The amplitude system for a simultaneous short-wave Turing and long-wave Hopf instability

Guido Schneider[1] and Matthias Winter[2]

[1]IADM, Universität Stuttgart, Pfaffenwaldring 57, 70569 Stuttgart, Germany

[2]Brunel University London, Department of Mathematics, Uxbridge UB8 3PH, United Kingdom

July 16, 2021

Abstract

We consider reaction-diffusion systems for which the trivial solution simultaneously becomes unstable via a short-wave Turing and a long-wave Hopf instability. Brusseletor, Gierer-Meinhardt system and Schnakenberg model are prototype biological pattern forming systems which show this kind of behavior for certain parameter regimes. In this paper we prove the validity of the amplitude system associated to this kind of instability. Our analytical approach is based on the use of mode filters and normal form transformations. The amplitude system allows us an efficient numerical simulation of the original multiple scaling problems close to the instability.

1 Introduction

We consider reaction-diffusion systems

$$\partial_t u = D\partial_x^2 u + f(u), \tag{1}$$

with space variable $x \in \mathbb{R}$, time variable $t \geq 0$, solution vector $u(x,t) \in \mathbb{R}^m$, diffusion matrix $D \in \mathbb{R}^{m\times m}$, and smooth nonlinearity $f : \mathbb{R}^m \to \mathbb{R}^m$. We assume that (1) depends on a control parameter $\alpha \in \mathbb{R}$. Moreover, we assume that System (1) possesses a trivial stationary spatially homogeneous solution $u = u^* \in \mathbb{R}^m$ for all values of $\alpha \in \mathbb{R}$. The linearization around $u = u^*$ is given by

$$\partial_t v = D\partial_x^2 v + \nabla f(u^*)v,$$

and it is generically solved by

$$v(x,t) = e^{\lambda_j(k)t}\varphi_j(k)e^{ikx},$$

with eigenvectors $\varphi_j(k) \in \mathbb{C}^m$ and eigenvalues $\lambda_j(k) \in \mathbb{C}$ for $j = 1, \dots, m$ and $k \in \mathbb{R}$. We assume that the trivial solution $u = u^*$ simultaneously becomes unstable via a short-wave Turing and a long-wave Hopf instability, i.e., we assume

Hypothesis A: There are $k_c > 0$, $\alpha = \alpha_c$, and $\omega_0 > 0$ with $\lambda_1(k_c, \alpha_c) = 0$, $\lambda_1(0, \alpha_c) = i\omega_0$, and $\lambda_2(0, \alpha_c) = -i\omega_0$. For all other values of (j, k) we have $\mathrm{Re}\lambda_j(k, \alpha_c) < 0$.

Remark 1.1. Hypothesis A includes systems with $m \geq 3$, too. We assume that the other eigenvalues (except for the three eigenvalues stated in Hypothesis A) have negative real part and are uniformly bounded away from the imaginary axis near the bifurcation point. A consequence of Hypothesis A is that in leading order the dynamics is in the direction of the corresponding eigenvectors, see (2). Other scenarios which are possible in the general case $m \geq 3$ have been excluded by assuming Hypothesis A.

Brusselator, Gierer-Meinhardt system and Schnakenberg model are examples of reaction-diffusion systems possessing this kind of spectral picture in certain parameter regimes, see Section 4. There are other examples falling into this class such as the the Gray-Scott model.

In order to derive the amplitude system for the description of the dynamics close to the bifurcation point $\alpha = \alpha_c$ we introduce the small bifurcation parameter $\varepsilon^2 = \alpha - \alpha_c$ and set $\alpha_2 = \partial_\alpha \lambda_1(k_c, \alpha_c)$ and $\beta_2 = \partial_\alpha \lambda_1(0, \alpha_c)$. Using the reflection symmetry of the problem, we can conclude from Hypothesis A that the spectral picture is as plotted in Figure 1.

Then for small $0 < \varepsilon^2 \ll 1$ we make the ansatz

$$u(x,t) - u^* \approx \varepsilon \Psi_{app}(x,t),$$

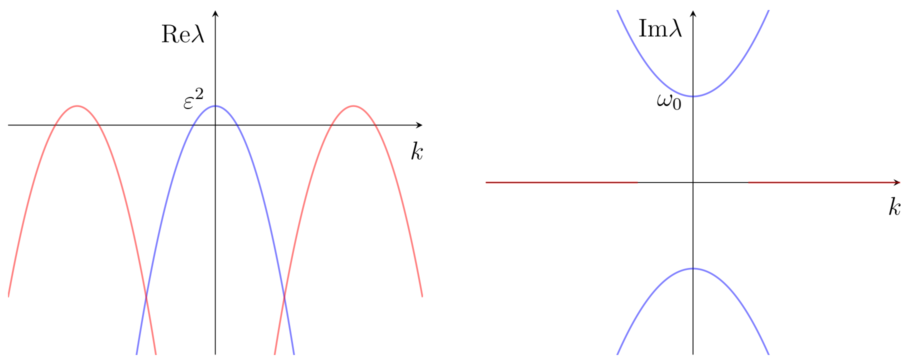

Figure 1: The spectrum of the linearization around the trivial solution becoming simultaneously unstable via a short-wave Turing instability (red) and a long-wave Hopf instability (blue).

with

$$\varepsilon\Psi_{app}(x,t) = \varepsilon A(\varepsilon x, \varepsilon^2 t)e^{ik_c x}\varphi_1(k_c) + \varepsilon B(\varepsilon x, \varepsilon^2 t)e^{i\omega_0 t}\varphi_1(0) + c.c., \tag{2}$$

with (short-wave Turing) amplitude function $A(X,T) \in \mathbb{C}$ and (long-wave Hopf) amplitude function $B(X,T) \in \mathbb{C}$. Inserting this ansatz into (1) and equating the coefficients in front of $\varepsilon^3 e^{ik_c x}\varphi_1(k_c)$ and in front of $\varepsilon^3 e^{i\omega_0 t}\varphi_1(0)$ to zero yields a system of coupled Ginzburg-Landau equations

$$\partial_T A = \alpha_1 \partial_X^2 A + \alpha_2 A + \alpha_3 A|A|^2 + \alpha_4 A|B|^2, \tag{3}$$
$$\partial_T B = \beta_1 \partial_X^2 B + \beta_2 B + \beta_3 B|B|^2 + \beta_4 B|A|^2, \tag{4}$$

with coefficients $\alpha_1, \ldots, \alpha_4 \in \mathbb{R}$ and $\beta_1, \ldots, \beta_4 \in \mathbb{C}$. It is the goal of this paper to prove the following approximation theorem:

Theorem 1.2. *Let $(A,B) \in C([0,T_0],(H^4_{l,u})^2)$ be a solution of (3)-(4). Then there exist $\varepsilon_0, C > 0$ such for all $\varepsilon \in (0,\varepsilon_0)$ we have solutions u of (1) with*

$$\sup_{t\in[0,T_0/\varepsilon^2]} \sup_{x\in\mathbb{R}} |u(x,t) - u^* - \varepsilon\Psi_{app}(x,t)| \leq C\varepsilon^2.$$

Remark 1.3. The space of uniformly local Sobolev functions $H^\theta_{l,u}$ is equipped with the norm

$$\|u\|_{H^\theta_{l,u}} = \sup_{y\in\mathbb{R}} \|u\|_{H^\theta(y,y+1)}$$

and is the completion of C^∞ w.r.t. this norm. In contrast to functions in Sobolev spaces H^θ, functions in $H^\theta_{l,u}$ need not decay to zero for $|x| \to \infty$. See [SU17, §8.3.1] for more details.

