A Finite Element Based Formulation for Sensitivity Studies of Piezoelectric Systems

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Abstract. Sensitivity Analysis is a branch of numerical analysis which aims to quantify the affects that variability in the parameters of a numerical model have on the model output. A finite element based sensitivity analysis formulation for piezoelectric media is developed here and implemented to simulate the operational and sensitivity characteristics of a piezoelectric based distributed mode actuator (DMA). The work acts as a starting point for robustness analysis in the DMA technology.

1. Introduction

Finite element methods and formulations for dynamic modelling of piezoelectric media have been the focus of many studies [1] [2] since the original work of Allik and Hughes [3].

Local sensitivity analysis is a branch of numerical modeling which aims to quantify the affects of variation in individual parameters on the solution variables of differential equation models by means of partial derivatives [4] [5] and has found widespread scientific applications [6] [7]. Finite element methods for performing sensitivity analysis methods are well established [8] [9] [10] [11] [12]. A common approach is to conduct sensitivity analysis on the discretised finite element model by differentiating the standard stiffness matrices with respect to a parameter. The adjoint sensitivity method is beneficial in cases where the number of parameters greatly exceeds the number of outputs. For example Kapadia et al [13] apply the adjoint method to a fuel cell system with 180,000 design variables. This is several orders of magnitude greater than the number of piezoelectric parameters considered in this paper and therefore the computational advantage of the adjoint method has to be weighed against it complexity of its implementation.

In the work presented here a finite element local sensitivity analysis formulation is applied to the governing equations of piezoelectric media and implemented to simulate the relative importance of design parameter variability in a piezoelectric based cantilever beam actuator application. The analysis is performed by directly differentiating the semi-discretised (time continuous-spatially discrete) governing equations of motion and integrating the resulting equations using a time stepping algorithm, following the methodology outlined in [8].

2. Governing Equations of Piezoelectric Theory

The dynamic electro-elastic response of a piezoelectric body of volume Ω and regular boundary surface S is governed by a coupled system of electrostatic and mechanical equilibrium boundary value partial differential equations.

The electrostatic boundary value system is defined by

$$\frac{\partial D_i}{\partial x_i} = q_v \quad \text{in} \quad \Omega \tag{1}$$

subject to the boundary condition

$$D_i n_i = -q_s \quad \text{on} \quad S \tag{2}$$

and the mechanical boundary value system is defined by

$$\frac{\partial \sigma_{ji}}{\partial x_j} = \rho \frac{\partial^2 u_i}{\partial t^2} \quad \text{in} \quad \Omega \tag{3}$$

subject to

$$\sigma_{ji}n_j = t_i \quad \text{on} \quad S \tag{4}$$

where q_s , q_v and ρ are surface charge, volume charge, and mass density respectively, x_i are spatial cartesian vector components, n_i are components of the outward normal of S and t_i are components of a traction vector applied to S. Following usual tensor convention, repeated subscripts indicate summation. σ_{ij} and D_i are components of the symmetric Cauchy stress tensor ($\sigma_{ji} = \sigma_{ij}$) and electrical flux vector respectively, and are related to those of the strain tensor ϵ_{ij} and electric field vector E_i through the piezoelectric constitutive equations

$$\sigma_{ij} = C_{ijkl}\epsilon_{kl} - e_{kij}E_k \tag{5}$$

$$D_i = e_{ikl}\epsilon_{kl} + \kappa_{ij}E_j \tag{6}$$

where C_{ijkl} , e_{kij} , and κ_{ik} denote elastic, piezoelectric and dielectric material constants respectively. The strain and electrical field components are linked to mechanical displacement components u_i and electric field scalar potential ϕ by

$$\epsilon_{ij} = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \tag{7}$$

$$E_i = -\frac{\partial \phi}{\partial x_i} \tag{8}$$

For arbitrary virtual displacements δu_i and potential $\delta \phi$ the differential equations (1) and (3) can be written together the weak form variational principle representation for PZT media

$$-\int_{\Omega} \rho \frac{d^2 u_i}{dt^2} \delta u_i \partial \Omega + \int_S t_i \delta u_i \partial S - \int_{\Omega} \sigma_{ji} \delta \epsilon_{ij} \partial \Omega = \int_{\Omega} q_v \delta \phi \partial \Omega + \int_S q_s \delta \phi \partial S - \int_{\Omega} D_i \delta E_i \partial \Omega(9)$$

the details of whose derivation are outlined in [1]. Using the Rayleigh-Ritz method [14], (9) can be discretised to give a piezoelectric finite element formulation.

