 = p - iq$, we prove that $x_0^{-1} \bmod x_1$ is a rational integer (Proposition 2). Based on robust CRT of [14], a 2-stage memory-efficient reconstruction algorithm can be obtained. Simulation results in Section IV demonstrate the effectiveness of the proposed systems. As a quick summary, Table I lists the required total sampling rate F_s and maximum achievable signal amplitude g_{max} for different modulo samplers in noiseless measurements.

Notations: Throughout this paper, $i = \sqrt{-1}$ and $e = \exp(1)$. The set of rational integers is denoted as $\mathbb{Z}$. For a real number x, $\lceil x \rceil$ and $\lfloor x \rceil$ correspond to the ceiling and rounding operations, respectively. For a complex number x, $\text{Re}(x)$ and $\text{Im}(x)$ are its real and imaginary parts, respectively. The rounding operation is given by $\lceil x \rceil = \lceil \text{Re}(x) \rceil + i \lceil \text{Im}(x) \rceil$. For a non-zero complex modulus Δ, $\langle x \rangle_\Delta$ represents the modulo operation in (1). The fundamental domain of Δ is denoted as $\mathbb{S}(\Delta)$, which represents its remainder set [15]:

$$\mathbb{S}(\Delta) = \left\{ \Delta(\alpha + i\beta), \ -\frac{1}{2} < \alpha, \beta \le \frac{1}{2} \right\}. \quad (3)$$

Geometrically, $\mathbb{S}(\Delta)$ is a rotated square with 4 vertices at $\frac{\Delta}{2}(1 + i)$, $\frac{\Delta}{2}(1 - i)$, $\frac{\Delta}{2}(-1 - i)$ and $\frac{\Delta}{2}(-1 + i)$. As an example, Fig. 1 shows $\mathbb{S}(3 + 4i)$. For a complex signal $g(t) = g_I(t) + ig_Q(t)$, g_{max} is given by

$$g_{max} = \max\left(\|g_I(t)\|_\infty, \|g_Q(t)\|_\infty \right), \quad (4)$$

where the subscript ∞ denotes the infinity norm.

II. SELECTION OF COMPLEX MODULI

In this section, we first introduce basics of Gaussian integers and then present muli-channel modulo samplers constructed from complex moduli. A Gaussian integer x can be written as $x = p + iq$ with $p, q \in \mathbb{Z}$. Its norm $\mathbb{N}(x)$ is given by [16]

$$\mathbb{N}(x) = x\overline{x} = |x|^2 = p^2 + q^2, \quad (5)$$

where $\overline{x}$ is the complex conjugate of x. For two Gaussian integers x_0 and x_1, they are relatively prime (or co-prime) if and only if there exist two Gaussian integers α and β so that the Bézout's identity holds, i.e., $x_0\alpha + x_1\beta = 1$. Alternatively,

TABLE II: Comparisons of g_{max} for real- and complex-valued moduli for $|\Delta_l| \le 5$ and $\epsilon = 1$

L	Moduli	Δ_l	g_{max}
2	Real [13]	$5, 4$	< 10
	Complex	$3 + 4i, 3 - 4i$	< 12.5
3	Real [13]	$5, 4, 3$	< 30
	Complex	$3 + 4i, 3 - 4i, 4$	< 50
4	Real [13]	$5, 4, 3, 1$	< 30
	Complex	$3 + 4i, 3 - 4i, 1 + 4i, 1 - 4i$	< 212.5

$x_0 = p_0 + iq_0$ and $x_1 = p_1 + iq_1$ are co-prime if and only if their greatest common divisor (gcd) satisfy [17]

$$\gcd\left(\mathbb{N}(x_0), \mathbb{N}(x_1), p_0p_1 + q_0q_1\right) = 1. \quad (6)$$

In the special case when $x_0 = p + iq$ and $x_1 = p - iq$, (6) can be simplified as [18]

$$\gcd(p, q) = 1 \quad \text{and} \quad |p - q| \bmod 2 = 1. \quad (7)$$

That is, p, q are co-prime with different parities, i.e., one odd and one even.

