CONVEXITY OF BÉZIER NETS ON SUB-TRIANGLES

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by

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Abstract: This note generalizes a result of Goodman[3], where it is shown that the convexity of Bèzier nets defined on a base triangle is preserved on sub-triangles obtained from a mid-point subdivision process. Here we show that the convexity of Bèzier nets is preserved on and only on sub-triangles that are "parallel" to the base triangle.

1. Introduction

Let T be a triangle, called the base triangle (see [1]), with vertices \( V_1, V_2, \text{and} V_3 \). (Here, and elsewhere in the paper, we assume that triangles are non-degenerate, that is, their vertices are not collinear.) Then each point \( P \) of the plane determined by \( V_1, V_2, \text{and} V_3 \) can be represented by its barycentric coordinates \((u,v,w)\) with respect to the base triangle \( T \) as

\[
P = uV_1 + vV_2 + wV_3, \quad u+v+w = 1.
\]

Denote \( f_n \) as a set of \((n+1)(n+2)/2\) values

\[
f_n = \{ f_{ijk} \in \mathbb{R} \mid i + j + k = n \ i \geq 0, j \geq 0, k \geq 0, \}
\]
The Bernstein polynomial of $f_n$ over $T$ is then given by

$$B(f_n; u, v, w) = \sum_{i+j+k=n} f_{ijk} B_{ijk}^n (u, v, w)$$

Where

$$B_{ijk}^n (u, v, w) = \frac{n!}{i!j!k!} u^i v^j w^k$$

are Bernstein basis functions. If we let shift operators $E_1, E_2$ and $E_3$ with respect to $T$ be defined by

$$E_1 f_{ijk} = f_{i+1,j,k}, \quad E_2 f_{ijk} = f_{i,j+1,k}, \quad E_3 f_{ijk} = f_{i,j,k+1},$$

then the Bernstein polynomial can be represented symbolically as

$$B(f_n; u, v, w) = (uE_1 + vE_2 + wE_3)^n f_{000}.$$ 

Now we consider $(n+1)(n+2)/2$ points of $T$ with barycentric coordinates $(i/n, j/n, k/n)$, namely,

$$P_{ijk} = (i/n)V_1 + (j/n)V_2 + (k/n)V_3,$$

where $i + j + k = n$.

Connecting

$$P_{i+1,j,k}, P_{i,j+1,k}, P_{i,j,k+1},$$

we obtain a triangle, denoted $U_{ijk}$ for $i+j+k=n-1$. Similarly, a triangle $W_{ijk}$ is obtained for $i+j+k=n+1$ with

$$P_{i-1,j,k}, P_{i,j-1,k}, P_{i,j,k-1},$$

as its vertices. A piecewise linear function $L(f_n; u, v, w)$, or $L(f_n)$ for short, is defined on $T$ such that it satisfies

$$L(f_n; P_{ijk}) = f_{ijk}, \quad i + j + k = n, $$

and is linear on each triangle $U_{ijk}$ or $W_{ijk}$. We call $L(f_n; u, v, w)$ the Bézier net of $f_n$.

The approximation theory of Bernstein polynomials and their applications in CAGD, Computer-aided Geometric Design, indicate that the Bézier net $L(f_n)$ is closely related to $B(f_n; u, v, w)$ and reflects certain features of $B(f_n; u, v, w)$. As this note is concerned with the convexity, we state the following results (see [1,2]):
Theorem 1  (i) If the Bézier net $L(f_n;u,v,w)$ is convex with respect to $T$, so is the Bernstein polynomial $B(f_n;u,v,w)$.

(ii) The Bézier net $L(f_n;u,v,w)$ is convex with respect to $T$ if and only if

$$
\begin{align*}
&f_{i+2,j,k} + f_{i,j+1,k+1} \geq f_{i+1,j+1,k} + f_{i+1,j,k+1}, \\
f_{i,j+2,k} + f_{i+1,j,k+1} \geq f_{i+1,j+1,k} + f_{i,j+1,k+1}, \\
f_{i,j,k+2} + f_{i+1,j+1,k} \geq f_{i+1,j,k+1} + f_{i,j,k+1},
\end{align*}
$$

for $i+j+k=n-2$.

Using the shift operators, one may rewrite (8) as

$$
\begin{align*}
\left( E_{i_1} - E_{i_2} \right) \left( E_{i_2} - E_{i_3} \right) f_{ijk} \geq 0, \quad i + j + k = n - 2,
\end{align*}
$$

for any permutation $\{i_1,i_2,i_3\}$ of $\{1,2,3\}$.