3. Piezoelectric Finite Element Formulation

For 1st order 3-d tetrahedral elements each of volume Ω^e with corresponding linear shape functions N^e_i i = 1, ..., 4, the displacement vector u is approximated elementally by

$$\{u^e\} = [N^u]\{u^n\}$$
(10)

where $[N^u]$ is a matrix of elemental shape functions

and $\{u^n\}$ is the vector of elemental nodal displacement components

$$\{u^n\} = [u^e_{x_1}, u^e_{x_2}, u^e_{x_3}, u^e_{x_4}, u^e_{y_1}, u^e_{y_2}, u^e_{y_3}, u^e_{y_4}, u^e_{z_1}, u^e_{z_2}, u^e_{z_3}, u^e_{z_4}]^T$$
(12)

where for clarity the mechanical displacement components u_i described are now termed $u_x u_y$ and u_z , and $u_{x_i}^e$ corresponds to the value of u_x at the i^{th} node of element e. The elemental scalar potential is similarly approximated across each element as

$$\phi^e = [N^{\phi}]\{\phi^n\} \tag{13}$$

where

$$[N^{\phi}] = [N_1^e N_2^e N_3^e N_4^e] \tag{14}$$

and $\{\phi^n\}$ is the vector of elemental nodal potentials

$$\{\phi^n\} = [\phi_1^e, \phi_2^e, \phi_3^e, \phi_4^e]^T \tag{15}$$

The mechanical strain tensor (7) and the electric field vector (8) then respectively take the discretised forms

$$\{\epsilon^e\} = [B^u]\{u^n\}$$
 and $\{E^e\} = -[B^{\phi}]\{\phi^n\}$ (16)

where

$$[B^{u}] = \begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0\\ 0 & \frac{\partial}{\partial y} & 0\\ 0 & 0 & \frac{\partial}{\partial z}\\ 0 & \frac{\partial}{\partial z} & \frac{\partial}{\partial y}\\ \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial x}\\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \end{bmatrix} [N^{u}] \quad \text{and} \quad [B^{\phi}] = \left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right]^{T} [N^{\phi}] \quad (17)$$

The discretised constitutive equations of piezoelectricity (5) and (6) are then

$$\{\sigma^e\} = [C]\{\epsilon^e\} - [e]\{E^e\}$$
(18)

$$\{D^e\} = [e]^T \{\epsilon^e\} + [\kappa] \{E^e\}$$
(19)

where [C], [e] and $[\kappa]$ are now matrices of elastic, piezoelectric and dielectric material parameters respectively.

The discretised elemental form of the variational principle (9) is then given by

$$-\int_{\Omega^{e}} \{\delta u^{e}\}^{T} \rho\{\ddot{u}^{e}\} \partial \Omega^{e} + \int_{S^{e}} \{\delta u^{e}\}^{T}\{t\} \partial S^{e} - \int_{\Omega^{e}} \left[\{\delta\epsilon^{e}\}^{T}[C]\{\epsilon^{e}\} - \{\delta\epsilon^{e}\}^{T}[e]\{E^{e}\}\right] \partial \Omega^{e} = \int_{\Omega^{e}} q_{v} \delta\phi^{e} \partial \Omega^{e} + \int_{S^{e}} q_{s} \delta\phi^{e} \partial S^{e} - \int_{\Omega^{e}} \left[\{\delta E^{e}\}^{T}[e]^{T}\{\epsilon^{e}\} + \{\delta E^{e}\}^{T}[\kappa]\{E^{e}\}\right] \partial \Omega^{e}$$
(20)

where \ddot{u} represents a double time derivative and $\{t\}$ is a vector of the traction components t_i . (20) can be expanded to give

$$-\{\delta u^{n}\}^{T} \int_{\Omega^{e}} [N^{u}]^{T} \rho[N^{u}]\{\ddot{u}^{n}\}\partial\Omega^{e} + \{\delta u^{n}\}^{T} \int_{S^{e}} [N^{u}]^{T}\{t\}\partial S^{e}$$

$$-\{\delta u^{n}\}^{T} \int_{\Omega^{e}} [B^{u}]^{T}[C][B^{u}]\{u^{n}\}\partial\Omega^{e} - \{\delta u^{n}\}^{T} \int_{\Omega^{e}} [B^{u}]^{T}[e][B^{\phi}]\{\phi^{n}\}\partial\Omega^{e} =$$