Next, we propose a multi-channel modulo sampler constructed from L complex divisors Δ_l ($0 \le l \le L - 1$), where τ_l in (2) are pairwise co-prime Gaussian integers. Besides, in the first $2L_1$ divisors, τ_{2l} and τ_{2l+1} ($0 \le l \le L_1 - 1$) are complex conjugate pairs:

$$\tau_{2l} = p_l + iq_l \quad \text{and} \quad \tau_{2l+1} = p_l - iq_l, \quad (8)$$

where p_l and q_l are positive integers. For the remaining $L - 2L_1$ divisors, $\tau_{l'}$ ($l' = 2L_1, \cdots, L - 1$) are positive integers. According to the generalized CRT [14], a complex number x can be perfectly recovered from L remainders $\langle x \rangle_{\Delta_l}$ ($l = 0, \cdots, L - 1$) if $x \in \mathbb{S}(P\epsilon)$, or equivalently,

$$|\text{Re}(x)| < P\epsilon/2 \quad \text{and} \quad |\text{Im}(x)| < P\epsilon/2, \quad (9)$$

where $P = \prod_{l=0}^{L_1-1} \left(p_l^2 + q_l^2\right) \cdot \prod_{l'=2L_1}^{L-1} \tau_{l'}$.

Hereafter, we will assume that $|\Delta_l| \le \Delta_{max}$ ($l = 0, 1, \cdots, L - 1$), i.e., their magnitudes are bounded by Δ_{max}. The following example shows that our proposed system can offer a larger signal dynamic range than that of the multi-channel system with only real-valued moduli [13].

Example 1: In this example, we compare the upper bound of g_{max} for complex and real moduli [13]. Suppose that $\epsilon = 1$ and we restrict $|\Delta_l| \le 5$. For $L = 2, 3$ and 4, Table II lists the corresponding Δ_l ($l = 0, \cdots, L - 1$) that can achieve maximum g_{max}. This table implies that by extending τ_l to Gaussian integers, one can sample a signal with a much higher dynamic range.

The next proposition proves that with a careful selection of τ_l and ϵ, a band-limited signal $g(t)$ with known g_{max} can be recovered at $F_s = 2f_{NYQ}$ and $|\Delta_0| = |\Delta_1| = \Delta_{max}$.

Proposition 1: Suppose that $g(t)$ is a finite-energy, band-limited complex-valued signal with g_{max} defined as in (4). Assume that the maximum dynamic range of modulo ADC is Δ_{max}. Consider a 2-channel modulo sampler with $\Delta_l = \tau_l\epsilon$ ($l = 0, 1$) and $\tau_0 = p + iq$ and $\tau_1 = p - iq$, in which

$$p = \left\lceil \frac{2g_{max}}{\Delta} \right\rceil, \quad q = \begin{cases} 1; & p \text{ even} \\ 2; & p \text{ odd} \end{cases}, \quad \epsilon = \frac{\Delta_{max}}{\sqrt{p^2 + q^2}}. \quad (10)$$
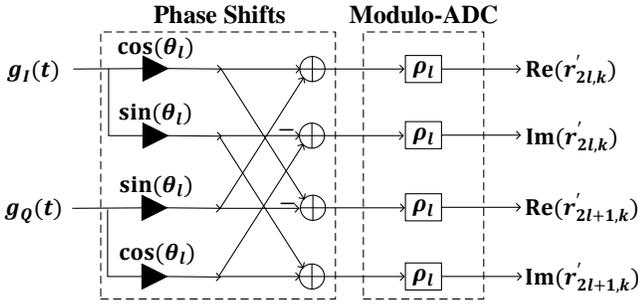
Fig. 2: Block diagram to generate $r'_{2l,k} = \left\langle g_k e^{-j\theta_l} \right\rangle_{\rho_l} + \delta'_{2l,k}$ and $r'_{2l+1,k} = \left\langle g_k e^{j\theta_l} \right\rangle_{\rho_l} + \delta'_{2l+1,k}$.

Then, $g(t)$ can be completely recovered from noise-free modulo samples $\langle g_k \rangle_{\Delta_l}$ ($l = 0, 1$) with $g_k = g(k/f_{NYQ})$ ($k \in \mathbb{Z}$).

Proof: From (10), it is clear that $\gcd(p, q) = 1$ and p, q have different parities. Thus, τ_0 and τ_1 are co-prime. Besides, it is straightforward $|\Delta_0| = |\Delta_1| = \Delta_{max}$. By (9)-(10), the maximum achievable range is $|\tau_0|\Delta_{max}/2 = \sqrt{p^2 + q^2}\Delta_{max}/2 > p\Delta_{max}/2 \geq g_{max}$. Thus, if each channel is sampled at the Nyquist rate f_{NYQ}, $g_k = g(k/f_{NYQ})$ can be reconstructed perfectly from $\langle g_k \rangle_{\Delta_l}$ ($l = 0, 1$). That is, the total sampling rate is $F_s = 2f_{NYQ}$. ∎

Remarks: *i)* The selection of p and q in (10) is not unique. *ii)* Recall that the minimum F_s is also $2f_{NYQ}$ for multi-channel system with real-valued moduli [13]. However, the modulo ADCs in [13] need to have different dynamic ranges, which makes it difficult for ADC calibration.