Remark 1.4. The approximation result is non-trivial because solutions of order $\mathcal{O}(\varepsilon)$ have to be controlled on a time scale of order $\mathcal{O}(1/\varepsilon^2)$. There exist various counter-examples where amplitude equations make wrong predictions although they are derived through a correct expansion w.r.t. a small perturbation parameter $0 < \varepsilon^2 \ll 1$, cf. [Sch95, SSZ15, BSSZ20, HS20].

Remark 1.5. The introduction of only one small parameter $0 < \varepsilon \ll 1$ is no restriction. The parameters $\alpha_1, \ldots, \beta_4$ appearing in the amplitude system (3)-(4) allow us to exhaust a neighborhood of the origin.

Remark 1.6. The situation of a trivial solution, which becomes simultaneously unstable via a short-wave Turing and a long-wave Hopf instability, has not been considered before in the mathematical literature about the justification of modulation equations. The validity of the Ginzburg-Landau approximation in case of a short-wave Turing instability has been considered for instance in [CE90, vH91, Sch94a, Sch94b]. The validity of the Ginzburg-Landau approximation in case of a long-wave Hopf instability has been considered in [Sch98]. In fact, the proof of Theorem 1.2 is a combination of the proofs given in [Sch94a] and [Sch98] based on the use of mode filters and normal form transformations.

Remark 1.7. The kind of instability considered in this paper appears in all systems, where the parameter regime, in which a short-wave Turing instability occurs, and the parameter regime, in which a long-wave Hopf instability occurs, have a common boundary. Besides Brusselator, Gierer-Meinhardt system and Schnakenberg model, which are discussed in Section 4, various other reaction-diffusion systems show this kind of instability. The scenario around these bifurcation points, which are sometimes called Turing-Hopf points, has been analyzed in the case of systems on bounded spatial domains, see for instance [LI80, ADG85, DWLDB96, MDWBS97, JBB$^+$01, PLNW20]. We remark that in case $x \in \mathbb{R}^d$ with $d > 1$ for the modes at $k = k_c$ no degenarated Ginzburg-Landau equation can be derived due to the rotational symmetry of (1).

Remark 1.8. Similar to the analysis which we will make subsequently at the beginning of Section 3.1, by adding higher order terms to the approximation, the approximation error can be made arbitrarily small. The construction of such an improved approximation is possible because the group velocities at the wave numbers $k = 0$ and $k = k_c$ are the same, namely zero. Making the approximation error arbitrarily small is for instance not possible in case of a Hopf bifurcation at $k = k_c$ because then the group velocities in general are different leading to a non-consistent perturbation ansatz.

Remark 1.9. In case that the coefficients α_3, α_4, β_3, and β_4 in (3)-(4) allow to prove the existence of an exponentially absorbing ball for (3)-(4) in some $H^{\theta}_{l,u}$-space, the combination of approximation results, like Theorem 1.2, with attractivity results allows to prove the global existence of the solutions of (1) in a small $\mathcal{O}(1)$-neighborhood of u^* in some $H^{\theta}_{l,u}$-space. See [MS95, SU17] for the description of the approach for a single Ginzburg-Landau equation as amplitude equation. Attractivity results show that the set of solutions which is described by (3)-(4) is attractive for initial conditions of order $\mathcal{O}(\delta)$ with $0 < \varepsilon \leq \delta$ small. The first attractivity result has been established in [Eck93].

Remark 1.10. The system of coupled Ginzburg-Landau equations (3)-(4) allows an efficient numerical simulation of the original reaction-diffusion system (1) close to the threshold of instability through a discretization of (3)-(4). Error estimates for this numerical approximation of the multiple scaling problem (1) close to the threshold of instability can be obtained through the triangle inequality, similar to the analysis given in [FS14]. This approach based on the amplitude equations (3)-(4) avoids a very expensive discretization of the original system. The oscillations in time and space occurring in the original system no longer have to be resolved and so instead of discretizing a spatial domain of size $\mathcal{O}(1/\varepsilon)$ and a time domain of size $\mathcal{O}(1/\varepsilon^2)$ a discretization of the amplitude system is sufficient, i.e., a spatial domain and a time domain of size $\mathcal{O}(1)$.

Notation. The Fourier transform of a function u is denoted by $\widehat{u}$. Constants which can be chosen independently of the small perturbation parameter $0 < \varepsilon \ll 1$ are denoted with the same symbol C.

Acknowledgement. The work of Guido Schneider is partially supported by the Deutsche Forschungsgemeinschaft DFG through the cluster of excellence 'SimTech' under EXC 2075-390740016. Matthias Winter thanks the IADM at Universität Stuttgart for their kind hospitality. We thank the referees for carefully reading the paper and making valuable suggestions for improvement.

2 Some preparations

In order to prove Theorem 1.2 we introduce the deviation v of the stationary solution u^* through

$$u = u^* + v.$$

The deviation v satisfies

$$\partial_t v = D\partial_x^2 v + g(v) = D\partial_x^2 v + B_1 v + B_2(v,v) + B_3(v,v,v) + N_4(v), \qquad (5)$$

with $g(v) = f(u^*+v)$, B_1 a linear mapping, B_2 a symmetric bilinear mapping, B_3 a symmetric trilinear mapping, and N_4 satisfying $N_4(v) = \mathcal{O}(\|v\|^4)$ for $v \to 0$. The major difficulty to come to the long time scale of order $\mathcal{O}(1/\varepsilon^2)$ with our error estimates are the quadratic terms B_2 which could lead to an exponential growth of order $\mathcal{O}(e^{\varepsilon t})$ for solutions of order $\mathcal{O}(\varepsilon)$. In order to prove that such a growth does not occur, we have to use the oscillatory character of the modes at $k = 0$ associated to the eigenvalues $\pm i\omega_0$ and the fact that the quadratic interaction of the modes with wave numbers $k = \pm k_c$ gives non-resonant terms at $k = 0$ and exponentially damped modes at $k = \pm 2k_c$, but no modes at $k = \pm k_c$. In order to use these properties, in Section 2.1 we separate the critical modes, i.e., the modes with positive or only weakly negative growth rates, from the exponentially damped modes. Then, in Section 2.2 we make a number of near identity change of variables eliminating various quadratic interactions of these critical modes. Background on the functional analytic approach will be provided in Section 2.3.

2.1 Separation of the modes

In Fourier space (5) is given by

$$\begin{aligned} \partial_t \widehat{v} &= -Dk^2\widehat{v} + \widehat{g}(\widehat{v}) \\ &= -Dk^2\widehat{v} + B_1\widehat{v} + \widehat{B}_2(\widehat{v},\widehat{v}) + \widehat{B}_3(\widehat{v},\widehat{v},\widehat{v}) + \widehat{N}_4(\widehat{v}). \end{aligned} \qquad (6)$$

The spectral assumptions from the introduction allow us to make some normal form transformations which simplify (6) near $k = 0$ and near $k = \pm k_c$. In order to extract the modes in a neighborhood of the wave numbers $k = 0$ and $k = \pm k_c$, we introduce a C_0^∞-function

$$\chi(k) \begin{cases} = 1, & \text{for } |k| \leq 1, \\ \in [0,1], & \text{for } |k| \in (1,2), \\ = 0, & \text{for } |k| \geq 2. \end{cases}$$

Near $k=0$ we will use new scalar variables $\widehat{w}_1(k,t) \in \mathbb{C}$ and $\widehat{w}_{-1}(k,t) \in \mathbb{C}$, corresponding to the eigenvalues $i\omega_0$ and $-i\omega_0$ and to the eigenvectors $\varphi_1(k)$ and $\varphi_2(k)$, where $\widehat{w}_{-1}(k,t) = \overline{\widehat{w}_1(-k,t)}$ and $\varphi_2(k) = \overline{\varphi_1(-k)}$ near $k=0$. Near $k=k_c$ and $k=-k_c$ we will use the new scalar variables $\widehat{z}_1(k,t) \in \mathbb{C}$ and $\widehat{z}_{-1}(k,t) \in \mathbb{C}$ corresponding to the eigenvalues $\lambda_1(k)$ and the eigenvectors $\varphi_1(k)$, where $\widehat{z}_{-1}(k,t) = \overline{\widehat{z}_1(-k,t)}$ and $\varphi_1(k) = \overline{\varphi_1(-k)}$ near $k=-k_c$. Since we consider a real-valued system, by controlling $\widehat{w}_1$ and $\widehat{z}_1$ we also get control of $\widehat{w}_{-1}$ and $\widehat{z}_{-1}$.

