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A Pentagonal Surface Patch for Computer-Aided Geometric Design

by

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Running Head: Pentagonal Patch

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Abstract

A vector valued interpolation scheme for a pentagon is described which is compatible with surface patches which have a rectangular domain of definition. Such a scheme could be useful in computer-aided geometric design problems, where a pentagonal patch occurs within a rectangular patch framework.

1. Introduction

The representation of a curved surface by piecewise defined vector valued surface patches is a technique which is widely used in computer-aided geometric design. Such patches usually have a rectangular domain of definition as, for example, in the theory developed by Steve Coons, see [3] and [4]. However, non-standard patches can occur within a rectangular patch framework, for example in a previous paper the authors consider the problem of constructing a surface patch with a triangular domain of definition [7].

In this paper we turn our attention to the problem of constructing a surface patch with a pentagonal domain of definition. The occurrence of such a patch for a model problem is illustrated in Fig. 1.1 and further examples are considered in Section 4.

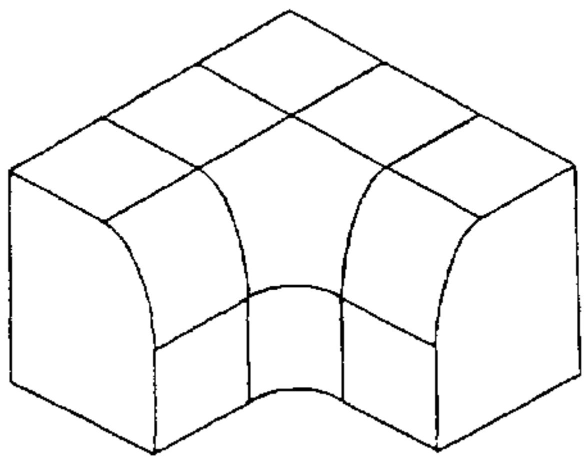


Fig. 1.1, Model Problem 1 : A corner moulding

Our pentagonal surface patch scheme is an example of a <u>blending function</u> interpolant, by which we mean that the interpolant matches data given everywhere on the boundary of the patch. The patch, as described later in Section 3, is a convex combination or blend of five component interpolants, each of which match given boundary curves and cross boundary slope conditions on two sides of a regular pentagon.

The resulting surface patch can thus be joined with positional and slope continuity to adjacent rectangular patches and is thus called a C^1 scheme. A corresponding C^0 scheme is also described, which can be joined with positional continuity to adjacent rectangular patches.

The component interpolants of our pentagonal surface patch can be described in terms of a parametric coordinates system (s,t) as follows. Let

$$F(s,t) = [x(s,t),\ y(s,t),\ z(s,t)] \tag{1.1}$$

be a given vector valued function. Then

$$P(s,t) = F(0,t) + F(s,0) - F(0,0) \tag{1.2}$$

defines an interpolant which is such that

$$P(0,t) = F(0,t) \quad \text{and} \quad P(s,0) = F(s,0) \tag{1.3}$$

More generally,

$$\begin{aligned} \underset{\sim}{p}(s,t) &= [1\ \ s]\begin{bmatrix} \underset{\sim}{F}(0,t) \\ \underset{\sim}{F}_s(0,t) \end{bmatrix} + \begin{bmatrix} \underset{\sim}{F}(s,0) & \underset{\sim}{F}_t(s,0) \end{bmatrix}\begin{bmatrix} 1 \\ t \end{bmatrix} \\ &= [1\ \ s]\begin{bmatrix} \underset{\sim}{F}(0,0) & \underset{\sim}{F}_t(0,0) \\ \underset{\sim}{F}_s(0,0) & \underset{\sim}{F}_{s,t}(0,0) \end{bmatrix}\begin{bmatrix} 1 \\ t \end{bmatrix} \end{aligned} \tag{1.4}$$

defines an interpolant which satisfies (1.3) together with the cross boundary slope properties that

$$\underset{\sim}{P}_S(0,t) = \underset{\sim}{F}_s(0,t) \quad \text{and} \quad \underset{\sim}{P}_t(s.0) = \underset{\sim}{F}_t(s.0) \tag{1.5}$$

,

where $\underset{\sim}{F}_s = \partial \underset{\sim}{F}/\partial s$, $\underset{\sim}{F}_t = \partial \underset{\sim}{F}/\partial t$, and $\underset{\sim}{F}_{s,t} = \partial^2 \underset{\sim}{F}/\partial s \partial t$.