$$\{\delta\phi^{n}\}^{T} \int_{\Omega^{e}} [N^{\phi}]^{T}q_{v}\partial\Omega^{e} + \{\delta\phi^{n}\}^{T} \int_{S^{e}} [N^{\phi}]^{T}q_{s}\partial S^{e}$$

$$+\{\delta\phi^{n}\}^{T} \int_{\Omega^{e}} [B^{\phi}]^{T}[e]^{T}[B^{u}]\{u^{n}\}\partial\Omega^{e} - \{\delta\phi^{n}\}^{T} \int_{\Omega^{e}} [B^{\phi}]^{T}[\kappa][B^{\phi}]\{\phi^{n}\}\partial\Omega^{e}$$

$$(21)$$

Imposing the stationary requirement of the variational principle on (21), that the integral terms corresponding to $\{\delta\phi^n\}$ and $\{\delta u^n\}$ must vanish, results in two independent

equations that define system equilibrium:

$$\int_{\Omega^{e}} \rho[N^{u}]^{T}[N^{u}]\{\ddot{u^{n}}\}\partial\Omega^{e} - \int_{S^{e}} [N^{u}]^{T}\{t\}\partial S^{e} + \int_{\Omega^{e}} [B^{u}]^{T}[C][B^{u}]\{u^{n}\}\partial\Omega^{e} + \int_{\Omega^{e}} [B^{u}]^{T}[e][B^{\phi}]\{\phi^{n}\}\partial\Omega^{e} = 0$$

$$(22)$$

$$\int_{\Omega^e} [N^{\phi}]^T q_v \partial \Omega^e + \int_{S^e} [N^{\phi}]^T q_s \partial S^e + \int_{\Omega^e} [B^{\phi}]^T [e]^T [e]^T [B^u] \{u^n\} \partial \Omega^e - \int_{\Omega^e} [B^{\phi}]^T [\kappa] [B^{\phi}] \{\phi^n\} \partial \Omega^e = 0$$
(23)

Equations (22) and (23) yield the elemental system of equilibrium finite element equations

$$[m]\{\ddot{u^n}\} + [K^{uu}]\{u^n\} + [K^{u\phi}]\{\phi^n\} = \{f\}$$
(24)

$$[K^{\phi u}]\{u^n\} + [K^{\phi \phi}]\{\phi^n\} = \{g\}$$
(25)

where the stiffness matrices are given by

$$[m] = \int_{\Omega^{e}} \rho[N^{u}]^{T}[N^{u}]\partial\Omega^{e}$$
$$[K^{uu}] = \int_{\Omega^{e}} [B^{u}]^{T}[C][B^{u}]\partial\Omega^{e}$$
$$[K^{u\phi}] = \int_{\Omega^{e}} [B^{u}]^{T}[e]^{T}[B^{\phi}]\partial\Omega^{e}$$
$$[K^{\phi u}] = \int_{\Omega^{e}} [B^{\phi}]^{T}[e][B^{u}]\partial\Omega^{e} = [K^{u\phi}]^{T}$$
$$[K^{\phi\phi}] = -\int_{\Omega^{e}} [B^{\phi}]^{T}[\kappa][B^{\phi}]\partial\Omega^{e}$$

and the excitation vectors by

$$\{f\} = \int_{S^e} [N^u]^T \{t\} \partial S^e \tag{26}$$
$$\{a\} = \int [N^{\phi}]^T a \, \partial \Omega^e \int [N^{\phi}]^T a \, \partial S^e \tag{27}$$

$$\{g\} = -\int_{\Omega^e} [N^{\phi}]^T q_v \partial \Omega^e - \int_{S^e} [N^{\phi}]^T q_s \partial S^e$$
(27)

The elemental system is summed over all elements to assemble a final global system. The parametric matrices [C], [e], and $[\kappa]$ in the formulation are given in full by

$$[C] = \begin{bmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & 0 \\ c_{21} & c_{22} & c_{23} & 0 & 0 & 0 \\ c_{31} & c_{32} & c_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & c_{66} \end{bmatrix}$$
(28)
$$[e] = \begin{bmatrix} 0 & 0 & 0 & 0 & e_{15} & 0 \\ 0 & 0 & 0 & e_{24} & 0 & 0 \\ e_{31} & e_{32} & e_{33} & 0 & 0 & 0 \end{bmatrix}$$
(29)

$$[\kappa] = \begin{bmatrix} \kappa_{11} & 0 & 0\\ 0 & \kappa_{22} & 0\\ 0 & 0 & \kappa_{33} \end{bmatrix}$$
(30)

