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John A. Gregory and Jianwei Zhou

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Convexity of Bȇzier Nets on Sub-triangles

by

John A. Gregory and Jianwei Zhou

Department of Mathematics and Statistics, Brunei University,

Uxbridge, UB8 3PH, England

Abstract: This note generalizes a result of Goodman[3], where it is shown that the convexity of Bèzier nets defined on a base triangle is preserved on sub-triangles obtained from a mid-point subdivision process. Here we show that the convexity of Bèzier nets is preserved on and only on sub-triangles that are "parallel" to the base triangle.

1.Introduction

Let T be a triangle, called the base triangle (see [1]), with vertices $\mathbf{V_1,V_2}$**and**$\mathbf{V_3}$**.** (Here, and elsewhere in the paper, we assume that triangles are non-degenerate, that is, their vertices are not colinear.) Then each point P of the plane determined by $\mathbf{V_1,V_2}$**and**$\mathbf{V_3}$**.**can be represented by its barycentric coordinates (u,v,w) with respect to the base triangle T as

(1) $$P = uV_1 + vV_2 + wV_3, \quad u+v+w = 1.$$

Denote f_n as a set of (n+1)(n+2)/2 values

(2) $$f_n = \{f_{ijk} \in R \mid i+j+k = n \;\; i \geq 0, j \geq 0, k \geq 0,\}.$$

The Bernstein polynomial of f_n over T is then given by

(3) $$B(f_n; u, v, w) = \sum_{i+j+k=n} f_{ijk} B^n_{ijk}(u, v, w) \quad ,$$

Where

(4) $$B^n_{ijk}(u, v, w) = \frac{n!}{i!\,j!\,k!} u^i v^j w^k$$

are Bernstein basis functions. If we let shift operators $\mathbf{E_1}, \mathbf{E_2}$ **and** $\mathbf{E_3}$ with respect to T be defined by

(5) $$E_1 f_{ijk} = f_{i+1,j,k} \quad , \quad E_2 f_{ijk} = f_{i,j+1,k} \quad , \quad E_3 f_{ijk} = f_{i,j,k+1} \quad ,$$

then the Bernstein polynomial can be represented symbolically as

(6) $$B(f_n; u, v, w) = (uE_1 + vE_2 + wE_3)^n f_{000} \quad .$$

Now we consider (n+l)(n+2)/2 points of T with barycentric coordinates (i/n,j/n,k/n), namely,

$$P_{ijk} = (i/n)V_1 + (j/n)V_2 + (k/n)V_3 \quad , i + j + k = n \; .$$

Connecting

$$P_{i+1,j,k} \; , \; P_{i,j+1,k} \; , \; P_{i,j,k+1} \quad ,$$

we obtain a triangle, denoted U_{ijk} for i+j+k=n-l. Similarly, a triangle W_{ijk} is obtained for

i+j+k=n+l with

$$P_{i-1,j,k} \; , \; P_{i,j-1,k} \; , \; P_{i,j,k-1}$$

as its vertices. A piecewise linear function $L(f_n; u, v, w)$, or $L(f_n)$ for short, is defined on T such that it satisfies

(7) $$L(f_n; P_{ijk}) = f_{ijk} \quad , i + j + k = n \; ,$$

and is linear on each triangle U_{ijk} or W_{ijk}. We call $L(f_n; u, v, w)$ the Bėzier net of f_n.

The approximation theory of Bernstein polynomials and their applications in CAGD, Computer-aided Geometric Design, indicate that the Bėzier net $L(f_n)$ is closely related to $B(f_n; u, v, w)$ and reflects certain features of $B(f_n; u, v, w)$. As this note is concerned with the convexity, we state the following results (see [1,2]):

Theorem 1 (i) If the Bėzier net $L(f_n;u,v,w)$ is convex with respect to T, so is the Bernstein polynomial $B(f_n;u,v,w)$.