The interpolants (1,2) and (1.4) will be used in the construction of the C^0 and C^1 patches respectively. They are derived using Boolean sum blending function theory, a theory suggested by the work of Coons [3] and formalised later by Gordon [5]. Here, the interpolants are

Boolean sums of Taylor interpolation operators, where in (1.4) we assume the compatitiIity condition that

$$\underset{\sim}{F}_{s,t}(0,0) = \underset{\sim}{F}_{t,s}(0,0)$$

i.e. that the term called the 'twist' is independent of the order of differentiation. If this is not so the function (1.4) does not interpolate the cross boundary slope condition $\underset{\sim}{F}_t$ on t = 0 although the interpolant could be corrected by the addition of the rational term

$$\frac{s^2 t}{s+t}\left[\underset{\sim}{F}_{s,t}(0,0) - \underset{\sim}{F}_{t,s}(0,0)\right]$$

cf. [1] and [6], However, in this paper we do not consider such cases.

The Boolean sum Taylor interpolants (1.2) and (1.4) are applied on the pentagon by making an appropriate choice for the parametric coordinate system (s,t). This system is introduced in the following section.

2. The Pentagonal Domain

Let Ω be a regular pentagon of height unity and with vertices $V_j = (u_j,v_j)$, i = 1,...,5 , in the (u,v) plane, The surface patch will be represented in terms of the variables λ_i , i = 1,...,5 , which denote the perpendicular distances of the general point V = (u,v) from the sides opposite the vertices V_i , i = 1,...,5 , see Fig. 2.1.

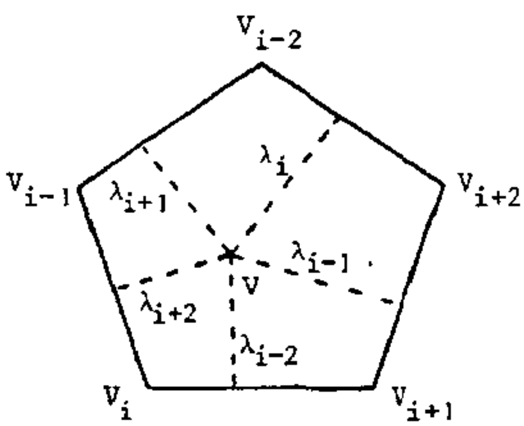


Fig. 2.1. The Pentagon Ω

<u>Note</u> Here, and elsewhere in this paper, a suffix i is interpreted, (i-l)mod 5 + 1 to bring it into the range $1 \leq i \leq 5$.

The variables λ_i. are clearly linearly dependent and we have that

$$\left.\begin{aligned} \lambda_{i-1} &= 2(1+\lambda_{i-2})\sin\theta - \lambda_{i+2} \\ \lambda_i &= 1 - 2\lambda_{i-2}\sin\theta - 2\lambda_{i+2}\sin\theta \,, \\ \lambda_{i+1} &= 2(1+\lambda_{i+2})\sin\theta - \lambda_{i-2} \,, \end{aligned}\right\} \tag{2.1}$$

where θ - $\pi/10$ and sin9 = $(\sqrt{5} - 1)/4$. From (2.1) it follows that

$$\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 = 1 + 4\sin\theta = \sqrt{5}. \tag{2.2}$$

We now proceed to construct a local parametric coordinate system (s_i, t_i), to be associated with each vertex V_i, i = 1,...,5. Let the sides λ_{i+1}, = 0 and $\lambda_{i-1} = 0$ intersect at the point R_i, i = 1.....5 and let E_i denote the intersection of $\lambda_i = 0$ with the radial line joining V and R_i. In particular, we consider the points E_{i-2} and E_{1+2} as shown in Fig, 2.2.