If the piezoelectric body in question is loaded by an electric potential in the form

$$\phi = \phi_0 \cos wt \tag{31}$$

the displacement response will adopt the form

$$u = u_0 \cos wt \tag{32}$$

where ω represents the system operational frequency. With

$$\ddot{u} = -\omega^2 u \tag{33}$$

the governing system (24) and (25) is written in the single stiffness matrix from

$$\begin{bmatrix} K^{uu} - \omega^2 m & K^{u\phi} \\ K^{\phi u} & K^{\phi\phi} \end{bmatrix} \begin{cases} u^n \\ \phi^n \end{cases} = \begin{cases} f \\ g \end{cases}$$
(34)

For a static (constant input) analysis with $\omega = 0$ then

$$\begin{bmatrix} K^{uu} & K^{u\phi} \\ K^{\phi u} & K^{\phi \phi} \end{bmatrix} \begin{cases} u^n \\ \phi^n \end{cases} = \begin{cases} f \\ g \end{cases}$$
(35)

When analysing a piezoelectric system in n-dimensions, with N finite element nodes, the global stiffness matrix of (34) will be symmetric and square with dimensions (n + 1)N, since at each of the N nodes there are n nodal displacement components and 1 nodal scalar potential value.

4. Finite Element Sensitivity Analysis Formulation

The matrices (28)-(30) contain the piezoelectric material system parameters. A system sensitivity analysis formulation based on the governing finite element equations is developed here whose solution will identify the system parameters whose value changes will have most affect on system operation by way of partial derivatives (sensitivity coefficients).

The equations to describe the nodal system sensitivity coefficients can be derived by directly differentiating the governing system finite element formulation with respect to a chosen parameter.

The finite element sensitivity formulation for the elastic parameters in the matrix [C] are defined directly from (34) by

$$\begin{bmatrix} K^{uu} - \omega^2 m & K^{u\phi} \\ K^{\phi u} & K^{\phi\phi} \end{bmatrix} \begin{cases} \frac{\partial u^n}{\partial c_{ij}} \\ \frac{\partial \phi^n}{\partial c_{ij}} \end{cases} = \begin{bmatrix} -K^{uu}_c & 0 \\ 0 & 0 \end{bmatrix} \begin{cases} u^n \\ \phi^n \end{cases}$$
(36)

where c_{ij} is the chosen element of the matrix [C], $\frac{\partial u^n}{\partial c_{ij}}$ and $\frac{\partial \phi^n}{\partial c_{ij}}$ are vectors of the nodal sensitivity coefficient values and

$$[K_c^{uu}] = \int_{\Omega^e} [B^u]^T \frac{\partial [C]}{\partial c_{ij}} [B^u] \partial \Omega^e$$
(37)

Similarly, the nodal sensitivity coefficient vector corresponding to any piezoelectric parameter e_{ij} in the matrix [e] is defined by

$$\begin{bmatrix} K^{uu} - \omega^2 m & K^{u\phi} \\ K^{\phi u} & K^{\phi\phi} \end{bmatrix} \begin{cases} \frac{\partial u^n}{\partial e_{ij}} \\ \frac{\partial \phi^n}{\partial e_{ij}} \end{cases} = \begin{bmatrix} 0 & -K^{u\phi}_e \\ -K^{\phi u}_e & 0 \end{bmatrix} \begin{cases} u^n \\ \phi^n \end{cases}$$
(38)

where

$$[K_e^{u\phi}] = \int_{\Omega^e} [B^u]^T \frac{\partial[e]}{\partial e_{ij}} [B^\phi] \partial \Omega^e = [K_e^{\phi u}]^T$$
(39)

The dielectric nodal sensitivity coefficients are given by

$$\begin{bmatrix} K^{uu} - \omega^2 m & K^{u\phi} \\ K^{\phi u} & K^{\phi\phi} \end{bmatrix} \begin{cases} \frac{\partial u^n}{\partial \kappa_{ij}} \\ \frac{\partial \phi^n}{\partial \kappa_{ij}} \end{cases} = \begin{bmatrix} 0 & 0 \\ 0 & K^{\phi\phi}_{\kappa} \end{bmatrix} \begin{cases} u^n \\ \phi^n \end{cases}$$
(40)

where

$$[K_{\kappa}^{\phi\phi}] = \int_{\Omega^{e}} [B^{\phi}]^{T} \frac{\partial[\kappa]}{\partial\kappa_{ij}} [B^{\phi}] \partial\Omega^{e}$$
(41)

and κ_{ij} is any element of the dielectric matrix $[\kappa]$.