(ii) The Bėzier net $L(f_n;u,v,w)$ is convex with respect to T if and only if

(8) $$\begin{cases} f_{i+2,j,k} + f_{i,j+1,k+1} \geq f_{i+1,j+1,k} + f_{i+1,j,k+1} & , \\ f_{i,j+2,k} + f_{i+1,j,k+1} \geq f_{i+1,j+1,k} + f_{i,j+1,k+1} & , \\ f_{i,j,k+2} + f_{i+1,j+1,k} \geq f_{i+1,j,k+1} + f_{i,j+1,k+1} & , \end{cases}$$

for i+j+k=n-2.

Using the shift operators, one may rewrite (8) as

(9) $$(E_{i_1} - E_{i_2})(E_{i_1} - E_{i_3}) f_{ijk} \geq 0 , \quad i+j+k = n-2 ,$$

for any permutation $\{i_1, i_2, i_3\}$ of $\{1,2,3\}$.

Let V_1^*, V_2^* and V_3^* be the vertices of another triangle T^* in the same plane as T and let

(10) $$P = u^* V_1^* + v^* V_2^* + w^* V_3^* , \quad u^* + v^* + w^* = 1 ,$$

define the barycentric coordinates (u^*, v^*, w^*) of P with respect to T^*. Assume that V_i^* has barycentric coordinates (u_i, v_i, w_i) with respect to T, that is,

(11) $$V_i^* = u_i V_1 + v_i V_2 + w_i V_3 , \quad u_i + v_i + w_i = 1$$

for i=l, 2, 3. Then

(12) $$\begin{cases} u = u^* u_1 + v^* u_2 + w^* u_3 & , \\ v = u^* v_1 + v^* v_2 + w^* v_3 & , \\ w = u^* w_1 + v^* w_2 + w^* w_3 & . \end{cases}$$

A Bernstein polynomial on T^* is defined symbolically by

(13) $$B(f_n^*; u^*, v^*, w^*) = (u^* E_1^* + v^* E_2^* + w^* E_3^*)^n f_{000}^* ,$$

Where

(14) $$f_n^* = \{ f_{ijk}^* \in R \mid i+j+k = n, i \geq 0, j \geq 0, k \geq 0 \}$$

and E_i^* define the shift operators on T^*. We then have:

Theorem 2 (Chang and Davis[1]) Let

(15) $$f_{ijk}^* = (u_1 E_1 + v_1 E_2 + w_1 E_3)^i (u_2 E_1 + v_2 E_2 + w_2 E_3)^j (u_3 E_1 + v_3 E_2 + w_3 E_3)^k f_{000},$$

for i+j+k=n. Then $B\left(f_n^*; u^*, v^*, w^*\right)$ is the Bernstein representation of $B(f_n; u,v,w)$ with respect to

T^*.

Proof: From (12) and (15) we obtain, by equating coefficients,

$$
\begin{aligned}
B\left(f_n; u, v, w\right) &= \left(uE_1 + vE_2 + wE_3\right)^n f_{000} \\
&= \left[u^*\left(u_1E_1 + v_1E_2 + w_1E_3\right) + v^*\left(u_2E_1 + v_2E_2 + w_2E_3\right) + w^*\left(u_3E_1 + v_3E_2 + w_3E_3\right)\right]^n f_{000} \\
&= \left(u^*E_1^* + v^*E_2^* + w^*E_3^*\right)^n f_{000}^*
\end{aligned}
$$

if and only if

$$
E_1^{*i}E_2^{*j}E_3^{*k}f_{000}^* = \left(u_1E_1 + v_1E_2 + w_1E_3\right)^i + \left(u_2E_1 + v_2E_2 + w_2E_3\right)^j + \left(u_3E_1 + v_3E_2 + w_3E_3\right)^k f_{000}.
$$

Remark In the above proof we note the identities

(16) $$E_i^* = u_iE_1 + v_iE_2 + w_iE_3, \ \ i = 1,2,3 ,$$

where, in symbolic manipulation involving these operators, indices in operator expressions must sum to n and the operator expressions are applied to $f_{000}^* = f_{000}$.