Let $V_i^*$, $V_2^*$ and $V_3^*$ be the vertices of another triangle $T^*$ in the same plane as $T$ and let

$$
P = u^* V_i^* + v^* V_2^* + w^* V_3^*, \quad u^* + v^* + w^* = 1,
$$

define the barycentric coordinates $(u^*, v^*, w^*)$ of $P$ with respect to $T^*$. Assume that $V_i^*$ has barycentric coordinates $(u_i, v_i, w_i)$ with respect to $T$, that is,

$$
V_i^* = u_i V_i + v_i V_2 + w_i V_3, \quad u_i + v_i + w_i = 1
$$

for $i=1, 2, 3$. Then

$$
\begin{align*}
&u = u^* u_i + v^* u_2 + w^* u_3, \\
v = u^* v_i + v^* v_2 + w^* v_3, \\
w = u^* w_i + v^* w_2 + w^* w_3.
\end{align*}
$$

A Bernstein polynomial on $T^*$ is defined symbolically by

$$
B(f_n^*;u^*, v^*, w^*) = \left( u^* E_1^* + v^* E_2^* + w^* E_3^* \right)^k f_{000}^*,
$$

Where

$$
f_n^* = \left\{ f_{ijk}^* \in R \mid i + j + k = n, i \geq 0, j \geq 0, k \geq 0 \right\}
$$

and $E_i^*$ define the shift operators on $T^*$. We then have:

Theorem 2  (Chang and Davis[1]) Let

$$
f_{ijk}^* = \left( u_i E_i + v_i E_2 + w_i E_3 \right) \left( u_j E_i + v_j E_2 + w_j E_3 \right) \left( u_k E_i + v_k E_2 + w_k E_3 \right)^k f_{000},
$$

-3-
for \(i+j+k=n\). Then \(B\left(f_n^*; u^*, v^*, w^*\right)\) is the Bernstein representation of \(B(f_n; u, v, w)\) with respect to \(T^*\).

**Proof:** From (12) and (15) we obtain, by equating coefficients,

\[
B(f_n; u, v, w) = (uE_1 + vE_2 + wE_3)^b f_{000}
\]

\[
= \left[u^*(u_1E_1 + v_1E_2 + w_1E_3) + v^*(u_2E_2 + v_2E_2 + w_2E_3) + w^*(u_3E_1 + v_3E_2 + w_3E_3)\right] f_{000}
\]

\[
= (u^*E_1^* + v^*E_2^* + w^*E_3^*) f_{000}
\]

if and only if

\[
E_i^*E_j^*E_k^* f_{000}^* = (u_1E_1 + v_1E_2 + w_1E_3)^i (u_2E_2 + v_2E_2 + w_2E_3)^j (u_3E_1 + v_3E_2 + w_3E_3)^k f_{000}^*.
\]

**Remark** In the above proof we note the identities

\[(16) \quad E_i^* = u_iE_i + v_iE_2 + w_iE_3, \quad i = 1, 2, 3,\]

where, in symbolic manipulation involving these operators, indices in operator expressions must sum to \(n\) and the operator expressions are applied to \(f_{000}^* = f_{000}\).

The set \(f_n^*\) determines a new Bezier net \(L(f_n^*; u^*, v^*, w^*)\) which is a piecewise linear function on \(T^*\). We call \(L(f_n^*)\) the restricted Bezier net of \(L(f_n)\) on \(T^*\). Naturally, \(T^*\) is called a sub-triangle of \(T\) if \(V_1^*, V_2^*, V_3^*\), the vertices of \(T^*\) are all inside or on the boundary of \(T\). In this case \(u_i, v_i, w_i \geq 0, \quad i = 1, 2, 3\).

Let \(T^*\) be a sub-triangle of \(T\). Then if \(B(f_n; u, v, w)\) is convex with respect to \(T\), so is \(B(f_n^*; u^*, v^*, w^*)\) with respect to \(T^*\). One would ask whether or not a similar result holds for the Bézier nets. Grandine[4] showed a negative result but in [3], Goodman shows that if \(T^*\) is a sub-triangle obtained from a "mid-point" subdivision process, then the convexity of \(L(f_n)\) does guarantee the convexity of the restricted Bézier net \(L(f_n^*)\) with respect to \(T^*\). In the next section, we generalize this result and show that only a very limited class of sub-triangles have the property of preserving the convexity of Bézier nets.
2. Main Result

Let \( T^* \) be a non-degenerate triangle that lies on the plane determined by the base triangle \( T \). We say \( T^* \) is parallel to \( T \) if each of the edges of \( T^* \) is parallel to one of those of \( T \). We now make the following definition:

**Definition** A non-degenerate sub-triangle \( T^* \) is called "Bézier net convexity preserving" if, for all Bézier nets \( L(f_n) \) that are convex with respect to \( T \), the restricted Bézier nets \( L(f_{n}^{*}) \) on \( T^* \) are also convex with respect to \( T^* \).

Noting the barycentric coordinate representation of the vertices of \( T^* \) with respect to \( T \) in (11), we have the following lemma.

**Lemma** The following statements are equivalent,

(i) \( T^* \) is parallel to \( T \).