The mass density ρ is a further system parameter whose nodal sensitivity coefficient vector is given by

$$\begin{bmatrix} K^{uu} - \omega^2 m & K^{u\phi} \\ K^{\phi u} & K^{\phi\phi} \end{bmatrix} \begin{cases} \frac{\partial u^n}{\partial \rho} \\ \frac{\partial \phi^n}{\partial \rho} \end{cases} = \begin{bmatrix} \omega^2 m_\rho & 0 \\ 0 & 0 \end{bmatrix} \begin{cases} u^n \\ \phi^n \end{cases}$$
(42)

where

$$[m_{\rho}] = \int_{\Omega^e} [N^u]^T [N^u] \partial \Omega^e \tag{43}$$

Altogether the matrix systems (36) (38) (40) and (42) are the piezoelectric elemental finite element formulation for the nodal sensitivities of displacement and potential to each of the parameters in the elastic, piezoelectric and dielectric matrices and the system mass density. The sensitivity equations can be solved in parallel with the governing system with the nodal solution vectors determined from the governing solution used to construct the sensitivity formulation at each solution iteration.

4.1. Implementation Example

By way of example, for the case of sensitivity with respect to the parameter e_{31} , the elemental nodal sensitivity coefficient vector in (38) is given by

$$\left\{ \begin{array}{c} \frac{\partial u^n}{\partial e_{31}}\\ \frac{\partial \phi^n}{\partial e_{31}} \end{array} \right\} = \left[\frac{\partial u^e_{x_1}}{\partial e_{31}}, ..., \frac{\partial u^e_{x_4}}{\partial e_{31}}, \frac{\partial u^e_{y_1}}{\partial e_{31}}, ..., \frac{\partial u^e_{y_4}}{\partial e_{31}}, \frac{\partial u^e_{z_1}}{\partial e_{31}}, ..., \frac{\partial u^e_{z_4}}{\partial e_{31}}, \frac{\partial \phi^e_1}{\partial e_{31}}, ..., \frac{\partial \phi^e_4}{\partial e_{31}} \right]^T (44)$$

with the right hand side vector in (38), returned as the nodal solution vector to the governing system, given by

$$\left\{ \begin{array}{c} u^{n} \\ \phi^{n} \end{array} \right\} = \left[u^{e}_{x_{1}}, ..., u^{e}_{x_{4}}, u^{e}_{y_{1}}, ..., u^{e}_{y_{4}}, u^{e}_{z_{1}}, ..., u^{e}_{z_{4}}, \phi^{e}_{1}, ..., \phi^{e}_{4} \right]^{T}$$
(45)

The stiffness matrix in (38) has already been determined and decomposed in the governing solution and the matrix in (39) is determined numerically or otherwise where

Upon constructing and solving the system (38) the sensitivity coefficients are approximated across each element in terms of their nodal values using

$$\left\{ \begin{array}{c} \frac{\partial u^e}{\partial e_{31}} \\ \frac{\partial \phi^e}{\partial e_{31}} \end{array} \right\} = \left[\begin{array}{c} N^u & 0 \\ 0 & N^\phi \end{array} \right] \left\{ \begin{array}{c} \frac{\partial u^n}{\partial e_{31}} \\ \frac{\partial \phi^n}{\partial e_{31}} \end{array} \right\}$$
(47)

where $\frac{\partial u^e}{\partial e_{31}} = \frac{\partial u^e}{\partial e_{31}}(x, y, z)$, $\frac{\partial \phi^e}{\partial e_{31}} = \frac{\partial \phi^e}{\partial e_{31}}(x, y, z)$ and the interpolating matrix in (47) has dimension 4x16 in accordance with (11) and (14).

The elemental sensitivity equations outlined here can be summed over all elements for a global piezoelectric finite element sensitivity analysis, and the analysis is readily extendable to any system parameter other than e_{31} using the same formulation.

4.2. Normalized Sensitivity Coefficients

The sensitivity coefficient $\frac{\partial u^e}{\partial e_{31}}$ computed above represents a linear estimate of the percentage change in u^e as a result of a unit change in e_{31} . With many different physical units involved between system outputs and parameters, a more widespread measure of sensitivity is a normalized sensitivity coefficient defined as

$$\overline{\frac{\partial u^e}{\partial e_{31}}} = \frac{e_{31}}{u^e} \frac{\partial u^e}{\partial e_{31}} \tag{48}$$

The normalized sensitivity coefficient indicated by the overline represents a linear estimate of the percentage change in the variable u^e given a 1% change in e_{31} . Being independent of the original system units in this way, normalized system sensitivity coefficients are readily comparable with each other and therefore offer a more informative description of parameter importance [7].