The set f_n^* determines a new Bezier net $L\left(f_n^*; u^*, v^*, w^*\right)$ which is a piecewise linear function on T^*. We call $L\left(f_n^*\right)$ the restricted Bezier net of $L(f_n)$ on T^*. Naturally, T^* is called a sub-triangle of T if V_1^*, V_2^* and V_3^*, the vertices of T^* are all inside or on the boundary of T. In this case $u_i, v_i, w_i \geq 0, \ i = 1,2,3$.

Let T^* be a sub-triangle of T. Then if $B\left(f_n; u, v, w\right)$ is convex with respect to T, so is $B\left(f_n^*; u^*, v^*, w^*\right)$ with respect to T^*. One would ask whether or not a similar result holds for the Bėzier nets. Grandine[4] showed a negative result but in [3], Goodman shows that if T^* is a sub-triangle obtained from a "mid-point" subdivision process, then the convexity of $L(f_n)$ does guarantee the convexity of the restricted Bėzier net $L\left(f_n^*\right)$ with respect to T^*. In the next section, we generalize this result and show that only a very limited class of sub-triangles have the property of preserving the convexity of Bėzier nets.

2. Main Result

Let T^* be a non-degenerate triangle that lies on the plane determined by the base triangle T. We say T^* is parallel to T if each of the edges of T^* is parallel to one of those of T. We now make the following definition:

Definition A non-degenerate sub-triangle T^* is called "Bėzier net convexity preserving" if, for all Bėzier nets $L(f_n)$ that are convex with respect to T, the restricted Bėzier nets $L(f_n^*)$ on T^* are also convex with respect to T^*

Noting the barycentric coordinate representation of the vertices of T^* with respect to T in (11), we have the following lemma.

Lemma The following statements are equivalent,

(i) T^* is parallel to T.

(ii) There exist a non-zero scalar ρ and a permutation $\{i_1, i_2, i_3\}$ of $\{1,2,3\}$ such that

(17) $$V_{i_1}^* - V_{i_2}^* = \rho(V_1 - V_2), \quad V_{i_3}^* - V_{i_1}^* = \rho(V_3 - V_1), \quad V_{i_2}^* - V_{i_3}^* = \rho(V_2 - V_3).$$

(iii) None of the sets $\{u_1, u_2, u_3\}, \{v_1, v_2, v_3\}$ and $\{w_1, w_2, w_3\}$ has all distinct members.

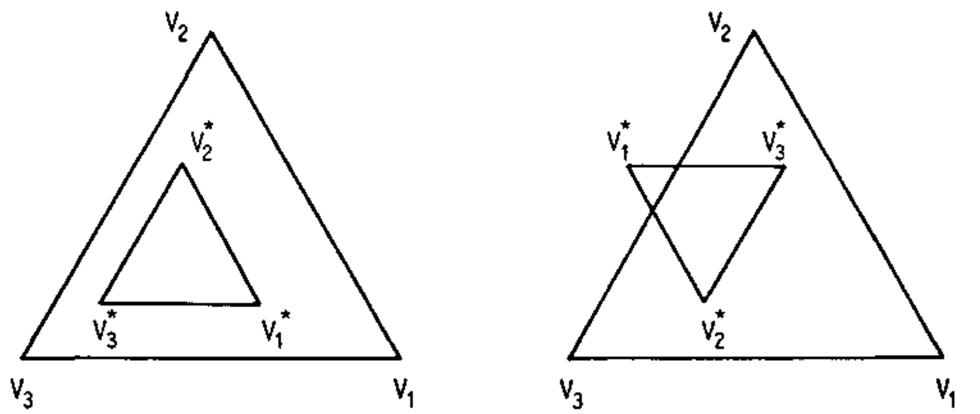

Figure. Examples of parallel triangles

This lemma is illustrated by the two examples of parallel triangles labelled as shown in the

figure. Property (ii) is simply a statement that the edges are parallel and property (iii) is a restatement of this property in terms of the barycentric coordinates of

$$V_i^* = u_i V_1 + v_i V_2 + w_i V_3, \quad i = 1,2,3.$$

Thus, in the examples $u_2 = u_3, v_3 = v_1$ and $w_1 = w_2$.