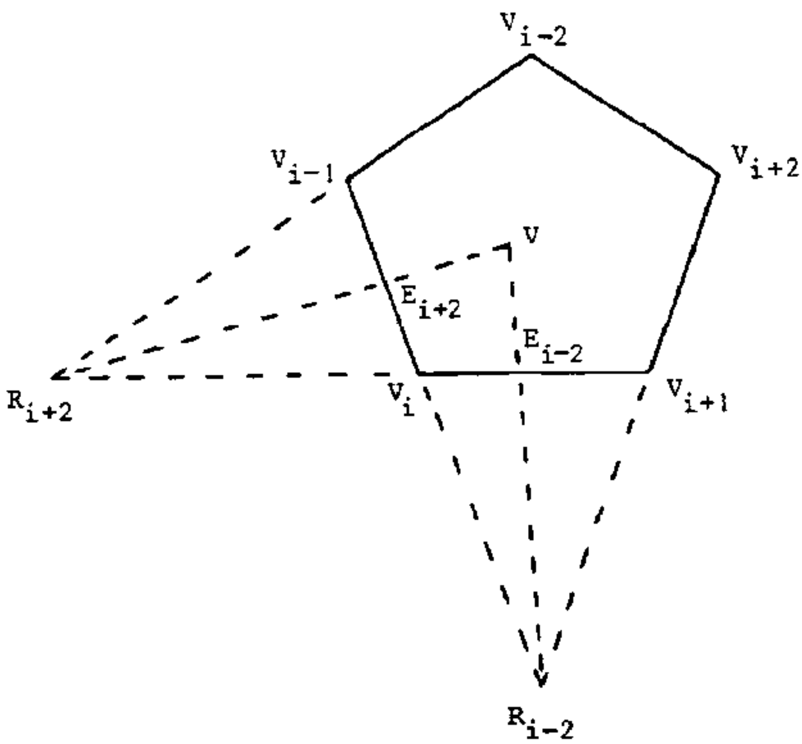


Fig. 2.2. Construction of the Local Coordinate System (s_i, t_i)

Then it can be shown that

$$\left.\begin{aligned} E_{i-2} &= (1-s_i)\, V_i + s_i\ V_{i+1} \ , \\ E_{i+2} &= (1-t_i)\, V_i + t_i\ V_{i+1} \ , \end{aligned}\right\} \qquad (2.3)$$

where

$$s_i = \lambda_{i+2} / (\lambda_{i-1} + \lambda_{i+2}) \ , \ t_i = \lambda_{i-2} \ / \ (\lambda_{i+1} + \lambda_{i-2}) \qquad (2.4)$$

The variables (s_i,t_i) define a local coordinate system for the pentagon with the property that s_i = constant corresponds to the radial line through R_{i-2} and E_{i-2} and t_i constant corresponds to the radial line through R_{i+2} and E_{i+2}. The singularities due to the rational terms in (2.4) lie well outside the pentagon Ω and for $V \in \Omega$ we have that

$$0 \le s_i \le 1 \qquad \text{and} \qquad 0 \le t_i \le 1$$

Furthermore, because of the radial construction, E_{i-2} and E_{i+2} remain on the boundary of the pentagon ft for all $V \in \Omega$.

Remark An alternative local coordinate system to (2,4) is

$$s_i = \lambda_{i+2} \ , \quad t_i - \lambda_{i-2} \qquad (2.5)$$

which gives parameter lines along parallels to the sides and corresponds to that used in [73. However, the points corresponding to (2.3) can now lie outside the pentagon ft for some $V \in \Omega$. Also, we have found that (2.4) gives a better interpretation of the cross boundary derivative which leads to a smoother surface. We thus prefer the system (2.4), although a full description of the pentagonal patch which uses (2.5) can be found in [2].

