TR/04/85 February 1985

The Complex Grothendieck for 2x2 matrices

by
Andrew Tonge

Summary. We show that the complex Grothendieck constant for 2x2 matrices is 1

Grothendieck's inequality asserts that if (a_{ij}) is an $n\times n$ matrix and if $x_1 \dots, x_n, y_1, \dots, y_n$ are elements of the unit ball of a Hilbert space, then there is an absolute constant K such that

$$\left|\sum_{i,j=1}^{n} a_{ij} < x_i , y_j >\right| \leq k \| a \|_{\infty ,1} .$$

We have written $\|a\|_{\infty,1}$ for the norm of the matrix (a_{ij}) considered as a linear operator from ℓ_∞^n to ℓ_1^n. Thus

$$\| a \|_{\infty,1} = \sup \left\{ \left| \sum_{i,j=1}^{n} a_{ij} s_i t_j \right| : | s_i | \leq 1, | t_j | \leq 1, 1 \leq i, j \leq n \right\}$$

When we say that K is an absolute constant, we mean that K is independent of n , of the matrix (a_{ij}) and of the vectors $x_1,\dots,x_n, y_1,\dots,y_n$. The reader may consult [2] for a proof of Grothendieck's inequality and for supplementary information.

The best value of K is unknown, and there are differences between the real and complex cases. However, in both cases it is known that $K > 1$ · More precise regults may be found in [1] and [3].

Allan Sinclair (Edinburgh) has suggested that it would be of interest to determine the best universal constants $K_{m,n}$ such that if (a_{ij}) is an $n\times n$ matrix and if $x_1,\dots,x_n, y_1,\dots,y_n$ are elements of the unit ball of a Hilbert space, then

$$\left|\sum_{i,j=1}^{n} a_{ij} < x_i , y_j >\right| \leq k_{m,n} \| a \|_{\infty ,1}$$

For example, this would be a way to obtain information about the best value of K ·

Sinclair has obtained various estimates for $K_{m,n}$ (private communication) and has conjectured that in the complex case $K_{2,2} = 1$. (Results from [l] show that in the real case $K_{2,n} = \sqrt{2}$ for every $n > 1$.) We prove this conjecture and show, more generally, that in the complex case $K_{m,2} = 1$ for every m .

THEOREM. Let (a_{ij}) be a complex 2×2 matrix and let x_1, x_2, y_1 and y_2 be elements of the unit ball of a complex Hilbert space. Then

$$\left|\sum_{i,j=1}^{2} a_{ij} < x_i , y_j >\right| \leq \| a \|_{\infty ,1}$$

Proof. Simple manipulations reduce the problem to that of proving

$$\sup\{\|a_{11}x_1 + a_{21}x_2\| + \|a_{12}x_1 + a_{22}x_2\| : \|x_1\| = \|x_2\| = 1\}$$
$$\le \sup\left\{|a_{11}s_1 + a_{21}s_2| + |a_{12}s_1 + a_{22}s_2| : |s_1| = |s_2| = 1\right\}.$$

Writing norms (and absolute values) in terms of inner products, this inequality becomes

$$\sup\left\{\sqrt{u + \mathrm{Re}\ v < x_1, x_2 >} + \sqrt{p + \mathrm{Re}\ q < x_1, x_2 >} : \|x_1\| = \|x_2\| = 1\right\}$$

$$\le \sup\left\{\sqrt{u + \mathrm{Re}\ v < s_1, \overline{s_2} >} + \sqrt{p + \mathrm{Re}\ q s_1, \overline{s_2} >} : |s_1| = |s_2| = 1\right\},$$

where $u = |a_{11}|^2 + |a_{21}|^2$, $v = 2a_{11}\overline{a_{21}}$, $p = |a_{12}|^2 + |a_{22}|^2$ and $q = 2a\overline{_{12}a_{22}}$. The result will therefore follow if we can show that

$$\sup\left\{\sqrt{u + \mathrm{Re}\ vz} + \sqrt{p + \mathrm{Re}\ qz} : |z| \le 1\right\}$$
$$= \sup\left\{\sqrt{u + \mathrm{Re}\ vz} + \sqrt{p + \mathrm{Re}\ qz} : |z| = 1\right\}.$$

Now write v, q and z in terms of their real and imaginary parts, so that $v = \alpha + i\beta$, $q = \lambda + i\mu$ and $z = x + iy$. Then if we set

$$f(x,y) = \sqrt{u + \alpha x - \beta y} + \sqrt{p + \lambda x - \mu y}$$

we have to prove that

$$\sup\left\{f(x,y) : x^2 + y^2 \le 1\right\} = \sup\left\{f(x,y) : x^2 + y^2 = 1\right\}.$$

[Note that when $x^2 + y^2 \le 1$, $f(x,y)$ is well-defined as a real number, since $u > |v|$ and $p > |q|$.]

This identity will certainly follow if we can show that f has no local maximum in the open disc $x^2 + y^2 < 1$. In this disc f is continuously differentiable, provided that $u \ge |v| > 0$ and $P \ge |q| > 0$. The partial derivatives are

$$\partial f/\partial x = \frac{1}{2}\alpha\,(u + \alpha x - \beta y)^{-\frac{1}{2}} + \frac{1}{2}\lambda\,(p + \lambda x - uy)^{-\frac{1}{2}} \text{ and}$$

$$\partial f/\partial y = -\frac{1}{2}\beta\,(u + \alpha x - \beta y)^{-\frac{1}{2}} - \frac{1}{2}u\,(p + \lambda x - uy)^{-\frac{1}{2}}.$$

A local maximum can only occur if $\partial f/\partial x = \partial f/\partial y = 0$, and this is impossible when the following conditions are satisfied.

(i) α , β , λ and μ are all non-zero, and

(ii) $\alpha u \neq \beta\lambda$.

However, if these conditions are not satisfied, then for any given $\varepsilon > 0$ we can select a matrix (a'_{ij}) such that

$$|a_{ij} - a'_{ij}| \leq \varepsilon/8 \qquad (1 \leq i, j \leq 2)$$

and such that the numbers α', β', λ' and μ' associated with (a'_{ij}) are all non-zero and do satisfy $\alpha'\mu' \neq \beta'\lambda'$. But then

$$\sup \left\{ \left| \sum_{i,j=1}^{2} a_{ij} < x_i, y_j > \right| : \| x_i \| \leq 1, \| y_j \| \leq 1, 1 \leq i, j \leq 2 \right\}$$

$$\leq \frac{1}{2}\varepsilon + \sup \left\{ \left| \sum_{i,j=1}^{2} a'_{ij} < x_i, y_j > \right| : \| x_i \| \leq 1, \| y_j \| \leq 1, 1 \leq i, j \leq 2 \right\}$$

$$\leq \frac{1}{2}\varepsilon + \| a' \|_{\infty,1}$$

$$\leq \varepsilon + \| a \|_{\infty,1}$$

Let $\varepsilon \to 0$ to get the result we want.

References.

[1] J.L .Krivine. Constantes de Grothendieck et fonctions de type positif sur les spheres. Adv. in Math, 31 (1979) 16-30.

[2] J. Lindenstrauss and L.Tzafriri. Classical Banach spaces I. Springer (1977).

[3] G.Pisier. Grothendieck' s theorem for non-cotntnutative C*-algebras with an appendix on Grothendieck's constants. J. Funct. Anal. 29 (1978) 397-415.