We define mode filters E_w and E_z to extract the critical modes, i.e., the modes with positive or weakly negative growth rates. Moreover, we define the complementary mode filter E_s which extracts the remaining part of the solution which is linearly damped with some exponential rate. In detail, the mode filters E_w and E_z are defined through their actions in Fourier space, namely

$$
\begin{aligned}
\widehat{E}_w(k)\widehat{v}(k) &= \chi(\frac{k}{r})\varphi_1^*(k)^T\widehat{v}(k),\\
\widehat{E}_z(k)\widehat{v}(k) &= \chi(\frac{k-k_c}{r})\varphi_1^*(k)^T\widehat{v}(k),\\
\widehat{E}_s(k)\widehat{v}(k) &= I-\chi(\frac{k}{r})\varphi_1^*(k)^T\widehat{v}(k)\varphi_1(k)-\chi(\frac{k}{r})\varphi_2^*(k)^T\widehat{v}(k)\varphi_2(k)\\
&\quad -\chi(\frac{k-k_c}{r})\varphi_1^*(k)^T\widehat{v}(k)\varphi_1(k)-\chi(\frac{k+k_c}{r})\varphi_1^*(k)^T\widehat{v}(k)\varphi_1(k),
\end{aligned}
$$

for an $r>0$ sufficiently small, but independent of the small perturbation parameter $0<\varepsilon \ll 1$. The $\varphi_j^*(k)$ are the adjoint eigenvectors. Since we work in $H^\theta_{l,u}$-spaces, we need operators which are smooth multipliers in Fourier space, like χ, cf. Lemma 2.1. Since the eigenvalues are simple for fixed $k \in \mathbb{R}$, the associated eigenvectors and adjoint eigenvectors depend smoothly on k near $k=0$ and near $k=\pm k_c$.

We introduce the new variables $\widehat{w}_1$ and $\widehat{z}_1$ as the solutions of the system

$$
\begin{aligned}
\partial_t\widehat{w}_1(k,t) &= \lambda_1(k)\widehat{w}_1(k,t)+\widehat{g}_w(\widehat{w}_1,\widehat{z}_1,\widehat{v}_s)(k,t), &(7)\\
\partial_t\widehat{z}_1(k,t) &= \lambda_1(k)\widehat{z}_1(k,t)+\widehat{g}_z(\widehat{w}_1,\widehat{z}_1,\widehat{v}_s)(k,t), &(8)\\
\partial_t\widehat{v}_s(k,t) &= \widehat{\Lambda}_s(k)\widehat{v}_s(k,t)+\widehat{g}_s(\widehat{w}_1,\widehat{z}_1,\widehat{v}_s)(k,t), &(9)
\end{aligned}
$$

where

$$\begin{aligned}
\widehat{g}_w(\widehat{w}_1, \widehat{z}_1, \widehat{v}_s)(k,t) &= \widehat{E}_w(k)(\widehat{g}(\widehat{v}) - B_1\widehat{v})(k,t), \\
\widehat{g}_z(\widehat{w}_1, \widehat{z}_1, \widehat{v}_s)(k,t) &= \widehat{E}_z(k)(\widehat{g}(\widehat{v}) - B_1\widehat{v})(k,t), \\
\widehat{g}_s(\widehat{w}_1, \widehat{z}_1, \widehat{v}_s)(k,t) &= \widehat{E}_s(k)(\widehat{g}(\widehat{v}) - B_1\widehat{v})(k,t), \\
\widehat{\Lambda}_s(k)\widehat{v}_s(k,t) &= \widehat{E}_s(k)(-Dk^2 + B_1)\widehat{v}_s(k,t).
\end{aligned}$$

We obtain the equations (7)-(9) by applying the mode filters $\widehat{E}_w(k)$, $\widehat{E}_z(k)$ and $\widehat{E}_s(k)$ to (6), respectively. The solution v can be reconstructed through

$$\widehat{v}(k,t) = \widehat{w}_1(k,t)\varphi_1(k)+\widehat{w}_{-1}(k,t)\varphi_2(k)+\widehat{z}_1(k,t)\varphi_1(k)+\widehat{z}_{-1}(k,t)\varphi_1(k)+\widehat{v}_s(k,t).$$

2.2 Change of variables

In order to prove our approximation result we have to control the quadratic interactions of the critical modes $\widehat{w}_{\pm 1}$ and $\widehat{z}_{\pm 1}$. We do so by using the fact that the quadratic interactions are non-resonant in time or in space. The terms which are non-resonant in time will be eliminated by a number of change of variables. A quadratic interaction term is called non-resonant in space if it is exponentially damped, for instance the quadratic interaction of $\widehat{z}_1$-modes is located in Fourier space at $k = 2k_c$ for which the linear modes are damped with an exponential rate.

In this section we explain the interaction structure and why near identity transformations allow to eliminate various quadratic interactions. Functional analytic details underpinning the approach will be provided in Section 2.3.

Since the nonlinear terms in (7)-(8) have a convolution structure w.r.t. $\widehat{z}_1$ and $\widehat{w}_1$ we need to investigate how terms of the type $\widehat{u}_{j_1}(k-l)\widehat{u}_{j_2}(l)$ in the equation for $\widehat{u}_j(k)$ can be eliminated by a near identity change of variables. It is well known that elimination is possible if the non-resonance condition

$$\lambda_j(k) - \lambda_{j_1}(k-l) - \lambda_{j_2}(l) \neq 0,$$

is satisfied where $\lambda_j(k)$ is the eigenvalue associated to the variable $\widehat{u}_j(k)$.

As an example we consider the term $\widehat{w}_1(k-l)\widehat{w}_1(l)$ in the equation for $\widehat{w}_1(k)$. The eigenvalue associated to $\widehat{w}_1(k)$ is $i\omega_0 + \mathcal{O}(k^2)$ for small $|k|$, such that

$$\begin{aligned}
&\lambda_j(k) - \lambda_{j_1}(k-l) - \lambda_{j_2}(l) \\
&= \quad i\omega_0 + \mathcal{O}(k^2) - i\omega_0 + \mathcal{O}((k-l)^2) - i\omega_0 + \mathcal{O}(l^2) \\
&= \quad -i\omega_0 + \mathcal{O}(k^2 + (k-l)^2 + l^2) \neq 0,
\end{aligned}$$

i.e., $\widehat{w}_1(k-l)\widehat{w}_1(l)$ can be eliminated in the equation for $\widehat{w}_1(k)$ for $k^2 + (k-l)^2 + l^2$ sufficiently small.