By defining new system sensitivity stiffness matrix terms as

$$\begin{split} \overline{[K^{uu} - \omega^2 m]}_{ij} &= \frac{1}{e_{31}} [K^{uu} - \omega^2 m]_{ij} \{u^n\}_j \ i = 1, ..., 12, j = 1, ..., 12, \\ \overline{[K^{u\phi}]}_{ij} &= \frac{1}{e_{31}} [K^{u\phi}]_{ij} \{\phi^n\}_j \ i = 1, ..., 12, j = 1, ..., 4 \\ \overline{[K^{\phi u}]}_{ij} &= \frac{1}{e_{31}} [K^{\phi u}]_{ij} \{u^n\}_j \ i = 1, ..., 4, j = 1, ..., 12 \\ \overline{[K^{\phi \phi}]}_{ij} &= \frac{1}{e_{31}} [K^{\phi \phi}]_{ij} \{\phi^n\}_j \ i = 1, ..., 4, j = 1, ..., 4 \end{split}$$

where subscript i, j denotes the (i, j)th element of the sub matrices in (38) and subscript j denoting the jth element of the vectors (12) and (15), then from (38) an elemental formulation for the normalized nodal sensitivity coefficients is given by

$$\begin{bmatrix} \overline{K^{uu} - \omega^2 m} & \overline{K^{u\phi}} \\ \overline{K^{\phi u}} & \overline{K^{\phi\phi}} \end{bmatrix} \begin{cases} \frac{\overline{\partial u^n}}{\partial e_{31}} \\ \frac{\overline{\partial \phi^n}}{\partial e_{31}} \end{cases} = \begin{bmatrix} 0 & -K_e^{u\phi} \\ -K_e^{\phi u} & 0 \end{bmatrix} \begin{cases} u^n \\ \phi^n \end{cases}$$
(49)

The normalized sensitivity coefficients can be interpolated across the element in terms of their nodal values derived from (49) in the usual way by

$$\left\{ \begin{array}{c} \frac{\overline{\partial u^e}}{\overline{\partial e_{31}}}\\ \frac{\overline{\partial \phi^e}}{\overline{\partial e_{31}}} \end{array} \right\} = \left[\begin{array}{c} N^u & 0\\ 0 & N^\phi \end{array} \right] \left\{ \begin{array}{c} \frac{\overline{\partial u^n}}{\overline{\partial e_{31}}}\\ \frac{\overline{\partial \phi^n}}{\overline{\partial e_{31}}} \end{array} \right\}$$
(50)

Modifying the original sensitivity formulation stiffness matrix to return normalized sensitivities directly in this way is a process that avoids the numerical complications often associated with explicitly multiplying the original sensitivity solution by a factor of $\frac{e_{31}}{u^e}$, say, at the point where $u^e = 0$.

5. Numerical Results

Distributed Mode Actuators (DMA) are piezoelectric based devices that can be used to excite a flat surface in order to produce sound. The DMA comprises one or more small piezoelectric crystal layers, each separated by an electrode layer and attached to a supporting beam. The overall assembly is clamped at one end to a common stub to make a cantilever beam. DMA's have become widespread in micro-engineering applications [15] and their optimisation has been the focus of much recent work [16].

A finite element model for simulation of a single piezoelectric layer DMA, as shown in figure 1, was developed in the commercial software package COMSOL Multiphysics. The discretised DMA system geometry of tetrahedral elements used for the finite element simulations is presented in figure 2. The accuracy of this DMA simulation model has been validated against experimental measurements [17].

A useful feature in COMSOL is the ability to export finite element simulation models into MATLAB. This COMSOL-MATLAB link allows models to be manipulated and solved from within the MATLAB environment, using special MATLAB functions provided by COMSOL. This feature was used to import the DMA simulation model into MATLAB where it was modified according to the formulation presented in Section 4 and solved to produce not only the standard system responses, but also the corresponding sensitivity solutions. In this way, the sensitivities are computed at reduced computational cost, when compared with traditional finite difference methods, as the sensitivity formulation presented in Section 4 is always linear.

Here we present simulation results on a DMA comprising of the lead zirconate titanate ceramic $PbZr_{0.53}Ti_{0.47}O_3$ (PZT-5H). This piezoelectric ceramic has relatively large characteristic piezoelectric coupling parameter values allowing for maximal and well controlled displacements; as such it is ideally suited for use in actuator applications.