III. IMPLEMENTATION AND RECONSTRUCTION

A. Implementation

The previous subsection shows that with complex divisors, one can reconstruct a signal with higher dynamic range. Note that the dynamic range of a modulo ADC is a positive real number. One may wonder how to implement modulo operations for complex divisors? To address this issue, we first express Δ_{2l} and Δ_{2l+1} ($0 \leq L_1 - 1$) in polar forms

$$\Delta_{2l} = \rho_l e^{j\theta_l}, \quad \Delta_{2l+1} = \rho_l e^{-j\theta_l}, \quad (11)$$

in which $\rho_l = \epsilon\sqrt{p_l^2 + q_l^2}$ and $\theta_l = \arctan(q_l/p_l)$. Denote $g_k = g(k/f_{NYQ})$ and define $y'_{2l,k}$ and $y'_{2l,k}$ as follows

$$y'_{2l,k} = \left\langle g_k e^{-j\theta_l} \right\rangle_{\rho_l}, \quad y'_{2l+1,k} = \left\langle g_k e^{j\theta_l} \right\rangle_{\rho_l}. \quad (12)$$

From (11) and (12), one can easily prove

$$y'_{2l,k} e^{j\theta_l} = \langle g_k \rangle_{\Delta_{2l}}, \quad y'_{2l+1,k} e^{-j\theta_l} = \langle g_k \rangle_{\Delta_{2l+1}}. \quad (13)$$

Note that $y'_{2l,k}$ and $y'_{2l+1,k}$ in (12) can be acquired by
1) Applying phase shifts to $g(t)$ to produce $g_{2l}(t) = g(t)e^{-j\theta_l}$ and $g_{2l+1}(t) = g(t)e^{j\theta_l}$;
2) Sampling $g_{2l}(t)$ and $g_{2l+1}(t)$ using modulo ADCs with dynamic range ρ_l.

Phase shifting of $g(t)$ by $-\theta_l$ and θ_l are equivalent to multiplying $\begin{bmatrix} g_I(t) & g_Q(t) \end{bmatrix}^T$ with $\mathbf{R}_l$ and $\hat{\mathbf{R}}_l$, respectively

$$\mathbf{R}_l = \begin{bmatrix} \cos(\theta_l) & \sin(\theta_l) \\ -\sin(\theta_l) & \cos(\theta_l) \end{bmatrix}, \quad \hat{\mathbf{R}}_l = \begin{bmatrix} \cos(\theta_l) & -\sin(\theta_l) \\ \sin(\theta_l) & \cos(\theta_l) \end{bmatrix}.$$

The corresponding implementation diagram is shown in Fig. 2. Here, the outputs $r'_{2l,k}$ and $r'_{2l+1,k}$ are given by

$$r'_{2l,k} = y'_{2l,k} + \delta'_{2l,k}, \quad r'_{2l+1,k} = y'_{2l+1,k} + \delta'_{2l+1,k} \quad (14)$$

in which $\delta'_{2l,k}$ and $\delta'_{2l+1,k}$ are errors introduced by modulo ADCs. Eq. (13) suggests we can use $r'_{2l,k}e^{j\theta}$ and $r'_{2l,k}e^{-j\theta}$ to approximate modulo samples $\langle g_k \rangle_{\Delta_{2l}}$ and $\langle g_k \rangle_{\Delta_{2l+1}}$ for reconstruction.

Note that in the special case with $L = 2$, Fig. 2 indicates that 4 modulo ADCs with the *same* dynamic range ρ_0 are required. If τ_0 and τ_1 are positive integers as proposed in [13], 4 modulo ADCs with 2 *different* dynamic ranges are needed. Thus, the proposed systems offer simpler ADC calibrations. The extra cost here is the analog phase rotations.

B. Reconstruction

We now consider the reconstruction of g_k from noisy modulo samples. Since CRT is applicable to Gaussian integers, one can use any existing recovery algorithms by replacing positive integers with Gaussian ones [14], [19]–[28]. We consider robust CRT (RCRT) [14] as it has a closed-form solution with fast implementation. Just as classical CRT, RCRT requires pre-calculated modular multiplicative inverses. Simply speaking, for two co-prime Gaussian integers a and b, a Gaussian integer $c \equiv a^{-1} \mod b$ is a's modular multiplicative inverse modulo b if $ca \mod b = 1$. For general Gaussian integers a and b, c is complex-valued. But for co-prime pairs $x_0 = p + iq$ and $x_1 = p - iq$, Proposition 2 shows that $x_0^{-1} \mod x_1$ is an integer in $\mathbb{Z}$.