We use this idea to eliminate a number of terms in the equations for (7) and (8). In the following table we collect the temporal and spatial wave numbers of the quadratic terms w.r.t. $\widehat{w}_{\pm 1}$ and $\widehat{z}_{\pm 1}$. Let us consider one example. Since the quadratic interaction $\widehat{w}_1(k-l)\widehat{w}_1(l)$ has a support in Fourier space located around $0k_c$ and oscillates in time approximately as $e^{2i\omega_0 t}$, we write $0k_c, 2i\omega_0$ at the position labelled horizontally and vertically with w_1.

	w_1	w_{-1}	z_1	z_{-1}
w_1	$0k_c, 2i\omega_0$	$0k_c, 0\omega_0$	$k_c, i\omega_0$	$-k_c, i\omega_0$
w_{-1}	$0k_c, 0\omega_0$	$0k_c, -2i\omega_0$	$k_c, -i\omega_0$	$-k_c, -i\omega_0$
z_1	$k_c, i\omega_0$	$k_c, -i\omega_0$	$2k_c, 0\omega_0$	$0k_c, 0\omega_0$
z_{-1}	$-k_c, i\omega_0$	$-k_c, -i\omega_0$	$0k_c, 0\omega_0$	$-2k_c, 0\omega_0$

i) In the $\widehat{w}_1$-equation by a near identity change of variables all terms can be eliminated if they do not have a $0k_c, i\omega_0$ in the previous table. In the following table the terms which can be eliminated in the $\widehat{w}_1$-equation by such a procedure are denoted with NF. The terms which are not present in the $\widehat{w}_1$-equation due to disjoint supports in Fourier space are denoted with DS:

	w_1	w_{-1}	z_1	z_{-1}
w_1	NF	NF	DS	DS
w_{-1}	NF	NF	DS	DS
z_1	DS	DS	DS	NF
z_{-1}	DS	DS	NF	DS

Hence, in the $\widehat{w}_1$-equation all quadratic interactions of critical modes are gone after such near identity changes of variables.

ii) In the $\widehat{z}_1$-equation by a near identity change of variables terms can be eliminated if they do not have a $k_c, 0\omega_0$ in the previous table. In the following table the terms which can be eliminated in the $\widehat{z}_1$-equation by such a procedure are denoted with NF. The terms which are not present in the $\widehat{z}_1$-equation due to disjoint supports in Fourier space are denoted with DS:

	w_1	w_{-1}	z_1	z_{-1}
w_1	DS	DS	NF	DS
w_{-1}	DS	DS	NF	DS
z_1	NF	NF	DS	DS
z_{-1}	DS	DS	DS	DS

Hence, in the $\widehat{z}_1$-equation all quadratic interactions of critical modes are gone after such near identity changes of variables.

After these change of variables, for details see Section 2.3.2, our system is of the form

$$\begin{aligned} \partial_t \widehat{W}_1(k,t) &= \lambda_1(k)\widehat{W}_1(k,t) + \widehat{h}_w(\widehat{W}_1, \widehat{Z}_1, \widehat{V}_s)(k,t), && (10) \\ \partial_t \widehat{Z}_1(k,t) &= \lambda_1(k)\widehat{Z}_1(k,t) + \widehat{h}_z(\widehat{W}_1, \widehat{Z}_1, \widehat{V}_s)(k,t), && (11) \\ \partial_t \widehat{V}_s(k,t) &= \widehat{\Lambda}_s(k)\widehat{V}_s(k,t) + \widehat{h}_s(\widehat{W}_1, \widehat{Z}_1, \widehat{V}_s)(k,t), && (12) \end{aligned}$$

where formally

$$\begin{aligned} \widehat{h}_*(\widehat{W}_1, \widehat{Z}_1, \widehat{V}_s) &= \mathcal{O}(|\widehat{W}_1|^3 + |\widehat{Z}_1|^3 + |\widehat{W}_1|\|\widehat{V}_s\| + |\widehat{Z}_1|\|\widehat{V}_s\| + \|\widehat{V}_s\|^2), \\ \widehat{h}_s(\widehat{W}_1, \widehat{Z}_1, \widehat{V}_s) &= \mathcal{O}(|\widehat{W}_1|^2 + |\widehat{Z}_1|^2 + \|\widehat{V}_s\|^2), \end{aligned}$$

for $* = w, z$. In order to simplify the notations in the following we introduce $V_c = (W_1, Z_1)$ such that (10)-(12) can be written, with a slight abuse of notation, as

$$\begin{aligned} \partial_t \widehat{V}_c(k,t) &= \widehat{\Lambda}_c(k)\widehat{V}_c(k,t) + \widehat{h}_c(\widehat{V}_c, \widehat{V}_s)(k,t), \\ \partial_t \widehat{V}_s(k,t) &= \widehat{\Lambda}_s(k)\widehat{V}_s(k,t) + \widehat{h}_s(\widehat{V}_c, \widehat{V}_s)(k,t), \end{aligned}$$

For estimating the solutions we consider this system in physical space

$$\begin{aligned} \partial_t V_c &= \Lambda_c V_c + h_c(V_c, V_s), && (13) \\ \partial_t V_s &= \Lambda_s V_s + h_s(V_c, V_s), && (14) \end{aligned}$$

where $\Lambda_j = \mathcal{F}^{-1}\widehat{\Lambda}_j\mathcal{F}$ for $j = c, s$ and where the nonlinear terms obey the estimates

$$\begin{aligned} \|h_c(V_c, V_s)\|_{H^\theta_{l,u}} &\leq C(\|V_c\|^3_{H^\theta_{l,u}} + \|V_c\|_{H^\theta_{l,u}}\|V_s\|_{H^\theta_{l,u}} + \|V_s\|^2_{H^\theta_{l,u}}), \\ \|h_s(V_c, V_s)\|_{H^\theta_{l,u}} &\leq C(\|V_c\|^2_{H^\theta_{l,u}} + \|V_s\|^2_{H^\theta_{l,u}}), \end{aligned}$$

in a neighborhood of the origin for $\theta \geq 1$.

2.3 Some functional analytic details

So far all calculations in Section 2 have been made in Fourier space. In order to work with (13)-(14) in physical space and to prove Theorem 1.2 we need estimates for the mode filters $E_j = \mathcal{F}^{-1}\widehat{E}_j\mathcal{F}$ in $H^\theta_{l,u}$-spaces. They are given in Section 2.3.1. Details and bounds for the normal form transformations used in Section 2.2 are provided in Section 2.3.2.

2.3.1 The mode filters

The mode filters E_w, E_z, and E_s are bounded linear mappings in every $H^\theta_{l,u}$-space due to the following multiplier lemma, cf. [Sch94b].

Lemma 2.1. *Let $m \in \mathbb{Z}$, $q \in \mathbb{N}_0$ with $m+q \geq 0$, and $k \mapsto (1+k^2)^{m/2}\widehat{M}(k) \in C^2_b(\mathbb{R}, L(\mathbb{R}^d, \mathbb{R}^d))$. Then $M = \mathcal{F}^{-1}\widehat{M}\mathcal{F} : H^q_{l,u} \to H^{q+m}_{l,u}$ is bounded with norm*

$$\leq C(q, m)\|k \mapsto (1 + k^2)^{m/2}\widehat{M}(k)\|_{C^2_b(\mathbb{R},L(\mathbb{R}^d,\mathbb{R}^d))},$$

where C(q,m) does not depend on M.