Room temperature PZT-5H has characteristic material parameter matrices [18]

$$[C] = \begin{bmatrix} 1.26e^{11} & 7.95e^{10} & 8.41e^{10} & 0 & 0 & 0 \\ 7.95e^{10} & 1.26e^{11} & 8.41e^{10} & 0 & 0 & 0 \\ 8.41e^{10} & 8.41e^{10} & 1.17e^{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & 2.3e^{10} & 0 \\ 0 & 0 & 0 & 0 & 2.3e^{10} & 0 \\ 0 & 0 & 0 & 0 & 0 & 2.33e^{10} \end{bmatrix}$$
(*GPa*) (51)
$$[e] = \begin{bmatrix} 0 & 0 & 0 & 0 & 17 & 0 \\ 0 & 0 & 0 & 17 & 0 & 0 \\ -6.5 & -6.5 & 23.3 & 0 & 0 & 0 \end{bmatrix}$$
(*C/m²*) (52)
$$[u] = \begin{bmatrix} 1.503e^{-8} & 0 & 0 \\ 0 & 0 & 1502e^{-8} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$
(*T/m*) (52)

$$[\kappa] = \begin{bmatrix} 1.503e^{-8} & 0 & 0 \\ 0 & 1.503e^{-8} & 0 \\ 0 & 0 & 1.3e^{-8} \end{bmatrix} (F/m)$$
(53)

and $\rho = 7500 kg/m^3$.

The simulated static displacement response of the single layer PZT-5H based DMA under the influence of an input DC voltage of 5V is presented in figure 3.

In the piezoelectric matrix (52), $e_{31} = e_{32}$ and $e_{24} = e_{15}$, such that there are three piezoelectric material parameters to consider in a sensitivity study of a PZT-5H based application. It is these three parameters that are subject to variation under the influence of temperature changes [19], and as such tolerance to these potential variations is of interest from a design perspective.

The sensitivity coefficients $\frac{\partial u}{\partial e_{ij}}$ for each of the piezoelectric parameters $e_{31} e_{33}$ and e_{15} , simulated by solving finite element formulation in section 4, are plotted in figure 4 against distance along the DMA. The sensitivity calculations are also performed using the finite difference method, that is by changing the relevant parameters by 1 unit and observing the simulated change in beam displacement returned by the governing model, are also presented in figure 4. If δ is considered to be a unit change in parameter e_{ij} , the finite difference calculations are performed using the standard approximation

$$\frac{\partial u}{\partial e_{ij}} \approx \frac{u(e_{ij} + \delta) - u(e_{ij})}{\delta} \tag{54}$$

The normalised sensitivity coefficients, simulated by solving the formulation outlined in section 49, are presented in figure 5 alongside the corresponding finite difference results. Normalised sensitivities represent the percentage change in output displacement given a 1% change in the parameters. These values are seen to be uniform along the length of the beam, indicating that beam displacement in figure 3 is proportional to the sensitivity coefficient values in figure 4.

The good agreement between the analytical and empirical sensitivity results in both figures 4 and 5 acts as a validation of the sensitivity analysis finite element formulation for piezoelectric media. However, analytical sensitivity analysis methods hold computational advantages over empirical studies since empirical studies often require repeated re-runs of a governing non-linear system where as the sensitivity equations are always linear [20] .

In figure 4 it can be seen that changing e_{31} by 1 unit has more impact on the output displacement of the DMA than a similar change in any of the other piezoelectric parameters. However, from the normalised sensitivity results in figure 5, as a percentage of nominal value, it is seen that changes in e_{33} will have most effect on system output. This comparison is evidence to the advantages in normalising the simulated sensitivity coefficients prior to interpretation.

From the sensitivity analysis results, changing the value of e_{15} for this particular design is seen to have no effect on DMA displacement. This is a significant result since, of all the parameters, e_{15} undergoes greatest variation under the effects of changing temperature [19], indicating that this particular design is robust to variation in e_{15} . Local sensitivity analysis results also provide important information for ranking the importance of parameters and for deciding effective parameter value ranges in design optimisation studies. They may also be used directly by gradient-based numerical optimisation algorithms.

6. Conclusions

A finite element based formulation for sensitivity analysis studies of piezoelectric media was developed and an existing finite element piezoelectric solver was extended to implement its solution. The solver was applied to simulate the static operational and sensitivity characteristics of a piezoelectric based distributed mode actuator. The finite element sensitivity solutions were verified against empirical results obtained using the original system model.