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We now present our main theorem:

<u>Theorem 3</u> Let T^* be a non-degenerate sub-triangle of T. Then T^* is Bėzier net convexity preserving if and only if it is parallel to T.

<u>Proof:</u> Suppose T^* is parallel to T and let $L(f_n)$ be any Bėzier net that is convex with respect to T. Comparing (16) and (11), it follows by statement (ii) of the lemma that there exist a non-zero scalar ρ and a permutation $\{i_1, i_2, i_3\}$ of {1,2,3} such that

$$E_{i_1}^* - E_{i_2}^* = \rho(E_1 - E_2), E_{i_3}^* - E_{i_1}^* = \rho(E_3 - E_1), E_{i_2}^* - E_{i_3}^* = \rho(E_2 - E_3).$$

Thus, by (16), we have for i+j+k=n-2

$$\left(E_{i_2}^* - E_{i_2}^*\right)\left(E_{i_2}^* - E_{i_3}^*\right)f_{ijk}^* = E_1^{*i} E_2^{*j} E_3^{*k} \left(E_{i_2}^* - E_{i_3}^*\right)\left(E_{i_2}^* - E_{i_3}^*\right)f^* 000$$
$$= \rho^2 (u_1 E_1 + v_1 E_2 + w_1 E_3)^i (u_2 E_1 + v_2 E_2 + w_2 E_3)^j (u_3 E_1 + v_3 E_2 + w_3 E_3)^k (E_1 - E_2)(E_1 - E_3) f_{000}$$
$$= \sum_{r+s+t=i} \sum_{\alpha+\beta+y=l} \sum_{\lambda+\mu+v=k} \rho^2 B_{rst}^i (u_1, v_1, w_1) B_{\alpha\beta\gamma}^j (u_2, v_2 w_2) B_{\lambda\mu\nu}^k (u_3, v_3, w_3)$$
$$(E_1 - E_2)(E_1 - E_3) f_{r+\alpha+\lambda, \delta+\beta+\mu, t+y+v}$$
$$\geq 0.$$

Similarly,

$$\left(E_{i_2}^* - E_{i_1}^*\right)\left(E_{i_2}^* - E_{i_3}^*\right)f_{ijk}^* \geq 0, \quad \left(E_{i_3}^* - E_{i_1}^*\right)\left(E_{i_3}^* - E_{i_2}^*\right)f_{ijk}^* \geq 0.$$

We therefore conclude that $L\left(f_n^*\right)$ is convex with respect to T*.

We now suppose that T^*. is not parallel to T. By statement (iii) of the lemma, at least one of the sets $\{u_1, u_2, u_3\}$, $\{v_1, v_2, v_3\}$ and $\{w_1, w_2, w_3\}$ has all distinct members. Without loss of

generality, we assume $u_2 > u_1 > u_3 \geq 0$. Then we have

$$(u_1 - u_2)(u_1 - u_3) < 0.$$

Consider the situation where

(18) $$f_{ijk} = \delta_{i,n-1}\ , \quad i + j + k = n\ ,$$

that is, $f_{n,0,0} = 1$ and f_{ijk}=0 if $i \le -1$. Obviously, the Bezier net $L(f_n)$ defined by (18) is convex with respect to T. Using (15) we obtain

$$f^*_{ijk} = u_1^i u_2^j u_3^k\ , \quad i + j + k = n$$

Hence for i+j+k=n-2 we have

$$(E_1^* - E_2^*)(E_1^* - E_3^*)f^*_{ijk} = u_i^i u_2^i u_3^k (u_1 - u_2)(u_1 - u_3)$$

and, particularly for $n \ge 2$,

$$(E_{i_1}^* - E_{i_2}^*)(E_{i_1}^* - E_{i_3}^*)f^*_{0,n-2,0} = u_2^{n-2}(u_1 - u_2)(u_1 - u_3) < 0.$$

Thus the restricted Bėzier net $L(f_n^*)$ is not convex with respect to T*.

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