2.3.2 The near identity change of variables

The near identity change of variables which transforms (7)-(9) into (10)-(12) is of the form

$$\begin{aligned} \widehat{w}_1 &= \widehat{W}_1 + \widehat{M}_1(\widehat{w}_1, \widehat{w}_1) + \widehat{M}_2(\widehat{w}_{-1}, \widehat{w}_{-1}) + \widehat{M}_3(\widehat{w}_1, \widehat{w}_{-1}) + \widehat{M}_4(\widehat{z}_1, \widehat{z}_{-1}), \\ \widehat{z}_1 &= \widehat{Z}_1 + \widehat{M}_5(\widehat{z}_1, \widehat{w}_{-1}) + \widehat{M}_6(\widehat{z}_1, \widehat{w}_1), \end{aligned}$$

and similarly for $\widehat{w}_{-1}$ and $\widehat{z}_{-1}$, where the $\widehat{M}_j$ are suitably chosen symmetric bilinear mappings. Let us consider one example. Let $\widehat{Q}_1(\widehat{w}_1, \widehat{w}_1)$ be the bilinear terms in the $\widehat{w}_1$-equation. In Fourier space this term is of the form

$$\widehat{Q}_1(\widehat{w}_1,\widehat{w}_1)(k,t) = \int q(k,k-l,l)\widehat{w}_1(k-l,t)\widehat{w}_1(l,t)dl,$$

with a bounded kernel $q = q(k,k-l,l)$. Following [SU17, §11.4] we have to choose

$$\widehat{M_1}(\widehat{w}_1,\widehat{w}_1)(k,t) = \int(\lambda_1(k)-\lambda_1(k-l)-\lambda_1(l))^{-1}q(k,k-l,l)\widehat{w}_1(k-l,t)\widehat{w}_1(l,t)dl$$

in order to eliminate $\widehat{Q}_1(\widehat{w}_1,\widehat{w}_1)$ in the $\widehat{w}_1$-equation. Since all bilinear terms in the transformations have compact support in Fourier space and since as an example $(\lambda_1(k)-\lambda_1(k-l)-\lambda_1(l))^{-1}q(k,k-l,l)$ is smooth w.r.t. the wave numbers k and l in a neighborhood of the Fourier support of $\widehat{w}_1$ the bilinear terms are arbitrarily smooth as a mapping from $H^\theta_{l,u}$ to $H^\theta_{l,u}$ and so in a neighborhood of the origin $(w_1,z_1,v_s) = (0,0,0)$ the normal form transform can be inverted using Neumann's series. Therefore, we have

Theorem 2.2. *For each $\theta \geq 1$ we have $\varrho, \varepsilon_0, C > 0$ such that for all $\varepsilon \in (0,\varepsilon_0)$ the following holds: There exists a smooth change of coordinates $\phi : H^\theta_{l,u} \cap \{\|u\|_{H^\theta_{l,u}} \leq \varrho\} \to H^\theta_{l,u}$ such that the inverse Fourier transform of* (7)-(9) *transforms into the inverse Fourier transform of* (10)-(12)*. The transform $V = \Phi(v)$ fulfills $\|\Phi(v) - v\|_{H^\theta_{l,u}} \leq C\|v\|^2_{H^\theta_{l,u}}$.*

3 The error estimates

By adding higher order terms to the approximation $\Phi(\varepsilon\Psi_{app})$, with $\varepsilon\Psi_{app}$ defined in (2), we can construct an approximation $\varepsilon\Psi$ for which the terms which do not cancel after inserting the improved approximation $\varepsilon\Psi$ into the equations (13)-(14), are sufficiently small for our purposes. The terms which do not cancel are collected in the residual

$$\begin{aligned} \mathrm{Res}_c(V_c,V_s) &= -\partial_t V_c + \Lambda_c V_c + h_c(V_c,V_s), \\ \mathrm{Res}_s(V_c,V_s) &= -\partial_t V_s + \Lambda_s V_s + h_s(V_c,V_s). \end{aligned}$$

For our purposes it is sufficient to have an approximation $\varepsilon\Psi$ with the following properties:

Lemma 3.1. *For* $(A,B) \in C([0,T_0],(H^4_{l,u}(\mathbb{R}))^2)$ *there exists an approximation* $\varepsilon\Psi = (\varepsilon\Psi_c, \varepsilon^2\Psi_s)$ *with*

$$\begin{array}{ll} \mathbf{i)} & \sup_{t\in[0,T_0/\varepsilon^2]} \|\mathrm{Res}_c(\varepsilon\Psi)\|_{H^1_{l,u}} \leq C_{res}\varepsilon^4, \\ \mathbf{ii)} & \sup_{t\in[0,T_0/\varepsilon^2]} \|\mathrm{Res}_s(\varepsilon\Psi)\|_{H^1_{l,u}} \leq C_{res}\varepsilon^3, \end{array}$$

and

$$\sup_{t\in[0,T_0/\varepsilon^2]} \|\varepsilon\Psi - \Phi(\varepsilon\Psi_{app})\|_{H^1_{l,u}} = \mathcal{O}(\varepsilon^2).$$

Proof. We start with the approximation for (7)-(9) and set $A_1 = A$, $A_{-1} = \overline{A}$, $B_1 = B$, and $B_{-1} = \overline{B}$. The approximation is then given by

$$\begin{aligned} \varepsilon\Psi_{z_1}(x,t) &= \varepsilon A_1(\varepsilon x,\varepsilon^2 t)e^{ik_cx} \\ &\quad +\varepsilon^2 A_{11}(\varepsilon x,\varepsilon^2 t)e^{ik_cx}e^{i\omega_0 t} + \varepsilon^2 A_{1-1}(\varepsilon x,\varepsilon^2 t)e^{ik_cx}e^{-i\omega_0 t} \\ &\quad +\varepsilon^3 A_{111}(\varepsilon x,\varepsilon^2 t)e^{ik_cx}e^{2i\omega_0 t} + \varepsilon^3 A_{1-1-1}(\varepsilon x,\varepsilon^2 t)e^{ik_cx}e^{-2i\omega_0 t}, \\ \varepsilon\Psi_{w_1}(x,t) &= \varepsilon B_1(\varepsilon x,\varepsilon^2 t)e^{i\omega_0 t} + \varepsilon^2 B_0(\varepsilon x,\varepsilon^2 t) \\ &\quad +\varepsilon^2 B_2(\varepsilon x,\varepsilon^2 t)e^{2i\omega_0 t} + \varepsilon^2 B_{-2}(\varepsilon x,\varepsilon^2 t)e^{-2i\omega_0 t} \\ &\quad +\varepsilon^3 B_3(\varepsilon x,\varepsilon^2 t)e^{3i\omega_0 t} + \varepsilon^3 B_{-1*}(\varepsilon x,\varepsilon^2 t)e^{-i\omega_0 t} + \varepsilon^3 B_{-3}(\varepsilon x,\varepsilon^2 t)e^{-3i\omega_0 t}, \\ \varepsilon^2\Psi_s(x,t) &= \varepsilon^2 S_2(\varepsilon x,\varepsilon^2 t)e^{2ik_cx} + \varepsilon^2 S_{-2}(\varepsilon x,\varepsilon^2 t)e^{-2ik_cx} + \varepsilon^2 S_0(\varepsilon x,\varepsilon^2 t) \\ &\quad +\varepsilon^2 S_{2*}(\varepsilon x,\varepsilon^2 t)e^{2i\omega_0 t} + \varepsilon^2 S_{-2*}(\varepsilon x,\varepsilon^2 t)e^{-2i\omega_0 t} \\ &\quad +\varepsilon^2 S_{11}(\varepsilon x,\varepsilon^2 t)e^{ik_cx}e^{i\omega_0 t} + \varepsilon^2 S_{1-1}(\varepsilon x,\varepsilon^2 t)e^{ik_cx}e^{-i\omega_0 t} \\ &\quad +\varepsilon^2 S_{-11}(\varepsilon x,\varepsilon^2 t)e^{-ik_cx}e^{i\omega_0 t} + \varepsilon^2 S_{-1-1}(\varepsilon x,\varepsilon^2 t)e^{-ik_cx}e^{-i\omega_0 t} \end{aligned}$$

and similarly for $\varepsilon\Psi_{z_{-1}}$ and $\varepsilon\Psi_{w_{-1}}$. Inserting this ansatz into (7)-(9) and equating the coefficients in front of $\varepsilon^2 e^{2ik_cx}$, $\varepsilon^2 e^{-2ik_cx}$, ε^2, $\varepsilon^2 e^{2i\omega_0 t}$ and $\varepsilon^2 e^{-2i\omega_0 t}$ to zero yields equations of the form