The sensitivity analysis was performed with respect to the material piezoelectric coupling parameters since it is these parameters that are subject to variability under operational conditions. As such, these sensitivity results are of interest from a robust design perspective. However, the analysis presented here is easily extended to other system parameters using the same basic formalism.

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- A. Benjeddou. Advances in piezoelectric finite element modeling of adaptive structural elements: a survey. *Computers and Structures*, 76:347–363, 2000.
- [2] J. Mackerle. Smart materials and structures a finite element approach: a bibliography (1986-1997). Modelling and Simulation in Materials Science and Engineering, 6:293-334, 1998.

- [3] H Allik and T. J. R. Hughes. Finite element method for piezoelectric vibration. International Journal for Numerical Methods on Engineering, 2:151–157, 1970.
- [4] A. M. Dunker. The decoupled direct method for calcuating sensitivity coefficients in chemical kinetics. J. Chem. Phys, 81:2385–2393, 1984.
- [5] Leis J. R. and Kramer M. A simultaneous solution and sensitivity analysis of systems described by ordinary differential equations. ACM transactions on mathematical software, 14(1):45–60, 1988.
- [6] Saltelli A., Tarantola S., Campolongon F., and Ratto M. Sensitivity Analysis in Practice. Wiley, 2004.
- [7] A. Saltelli, K. Chan, and E. M. Scott. Sensitivity Analysis (Chapter 5). Wiley, 2000.
- [8] J. P. Conte, P. K. Vijalapura, and M. Meghella. Consistent finite-element response sensitivity analysis. J. Eng. Mech, 129(12):1380–1393, 2003.
- T. Haukaas and A. Der Kiureghian. Parameter sensitivity and importance measures in nonlinear finite element reliability analysis. ASCE Journal of Engineering Mechanics, 131(10):1013–1026, 2005.
- [10] M. Kleiber, H. Antunez, T. D. Hein, and P. Kowalczyk. Parameter Sensitivity in non-linear mechanics: Theory and finite element computations. Wiley, New York., 1997.
- [11] J. J. Tsay and J. S. Arora. Nonlinear structural design sensitivity analysis for path dependent problems. part 1: General theory. *Comput. Methods Appl. Mech. Eng.*, 81:183–208, 1990.
- [12] J. J. Tsay, J. E. B. Cardoso, and J. S. Arora. Nonlinear structural design sensitivity analysis for path dependent problems. part 2: Analytical examples. *Comput. Methods Appl. Mech. Eng.*, 81:209–228, 1990.
- [13] S. Kapadia, W. K. Anderson, L. Elliot, and C. Burdyshaw. Adjoint method for solid-oxide fuel cell simulations. *Journal of Power Sources*, 166:376–385, 2007.
- [14] Jianming Jin. The Finite element Method in Electromagnetics. 2nd Edition. Wiley, 2002.
- [15] P Muralt. Ferroelectric thin films for micro-sensors and actuators: a review. Journal of Micromechanics and Microengineering., 10:136–146., 2000.
- [16] M.I. Frecker. Recent advances in optimization of smart structures and actuators. Journal of Intelligent Material Systems and Structures, 14(5):207–216., 2003.
- [17] Y. Shen, M.A. Atherton, M.A. Perry, R. A. Bates, and H. P. Wynn. Simulation of the electromechanical coupling in multilayered piezoelectric distributed mode actuator. *Submitted* to: Sensors and Actuators: A. Physical, 2006.
- [18] Piezo Systems Inc. Product Catalogue. Cambridge, MA., 1995.
- [19] D. Wang, Y Fotinich, and G. P. Carman. Influence of temperature on the electromechanical and fatigue behaviour of piezoelectric ceramics. *Journal of Applied Physics*, 83(10):5342–5350, 1998.
- [20] Dariusz Ucinski. Optimal Measurement Methods for Distributed Parameter System Identification (Chapter 2). CRC Press, 2005.



Figure 1. Geometry and dimensions of a single layer PZT based Distributed Mode Actuator



Figure 2. The discretized DMA system model as used for the finite element calculations



Figure 3. Simulated displacement response of the PZT-5H DMA shim at an applied input voltage of 5V DC alongside the original DMA position prior to application of the voltage.



Figure 4. Analytic and empirically simulated values of the sensitivity coefficients $\frac{\partial u}{\partial e_{ij}}$ plotted against distance along the shim of the PZT-5H DMA.



Figure 5. Analytic and empirically simulated values of the normalised sensitivity coefficients $\frac{\overline{\partial u}}{\partial e_{ij}}$ plotted against distance along the shim of the PZT-5H DMA.