Note Since

$$t_{i+1} = 1 - s_i \quad \text{and } s_{i-1} = 1 - t_i$$

it follows that (2.3) can be written in the cyclic form-

$$\left.\begin{array}{l} xi-2 = ti+1 \ \ Vi+Si\ Vi+1 \\ Ei+2 \ = ti \ \ Vi-1 \ + si-1Vi \end{array}\right\} \tag{2.6}$$

Thus

$$E_i = ti\text{-}2\ V_{i+2} + s_{\ i+2}\ V_{\ i-2}\ ,\ i = 1,\ldots\ldots.,5 \tag{2.7}$$

where

$$t_{i-2} = \lambda_{i+1}/(\lambda_{i-1} + \lambda_{i+2})\ ,\ s_{i+2} = \lambda_{\ i-1}/(\lambda_{i-1} + \lambda_{i+1}). \tag{2.8}$$

3. <u>The Pentagonal Surface Patch</u>

Let

$$\underset{\sim}{F}(V) = [x(V),\ y(V), z(V)]'\ ,\quad V \in \Omega \tag{3.1}$$

denote a vector valued function defined on the pentagon Ω, In particular, for the C^0 patch, we assume that $\underset{\sim}{F}$ is defined everywhere on the sides X- = 0 , i = 1,...,5, of the pentagon Ω. In addition, for the C^1 patch, we assume that the cross boundary slope denoted by $\underset{\sim}{F}_{n\,i}$ is specified on each side X. = 0, together with the twist condition $\underset{\sim}{F}\,n_{\,i+2}, n_{\,i-2}$ at each vertex V_i, where $\underset{\sim}{F}\,n_{\,i+2}, n_{\,i-2} = \underset{\sim}{F}\,n_{\,i-2}, n_{i+2}$. This boundary data will be defined by that of the adjoining rectangular patches. The slope condition is identified with the cross boundary derivative of the rectangular scheme, appropriately signed to be pointing in an inward direction to the side of the pentagon. Likewise the twist condition at each vertex must be appropriately signed. In the domain of the pentagon, the slope condition is interpreted as being defined along the direction of the radial line from R_i to E_i .

<u>The C^0 Patch</u>. From (1.2), with the local coordinate system (s_i, t_i), we can define the vector valued function

$$P_i(V) = \underset{\sim}{F}(E_{i+2}) + \underset{\sim}{F}(E_{i-2}) - \underset{\sim}{F}(V_i). \tag{3.2}$$

This function interpolates $\underset{\sim}{F}$ on the sides $\lambda_{i+2} = 0$ and $\lambda_{i-2} = 0$ of the pentagon, that is

$$\underset{\sim}{P}_i(E_{i+2}) = \underset{\sim}{F}(E_{i+2}) \text{ and } \underset{\sim}{P}_i(E_{i-2}) = \underset{\sim}{F}(E_{i-2}) \tag{3.3}$$

The pentagonal patch is now defined by the convex combination

$$\underset{\sim}{p}(v) = \sum_{i=1}^{5} \alpha_i(v)\, \underset{\sim i}{p}(v) \tag{3.4}$$

where

$$\alpha_i(V) = \lambda_{i-1}\ \lambda_i\ \lambda_{i+1} \Big/ \sum_{k=1}^{5} \lambda_{k-1}\ \lambda_k\ \lambda_{k+1} \tag{3.5}$$

Equation (3.4) is a convex combination since the α_i are positive on Ω and, by construction,

$$\sum_{i=1}^{5} \alpha_i(V) = 1 \tag{3.6}$$

Furthermore, since

$$\alpha_i(E_j) = 0 \quad , \quad i = j-1,\ j,\ j+1 \quad , \tag{3.7}$$

it follows that

$$\underset{\sim}{P}(E_j) = \underset{\sim}{F}(E_j) \quad , \quad j = 1.....5 \quad , \tag{3.8}$$

that is $\underset{\sim}{P}$ interpolates $\underset{\sim}{F}$ on the entire boundary of the pentagon. Finally, it should be noted that the rational weight functions (3.5) are well behaved on Ω, the denominator being always strictly positive