$$\begin{aligned} 0 &= B_0 + b_0(A_1,A_{-1}) + b_0(B_1,B_{-1}), \\ 0 &= B_2 + b_2(B_1,B_1), \\ 0 &= B_{-2} + b_{-2}(B_{-1},B_{-1}), \\ 0 &= A_{11} + a_{11}(A_1,B_1), \\ 0 &= A_{1-1} + a_{1-1}(A_1,B_{-1}), \\ 0 &= A_{-11} + a_{-11}(A_{-1},B_1), \\ 0 &= A_{-1-1} + a_{-1-1}(A_{-1},B_{-1}), \end{aligned}$$

where $a_2, \ldots, a_{-1-1}$ are smooth bilinear mappings,

$$\begin{array}{rcl} 0 & = & S_2 + s_2(A_1, A_1), \\ 0 & = & S_{-2} + s_{-2}(A_{-1}, A_{-1}), \\ 0 & = & S_0 + s_0(A_1, A_{-1}) + b_0(B_1, B_{-1}), \\ 0 & = & S_{2*} + s_{2*}(B_1, B_1), \\ 0 & = & S_{-2*} + s_{-2*}(B_{-1}, B_{-1}), \\ 0 & = & S_{11} + s_{11}(A_1, B_1), \\ 0 & = & S_{1-1} + s_{1-1}(A_1, B_{-1}), \\ 0 & = & S_{-11} + s_{-11}(A_{-1}, B_1), \\ 0 & = & S_{-1-1} + s_{-1-1}(A_{-1}, B_{-1}), \end{array}$$

where $s_2, \ldots, s_{-1-1}$ are smooth bilinear mappings and

$$\begin{array}{rcl} 0 & = & B_3 + b_3(B_1, B_1, B_1), \\ 0 & = & B_{-1*} + b_{-1*}(B_1, B_{-1}, B_{-1}) + b_{-1**}(A_1, A_{-1}, B_{-1}), \\ 0 & = & B_{-3} + b_{-3}(B_{-1}, B_{-1}, B_{-1}), \\ 0 & = & A_{111} + a_{111}(A_1, B_1, B_1), \\ 0 & = & A_{1-1-1} + a_{1-1-1}(A_1, B_{-1}, B_{-1}), \\ 0 & = & A_{-111} + a_{-111}(A_{-1}, B_1, B_1), \\ 0 & = & A_{-1-1-1} + a_{-1-1-1}(A_{-1}, B_{-1}, B_{-1}), \end{array}$$

where the $b_3, \ldots, a_{-1-1-1}$ are smooth trilinear mappings. By construction we canceled in the w_1-equation and in the z_1-equation all terms of order $\mathcal{O}(\varepsilon^3)$. Moreover, in the v_s-equation all terms of order $\mathcal{O}(\varepsilon^2)$ are canceled. The statement follows by defining $\varepsilon\Psi$ as Φ applied to the constructed approximation for (w_1, z_1, v_s). In order to derive the Ginzburg-Landau equations in Fourier space we have to expand λ_1 at $k = k_c$ and $k = 0$ up to quadratic terms. In order to estimate the error made by this approximation we use Lemma 2.1 and so there is a loss of three derivatives. □

Proof of Theorem 1.1. We set $\varepsilon\Psi_c = E_c(\varepsilon\Psi)$ and $\varepsilon^2\Psi_s = E_s(\varepsilon\Psi)$. By construction we have that $\varepsilon\Psi_c = \mathcal{O}(\varepsilon)$ and $\varepsilon^2\Psi_s = \mathcal{O}(\varepsilon^2)$ in $H^4_{l,u}$. By Sobolev's embedding theorem the same is true in C^1_b.

We introduce the error made by the improved approximation $(\varepsilon\Psi_c, \varepsilon^2\Psi_s)$ through

$$(V_c, V_s) = (\varepsilon\Psi_c, \varepsilon^2\Psi_s) + (\varepsilon^2 R_c, \varepsilon^3 R_s).$$

Inserting this into (13)-(14) shows that the error functions satisfy

$$\begin{aligned} \partial_t R_c &= \Lambda_c R_c + N_c(\varepsilon \Psi_c, \varepsilon^2 R_c, \varepsilon^2 \Psi_s, \varepsilon^3 R_s), & (15) \\ \partial_t R_s &= \Lambda_s R_s + N_s(\varepsilon \Psi_c, \varepsilon^2 R_c, \varepsilon^2 \Psi_s, \varepsilon^3 R_s), & (16) \end{aligned}$$

where

$$\begin{aligned} \|N_c\|_{H^1_{l,u}} &\leq C_{1,c}\varepsilon^2(\|R_c\|_{H^1_{l,u}} + \|R_s\|_{H^1_{l,u}}) \\ &\quad + C_{2,c}(M_c, M_s)\varepsilon^3(\|R_c\|_{H^1_{l,u}} + \|R_s\|_{H^1_{l,u}})^2 + C_{res}\varepsilon^2, \\ \|N_s\|_{H^1_{l,u}} &\leq C_{1,s}(\|R_c\|_{H^1_{l,u}} + \varepsilon\|R_s\|_{H^1_{l,u}}) \\ &\quad + C_{2,s}(M_c, M_s)\varepsilon(\|R_c\|_{H^1_{l,u}} + \|R_s\|_{H^1_{l,u}})^2 + C_{res}, \end{aligned}$$

as long as $\|R_c\|_{H^1_{l,u}} \leq M_c$ and $\|R_s\|_{H^1_{l,u}} \leq M_s$ for any ε-independent, but fixed constants M_c and M_s. We apply the variation of constants formula to the equations for the error (15)-(16) and obtain

$$\begin{aligned} R_c(t) &= e^{\Lambda_c t} R_c(0) + \int_0^t e^{\Lambda_c(t-\tau)} N_c(\varepsilon \Psi_c, \varepsilon^2 R_c, \varepsilon^2 \Psi_s, \varepsilon^3 R_s)(\tau) d\tau, & (17) \\ R_s(t) &= e^{\Lambda_s t} R_s(0) + \int_0^t e^{\Lambda_s(t-\tau)} N_s(\varepsilon \Psi_c, \varepsilon^2 R_c, \varepsilon^2 \Psi_s, \varepsilon^3 R_s)(\tau) d\tau, & (18) \end{aligned}$$

where the semigroups obey the estimates

$$\|e^{\Lambda_c t}\|_{H^1_{l,u} \to H^1_{l,u}} \leq C_\Lambda e^{\sigma_c \varepsilon^2 t} \qquad \text{and} \qquad \|e^{\Lambda_s t}\|_{H^1_{l,u} \to H^1_{l,u}} \leq C_\Lambda e^{-\sigma_s t},$$

with $\sigma_c > 0$ and $\sigma_s > 0$. We introduce

$$S_c(t) = \sup_{0 \leq \tau \leq t} \|R_c(\tau)\|_{H^1_{l,u}} \qquad \text{and} \qquad S_s(t) = \sup_{0 \leq \tau \leq t} \|R_s(\tau)\|_{H^1_{l,u}}.$$

Using the semigroup estimates we find the inequalities

$$\begin{aligned} \|R_c(t)\|_{H^1_{l,u}} &\leq C_\Lambda e^{\sigma_c T_0} S_c(0) + \int_0^t C_\Lambda e^{\sigma_c T_0}(C_{1,c}\varepsilon^2(S_c(\tau) + S_s(\tau)) \\ &\quad + C_{2,c}(M_c, M_s)\varepsilon^3(S_c(\tau) + S_s(\tau))^2) d\tau + T_0 C_\Lambda e^{\sigma_c T_0} C_{res}, \\ \|R_s(t)\|_{H^1_{l,u}} &\leq C_\Lambda S_s(0) + C_\Lambda C_\sigma(C_{1,s} S_c(t) + C_{1,s}\varepsilon S_s(t) \\ &\quad + C_{2,s}(M_c, M_s)\varepsilon(S_c(t) + S_s(t))^2) + C_\Lambda C_\sigma C_{res}, \end{aligned}$$

where $C_\sigma = \sup_{t\in[0,T_0/\varepsilon^2]} \int_0^t e^{-\sigma_s(t-\tau)} d\tau < \infty$ is independent of $0 \leq \varepsilon \ll 1$. The functions $C_{2,c}(M_c, M_s)$ and $C_{2,s}(M_c, M_s)$ are smooth functions w.r.t. M_c, M_s. This is now exactly the set of inequalities which appeared in [SU17, p.342], and so the rest of the proof of Theorem 1.2 follows line for line as in [SU17, §10.4]. The sup-estimate stated in Theorem 1.2 follows from the $H^1_{l,u}$-estimate by a generalization of Sobolev's embedding theorem. □