(ii) There exist a non-zero scalar \( \rho \) and a permutation \( \{i_1, i_2, i_3\} \) of \( \{1,2,3\} \) such that

\[
V_{i_1}^* - V_{i_2}^* = \rho(V_{i_1} - V_{i_2}), \quad V_{i_2}^* - V_{i_3}^* = \rho(V_{i_2} - V_{i_3}), \quad V_{i_3}^* - V_{i_1}^* = \rho(V_{i_3} - V_{i_1}).
\]

(iii) None of the sets \( \{u_1, u_2, u_3\} \), \( \{v_1, v_2, v_3\} \) and \( \{w_1, w_2, w_3\} \) has all distinct members.

Figure. Examples of parallel triangles

This lemma is illustrated by the two examples of parallel triangles labelled as shown in the figure. Property (ii) is simply a statement that the edges are parallel and property (iii) is a
restatement of this property in terms of the barycentric coordinates of
\[ V_i^* = u_i V_1 + v_i V_2 + w_i V_3, \quad i = 1,2,3. \]

Thus, in the examples \( u_2 = u_3, v_3 = v_1 \) and \( w_1 = w_2 \).

We now present our main theorem:

**Theorem 3** Let \( T^* \) be a non-degenerate sub-triangle of \( T \). Then \( T^* \) is Bèzier net convexity preserving if and only if it is parallel to \( T \).

**Proof:** Suppose \( T^* \) is parallel to \( T \) and let \( L(f_n) \) be any Bèzier net that is convex with respect to \( T \). Comparing (16) and (11), it follows by statement (ii) of the lemma that there exist a non-zero scalar \( \rho \) and a permutation \( \{i_1, i_2, i_3\} \) of \( \{1,2,3\} \) such that

\[ E_{i_1}^* - E_{i_2}^* = \rho (E_1 - E_2), \quad E_{i_3}^* - E_{i_1}^* = \rho (E_3 - E_1), \quad E_{i_2}^* - E_{i_3}^* = \rho (E_2 - E_3). \]

Thus, by (16), we have for \( i+j+k=n-2 \)

\[
\left( E_{i_1}^* - E_{i_2}^* \right) \left( E_{i_2}^* - E_{i_3}^* \right) \left( E_{i_3}^* - E_{i_1}^* \right) = \epsilon_0 \quad 0
\]

\[
= \rho^2 (u_1 E_1 + v_1 E_2 + w_1 E_3) (u_2 E_1 + v_2 E_2 + w_2 E_3) (u_3 E_1 + v_3 E_2 + w_3 E_3) (E_1 - E_2) (E_1 - E_3) f_{000}
\]

\[
= \sum_{r+s+t=i} \sum_{u+v+y=z} \sum_{\lambda+\mu+\nu=k} \rho^2 B^{r}_{\alpha}(u_1, v_1, w_1) B^{s}_{\beta}(u_2, v_2, w_2) B^{t}_{\gamma}(u_3, v_3, w_3)
\]

\[
(E_1 - E_2)(E_1 - E_3) f_{n} \geq 0.
\]

Similarly,

\[
\left( E_{i_2}^* - E_{i_1}^* \right) \left( E_{i_2}^* - E_{i_3}^* \right) \left( E_{i_3}^* - E_{i_2}^* \right) = \epsilon_0 \quad 0
\]

\[
\left( E_{i_1}^* - E_{i_2}^* \right) \left( E_{i_3}^* - E_{i_1}^* \right) \left( E_{i_2}^* - E_{i_3}^* \right) \geq 0.
\]

We therefore conclude that \( L(f_n^*) \) is convex with respect to \( T^* \).

We now suppose that \( T^* \) is not parallel to \( T \). By statement (iii) of the lemma, at least one of the sets \( \{u_1, u_2, u_3\} \), \( \{v_1, v_2, v_3\} \) and \( \{w_1, w_2, w_3\} \) has all distinct members. Without loss of generality, we assume \( u_2 > u_1 > u_3 \geq 0 \). Then we have

\[ (u_1 - u_2)(u_1 - u_3) < 0. \]

Consider the situation where
(18) \[ f_{ijk} = \delta_{i+n-j} , \quad i + j + k = n , \]
that is, \( f_{n,0,0} = 1 \) and \( f_{ijk} = 0 \) if \( i \leq -1 \). Obviously, the Bezier net \( L(f_n) \) defined by (18) is convex with respect to \( T \). Using (15) we obtain
\[ f_{ijk}^* = u_1^{i}u_2^{j}u_3^{k} , \quad i + j + k = n \]
Hence for \( i+j+k=n-2 \) we have
\[ \left( E_{i_1}^* - E_{i_2}^* \right) \left( E_{i_3}^* - E_{i_4}^* \right) f_{ijk}^* = u_1^{i}u_2^{j}u_3^{k} (u_1 - u_2)(u_1 - u_3) \]
and, particularly for \( n \geq 2 \),
\[ \left( E_{i_1}^* - E_{i_2}^* \right) \left( E_{i_3}^* - E_{i_4}^* \right) f_{0,n-2,0}^* = u_2^{n-2} (u_1 - u_2)(u_1 - u_3) < 0. \]
Thus the restricted Bèzier net \( L(f_n^*) \) is not convex with respect to \( T^* \).

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References