Proposition 2: Suppose that $x_0 = p + iq$ and $x_1 = p - iq$ are co-prime with $p, q \in \mathbb{Z}^+$. Then, the modular multiplicative inverse $c \equiv x_0^{-1} \mod x_1$ is a rational integer.

Proof: By (7), the co-prime condition of x_0 and x_1 implies $\gcd(p, q) = 1$ and p, q have different parities. Without loss of generality, we assume that p is odd and q is even. Under such an assumption, it is clear that $p^2 + q^2$ is an odd number, i.e., $\gcd(p^2 + q^2, 2) = 1$. Besides, as $\gcd(p, q) = 1$, $\gcd(p^2 + q^2, p) = 1$. Therefore, $\gcd(p^2 + q^2, 2p) = 1$. By Bézout's identity, there exist two integers $c, d \in \mathbb{Z}$ so that

$$c \cdot 2p + d \cdot (p^2 + q^2) = 1 \quad (15)$$

As $2p = x_0 + x_1$ and $p^2 + q^2 = x_0 x_1$, we have

$$cx_0 + cx_1 + dx_0 x_1 = 1,$$

which implies that $cx_0 \mod x_1 = 1$, i.e., c is the multiplicative inverse of x_0 modulo x_1. When q is odd and p is even, the proof can be done in a similar way. ∎

Based on the above proposition, we propose a 2-stage reconstruction algorithm as in [21]. In the first stage, RCRT [14] is applied to erroneous remainders of Δ_{2l} and Δ_{2l+1} for $l = 0, 1, \cdots, L_1 - 1$ to solve $\hat{x}_{l,k}$

$$\langle \hat{x}_{l,k} \rangle_{\Delta_{2l}} \approx r'_{2l,k} e^{j\theta_l}, \quad (16)$$

$$\langle \hat{x}_{l,k} \rangle_{\Delta_{2l+1}} \approx r'_{2l+1,k} e^{-j\theta_l}. \quad (17)$$

By Proposition 2, the modular multiplicative inverses $\tau_{2l}^{-1} \mod \tau_{2l+1}$ are integers in $\mathbb{Z}$. In the second stage, we aim

to get reconstructed signal sample $\hat{g}_k$ through RCRT [14]

$$\langle \hat{g}_k \rangle_{N_l\epsilon} \approx \hat{x}_{l,k}, \quad l = 0, \cdots, L_1 - 1 \quad (18)$$

$$\langle \hat{g}_k \rangle_{\Delta'_l} \approx r_{l',k}, \quad l' = 2L_1, \cdots, L - 1 \quad (19)$$

in which $N_l = p_l^2 + q_l^2$ and $r_{l'k} = \langle g_k \rangle_{\Delta_{l'}} + \delta_{l',k}$ with $\delta_{l',k}$ being the ADC error ($l' = 2L_1, \cdots L - 1$). As N_l and τ'_l are positive integers, all the modular multiplicative inverses are positive integers. Thus, the proposed algorithm is memory efficient as we only need to store rational integers. In addition, following a similar analysis as in [14], [21], one can show that if the errors from modulo samplers satisfy

$$|\delta'_{l,k}| < \epsilon/4, \quad |\delta_{l',k}| < \epsilon/4, \quad (20)$$

stable reconstruction can be achieved with $|\hat{g}_k - g_k| < \epsilon/4$.

Remark: For $L = 2$, Eq. (10) implies that ϵ becomes small when g_{max} gets large. To improve the system's robustness to ADC noises, over-sampling is required by increasing ϵ and L under the constraint $|\Delta_l| \le \Delta_{max}$.

IV. SIMULATIONS

We report simulation results for different modulo ADCs to sample complex-valued, band-limited signals $g(t)$ with maximum angular frequency $\Omega = \pi$ rad/s ($f_{NYQ} = 1$ Hz). We restrict $|\Delta_l| \le 5$ in all simulations. For multi-channel systems, $\epsilon = 1$ and each channel is sampled at 1 Hz. We run simulations with $L = 2$ ($F_s = 2$ Hz) and $L = 3$ ($F_s = 3$ Hz) for both real-valued [13] and complex-valued moduli. The corresponding values of Δ_l and achievable g_{max} can be found in Table II. For single-channel systems, we fix $\Delta = 5$ and consider i) HOD reconstruction [1], [3] at $F_s = 18$ Hz and ii) ChP-based reconstruction [5] for $F_s = 2$ Hz and $F_s = 3$ Hz, respectively. Each test signal $g(t)$ can be expressed as