Remark 3.2. For completeness we recall the underlying idea of the rest of the proof. Since the right-hand sides of the last estimates increase monotonically w.r.t. t we can replace $\|R_c(t)\|_{H^1_{l,u}}$ and $\|R_s(t)\|_{H^1_{l,u}}$ on the left-hand side by $S_c(t)$ and $S_s(t)$, respectively. From the second inequality we then obtain an estimate $S_s(t) \leq CS_c(t) + C$ for $\varepsilon > 0$ sufficiently small. Inserting this in the first inequality yields

$$S_c(t) \leq C + \varepsilon^2 \int_0^t CS_c(\tau) d\tau,$$

again for $\varepsilon > 0$ sufficiently small. Gronwall's inequality immediately yields the required estimate. We refer to [SU17, §10.4] for the missing details.

4 Examples

In this section we show that the Brusselator, the Gierer-Meinhardt system and the Schnakenberg model are three examples of systems for which the trivial solution becomes simultaneously unstable via a short-wave Turing and a long-wave Hopf instability. The Gierer-Meinhardt system is of activator-inhibitor type, whereas the Brusselator and Schnakenberg model are activator-substrate systems. Our presentation of the Brusselator is based on [Kur84] and our presentation of Gierer-Meinhardt system is based on [HO78].

4.1 The Brusselator

The Brusselator [Kur84] is a two component system given by

$$\begin{aligned} \partial_t \xi &= d_1 \partial_x^2 \xi + a - (b+1)\xi + \xi^2 \eta, \\ \partial_t \eta &= d_2 \partial_x^2 \eta + b\xi - \xi^2 \eta, \end{aligned}$$

where a, b, d_1, d_2 are non-negative constants. The two components ξ and η are real-valued functions of $t \geq 0$ and $x \in \mathbb{R}$. There exists a unique spatially homogeneous trivial solution $(\xi_0, \eta_0) = (a, b/a)$. The deviation

$$(u, v) = (\xi - \xi_0, \eta - \eta_0)$$

from (ξ_0, η_0) satisfies the system

$$\partial_t u = (b-1)u + a^2 v + d_1 \partial_x^2 u + f(u,v), \tag{19}$$
$$\partial_t v = -bu - a^2 v + d_2 \partial_x^2 v - f(u,v), \tag{20}$$

with nonlinear terms

$$f(u,v) = (b/a)u^2 + 2auv + u^2 v.$$

The stability of the trivial solution $(u, v) = (0, 0)$ is determined by the linearization of (19)-(20) which is solved by

$$(u, v) = (\widehat{u}, \widehat{v}) e^{ikx + \lambda(k)t}.$$

The eigenvalues λ are the roots of the quadratic equation

$$\lambda^2 + \alpha(k)\lambda + \beta(k) = 0,$$

where

$$\begin{aligned} \alpha(k) &= 1 + a^2 - b + (d_1 + d_2)k^2, \\ \beta(k) &= a^2 + (a^2 d_1 + (1-b)d_2)k^2 + d_1 d_2 k^4. \end{aligned}$$

In the following we fix d_1, d_2, and a, and take b as a control parameter.

i) The long-wave Hopf instability occurs at the critical wave vector $k = 0$ and for $b = b_{hopf}(a) = 1 + a^2$

ii) The short-wave Turing instability occurs at the critical wave number $k = k_c = \sqrt{a/\sqrt{d_1 d_2}}$ if $b = b_{turing}(a, D) = (1 + a/\sqrt{D})^2$ for the control parameter, where $D = d_2/d_1$.

In this paper we are interested in the case when both instabilities occur simultaneously, i.e., when $b_{turing}(a, D) = b_{hopf}(a)$, or equivalently when

$$\sqrt{D} = a/(\sqrt{1 + a^2} - 1).$$

We introduce the small bifurcation parameter $\varepsilon^2 = (b - b_{hopf})/b_{hopf}$. Drawing the curves of eigenvalues $\lambda_{1,2}$ gives a figure similar to Figure 1.

4.2 The Gierer-Meinhardt system

The normalized Gierer-Meinhardt model [GM72] is given by

$$\begin{aligned} \partial_t a &= d_1 \partial_x^2 a + \frac{a^2}{h} - \mu a + \rho, & (21) \\ \partial_t h &= d_2 \partial_x^2 h + a^2 - h, & (22) \end{aligned}$$

where d_1, d_2, μ, and ρ are non-negative constants. The activator a and the inhibitor h are real-valued functions of $t \geq 0$ and $x \in \mathbb{R}$. There exists a unique spatially homogeneous trivial solution $(a, h) = (a_0, h_0)$, with

$$a_0 = \frac{1}{\mu}(\rho + 1), \qquad h_0 = a_0^2.$$

The deviation $(q_1, q_2) = (a - a_0, h - h_0)$ from the trivial solution (a_0, h_0) satisfies

$$\begin{aligned} \partial_t q_1 &= d_1 \partial_x^2 q_1 + \mu(\frac{2}{\rho+1} - 1)q_1 - \frac{\mu^2}{(\rho+1)^2} q_2 + N_1(q_1, q_2), \\ \partial_t q_2 &= d_2 \partial_x^2 q_2 + \frac{2}{\mu}(\rho+1) q_1 - q_2 + N_2(q_1, q_2), \end{aligned}$$

with $|N_j(q_1, q_2)| = \mathcal{O}(q_1^2 + q_2^2)$. The linearization yields the eigenvalues

$$\lambda_\pm = \frac{\alpha(k)}{2} \pm \sqrt{\frac{\alpha^2(k)}{4} - \beta(k)},$$

where

$$\begin{aligned} \alpha(k) &= -(d_1 + d_2) k^2 + \frac{2\mu}{\rho+1} - \mu - 1, \\ \beta(k) &= \left(d_1 k^2 + \mu\right)\left(1 + d_2 k^2\right) - \frac{2\mu d_2 k^2}{\rho+1}. \end{aligned}$$

i) Therefore, a long-wave Hopf instability at $k = 0$ occurs if

$$\alpha(0) = 0 \qquad \text{and} \qquad \frac{\alpha^2(0)}{4} - \beta(0) = -\mu < 0.$$

This leads to

$$\rho_{hopf}(\mu) = \frac{\mu - 1}{\mu + 1} \qquad \text{and} \qquad \lambda_\pm(0) = \pm i\sqrt{\mu}.$$

ii) A short-wave Turing instability at $k = k_c$ occurs if $\lambda_+(k_c) = 0$. Hence, we immediately obtain the condition $\beta = 0$. Additionally we have to satisfy $\partial_k \lambda_+(k_c) = 0$. This finally leads to

$$\rho_{turing}(\mu, D) = \frac{2\sqrt{\mu D}}{2 + \sqrt{\mu D} + \dfrac{1}{\sqrt{\mu D}}} - 1 \qquad \text{and} \qquad k_c = \sqrt[4]{\frac{\mu}{d_1 d_2}},$$

where $D = d_2/d_1$. As before, we are interested in the case when both instabilities occur simultaneously, i.e., when $\rho_{turing}(\mu, D) = \rho_{hopf}(\mu)$, or equivalently when

$$\sqrt{D} = \sqrt{\mu} + \sqrt{\mu + 1}.$$

We introduce the small bifurcation parameter $\varepsilon^2 = (\rho - \rho_{hopf})/\rho_{hopf}$. Drawing the curves of eigenvalues $\lambda_\pm$ gives a figure similar to Figure 1.

4.3 The Schnakenberg model

As the third example we consider the Schnakenberg model [Sch79] (see also [GM72]) which is given by

$$\begin{aligned} \partial_t \xi &= d_1 \partial_x^2 \xi - a\xi + \xi^2 \eta, \\ \partial_t \eta &= d_2 \partial_x^2 \eta + b - \xi^2 \eta, \end{aligned}$$

where a, b, d_1, d_2 are non-negative constants. The two components ξ and η are real-valued functions of $t \geq 0$ and $x \in \mathbb{R}$. There exists a unique spatially homogeneous trivial solution $(\xi_0, \eta_0) = (b/a, a^2/b)$. The deviation

$$(u, v) = (\xi - \xi_0, \eta - \eta_0)$$

from (ξ_0, η_0) satisfies the system

$$\partial_t u = d_1 \partial_x^2 u + au + \frac{b^2}{a^2} v + f(u, v), \tag{23}$$

$$\partial_t v = d_2 \partial_x^2 v - 2au - \frac{b^2}{a^2} v - f(u, v), \tag{24}$$

with nonlinear terms

$$f(u, v) = (a^2/b)u^2 + 2(b/a)uv + u^2 v.$$

The stability of the trivial solution $(u,v) = (0,0)$ is determined by the linearization of (23)-(24) which is solved by

$$(u,v) = (\widehat{u},\widehat{v})e^{ikx+\lambda(k)t}.$$

The eigenvalues λ are the roots of the quadratic equation

$$\lambda^2 + \alpha(k)\lambda + \beta(k) = 0,$$

where

$$\begin{aligned}\alpha(k) &= \frac{b^2}{a^2} - a + (d_1+d_2)k^2 \\ \beta(k) &= \frac{b^2}{a} + (\frac{b^2}{a^2}d_1 - ad_2)k^2 + d_1d_2k^4.\end{aligned}$$

In the following we fix d_1, d_2, and a, and take b as a control parameter.

i) The long-wave Hopf instability occurs at the critical wave vector $k = 0$ and for $b = b_{hopf}(a) = a^{3/2}$.

ii) The short-wave Turing instability occurs at the critical wave number $k_c = \sqrt{a/(d_1(1+\sqrt{2}))}$ if

$$b = b_{turing}(a,D) = (1+\sqrt{2})^{-1}\sqrt{D}a^{3/2}$$

for the control parameter, where $D = d_2/d_1$.

In this paper we investigate the case when both instabilities occur simultaneously, i.e., when $b_{turing}(a,D) = b_{hopf}(a)$, or equivalently when

$$\sqrt{D} = 1+\sqrt{2}.$$

We introduce the small bifurcation parameter $\varepsilon^2 = -(b-b_{hopf})/b_{hopf}$. Drawing the curves of eigenvalues $\lambda_{1,2}$ gives a figure similar to Figure 1.

References

[ADG85] D. Armbruster, G. Dangelmayr, and W. Güttinger. Imperfection sensitivity of interacting Hopf and steady-state bifurcations and their classification. *Physica D*, 16:99–123, 1985.

[BSSZ20] Simon Baumstark, Guido Schneider, Katharina Schratz, and Dominik Zimmermann. Effective slow dynamics models for a class of dispersive systems. *J. Dyn. Differ. Equations*, 32(4):1867–1899, 2020.

[CE90] Pierre Collet and Jean-Pierre Eckmann. The time dependent amplitude equation for the Swift-Hohenberg problem. *Comm. Math. Phys.*, 132:139–152, 1990.

[DWLDB96] A. De Wit, D. Lima, G. Dewel, and P. Borckmans. Spatiotemporal dynamics near a codimension-two point. *Phys. Rev. E*, 54:261–271, Jul 1996.

[Eck93] W. Eckhaus. The Ginzburg-Landau manifold is an attractor. *J. Nonlinear Sci.*, 3(3):329–348, 1993.

[FS14] Erwan Faou and Katharina Schratz. Asymptotic preserving schemes for the Klein-Gordon equation in the non-relativistic limit regime. *Numer. Math.*, 126(3):441–469, 2014.

[GM72] Alfred Gierer and Hans Meinhardt. A theory of biological pattern formation. *Kybernetik*, 12:30–39, 1972.

[HO78] H. Haken and H. Olbrich. Analytical treatment of pattern formation in the Gierer-Meinhardt model of morphogenesis. *J. Math. Biol.*, 6:317–331, 1978.

[HS20] T. Haas and G. Schneider. Failure of the n-wave interaction approximation without imposing periodic boundary conditions. *ZAMM*, pages online, https://doi.org/10.1002/zamm.201900230, 2020.

[JBB+01] W. Just, M. Bose, S. Bose, H. Engel, and E. Schöll. Spatiotemporal dynamics near a supercritical Turing-Hopf bifurcation in a two-dimensional reaction-diffusion system. *Phys. Rev. E*, 64:026219, Jul 2001.

[Kur84] Yoshiki Kuramoto. *Chemical Oscillations, Waves, and Turbulence.* Springer-Verlag New York, NY, 1984.

[LI80] W. F. Langford and G. Iooss. Interactions of Hopf and Pitchfork bifurcations. Bifurcation problems and their numerical solution, Workshop Dortmund 1980, ISNM 54, 103-134, 1980.

[MDWBS97] M. Meixner, A. De Wit, S. Bose, and E. Schöll. Generic spatiotemporal dynamics near codimension-two Turing-Hopf bifurcations. *Phys. Rev. E*, 55:6690–6697, Jun 1997.

[MS95] Alexander Mielke and Guido Schneider. Attractors for modulation equations on unbounded domains – existence and comparison. *Nonlinearity*, 8(5):743–768, 1995.

[PLNW20] Frédéric Paquin-Lefebvre, Wayne Nagata, and Michael J. Ward. Pattern Formation and Oscillatory Dynamics in a Two-Dimensional Coupled Bulk-Surface Reaction-Diffusion System. https://arxiv.org/pdf/1810.00251.pdf, 2020.

[Sch79] J. Schnakenberg. Simple chemical reaction systems with limit cycle behaviour. *Journal of Theoretical Biology*, 81(3):389–400, 1979.

[Sch94a] G. Schneider. A new estimate for the Ginzburg-Landau approximation on the real axis. *J. Nonlinear Sci.*, 4(1):23–34, 1994.

[Sch94b] Guido Schneider. Error estimates for the Ginzburg-Landau approximation. *ZAMP*, 45:433–457, 1994.

[Sch95] Guido Schneider. Validity and limitation of the Newell-Whitehead equation. *Math. Nachr.*, 176:249–263, 1995.

[Sch98] Guido Schneider. Hopf bifurcation in spatially extended reaction-diffusion systems. *J. Nonlinear Sci.*, 8(1):17–41, 1998.

[SSZ15] Guido Schneider, Danish Ali Sunny, and Dominik Zimmermann. The NLS approximation makes wrong predictions for the water wave problem in case of small surface tension and spatially periodic boundary conditions. *J. Dyn. Differ. Equations*, 27(3-4):1077–1099, 2015.

[SU17] Guido Schneider and Hannes Uecker. *Nonlinear PDEs. A dynamical systems approach.*, volume 182. Providence, RI: American Mathematical Society (AMS), 2017.

[vH91] A. van Harten. On the validity of the Ginzburg-Landau equation. *J. Nonlinear Sci.*, 1(4):397–422, 1991.