$$g(t) = \sum_{k=-10}^{10} A \cdot (a_k + ib_k)\text{sinc}(t - k), \quad -40 \le t < 40 \quad (21)$$

where $\text{sinc}(t) = \frac{\sin(\pi t)}{\pi t}$ is an ideal low-pass filter in $[-\pi, \pi]$. Coefficients a_k and b_k are uniformly distributed in $[-1, 1]$ and A is a constant to obtain different g_{max}. The input signal to noise ratio (SNR) follows the same definition as that in [7] and varies from 10 dB to 40 dB with a step size of 2 dB. For each input SNR, we produce $Q = 10^4$ realizations of $g(t)$. The root relative square error (RRSE) is used to quantify reconstruction error:

$$\text{RRSE} = \sqrt{\frac{\sum_n |g(n/f_s) - \hat{g}(n/f_s)|^2}{\sum_n |g(n/f_s)|^2}}, \quad (22)$$

where $f_s = F_s/L$ is the sampling frequency for each channel.

Fig. 3a and Fig. 3b show the results of average RRSE for $g_{max} = 9.5$ and $g_{max} = 48$, respectively. In these figures, κ denotes the order of finite difference in HOD-based reconstruction [1], [3]. For ChP-based method [5], the filter order is $2K$, where K represents the order of Chebychev polynomial calculated from Eq. (9) in [5]. One can observe that for $g_{max} = 9.5$, HOD [1], [3] offers the best performance at $F_s = 18$ Hz for input SNR from 12 to 20 dB. In fact, HOD reconstruction can work at sampling rates lower than
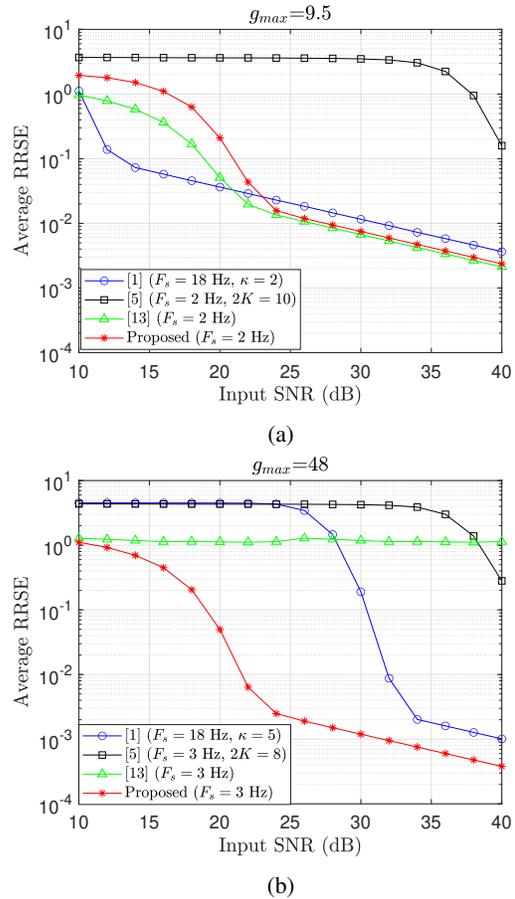
Fig. 3: Reconstruction results for different modulo samplers (a) $g_{max} = 9.5$ (b) $g_{max} = 48$.

$2e\pi f_{NYQ}$ for high SNR [3]. However, HOD's performance degrades for $g_{max} = 48$ and low SNR. For ChP-based reconstruction [5], it produces large reconstruction error at $F_s = 2$ Hz or $F_s = 3$ Hz even at high SNR. For multi-channel systems in [13], they work well for $g_{max} = 9.5$, but fail at $g_{max} = 48$ as the signal's amplitude exceeds the maximum achievable dynamic range. Our proposed systems offer the best performance at $g_{max} = 48$ and they are slightly worse than the system in [13] for $g_{max} = 9.5$. Thus, they hold great potential to sample a high dynamic range complex-valued signal at low sampling rates.

V. CONCLUSIONS

This paper proposes multi-channel modulo sampling systems based on Gaussian integers. In particular, we discuss the selection of complex divisors and propose implementations through modulo ADC systems. Besides, a memory efficient, closed-form reconstruction algorithm is presented for robust recovery. Compared with multi-channel modulo samplers with only real-valued divisors [13], the proposed systems can recover a signal with higher dynamic range. Compared with single-channel modulo samplers [3], [5], our system can offer stable reconstruction with a provable error bound at much lower sampling rates.

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