The C^1 Patch. From (1.4) we can define the vector valued function

$$\begin{aligned}\underset{\sim i}{p}(V) = [1\ s_i]\begin{bmatrix} \underset{\sim}{F}(E_{i+2}) \\ F_{\sim n_{i+2}}(E_{i+2}) \end{bmatrix} + \begin{bmatrix} \underset{\sim}{F}(E_{i+2}) & F_{\sim n_{i-2}}(E_{i+2}) \end{bmatrix}\begin{bmatrix} 1 \\ t_i \end{bmatrix} \\ [1\ s_i]\begin{bmatrix} \underset{\sim}{F}(v_i) & F_{\sim n_{i+2}}(v_i) \\ F_{\sim n_{i+2}}(v_i) & F_{\sim n_{i+2}, n_{i-2}}(v_i) \end{bmatrix}\begin{bmatrix} 1 \\ t_i \end{bmatrix}\end{aligned} \tag{3.9}$$

This function interpolates $\underset{\sim}{F}$ on the sides $\lambda_{i+2} = 0$ and $\lambda_{i-2} = 0$ of the pentagon, that is equations (3.3) hold. Furthermore the function interpolates the cross boundary slope conditions on these sides, namely

$$[\partial \underset{\sim i}{P} / \partial s_i](E_{i+2}) = F_{\sim n_{i+2}}(E_{i+2}) \text{ and } [\partial \underset{\sim i}{P} / \partial t_i](E_{i-2}) = F_{\sim n_{i-2}}(E_{i-2}) \tag{3.10}$$

The pentagonal patch is now defined by the convex combination

$$\underset{\sim}{p}(V) = \sum \alpha_i(V) P_i(V) \tag{3.11}$$

where
$$\alpha_i(V) = (\lambda_{i-1}\, \lambda_i\, \lambda_{i+1})^2 \Big/ \sum_{k=1}^{5} (\lambda_{k-1}\, \lambda_k\, \lambda_{k+1})^2 \tag{3.12}$$

Here, the squared terms are introduced in (3.12) in order that $\underset{\sim}{P}_i(V)$ does not contribute to the tangent plane of $\underset{\sim}{P}(V)$ on $\lambda_j = 0$, $i = j-1, j, j+1$. Thus, on A. = 0, the position and slope of $\underset{\sim}{P}$ is determined by a convex combination of $\underset{\sim}{P}_{j-2}$ and $\underset{\sim}{P}_{j+2}$. Hence $\underset{\sim}{P}$ interpolates $\underset{\sim}{F}$, and has the same tangent plane as $\underset{\sim}{F}$, on the sides A. = 0 , $j = 1,\ldots,5$, of the pentagon Ω

<u>Remark</u>. In (3.9), the cross boundary slope conditions Fn_{i+2} and Fn_{i-2} are interpreted locally as $\partial F/\partial s_i$ and $\partial F/\partial t_i$ respectively, where F is considered as a function of the local parameters s_i and t_i (In fact, for slope interpolation, we must have that

$$[\partial \underset{\sim}{F} / \partial s_i](v_i) = F_{\sim n_{i+2}}(v_i), \quad [\partial F / \partial t_i](V_i) = F_{\sim n_{i-2}}(V_i), \text{ and}$$

$$[\partial^2 F / \partial s_i](v_i) = [\partial^2 \underset{\sim}{F} / \partial t_i\, \partial s_i](V_i) = F_{\sim n_{i+2}, n_{i-2}}(V_i)$$

although these conditions follow as a consequence of using the rectangular patch data.) This interpretation of the slopes cannot be made for the final scheme, however, because the magnitudes of the derivatives along the radial lines are determined by a convex combination- Such magnitudes do not affect tangent plane continuity arguraents for vector valued schemes, but they would have to be considered in generalizing the pentagonal scheme for higher order derivative boundary conditions or in adapting the scheme to scalar valued interpolation.

4. Examples

We have applied the C^1 pentagonal scheme to three model problems, where the rectangular patches are defined by tensor product bicubic Hermite interpolants. Thus on the square S = [0,1] x [0,1] we have

$$\underset{\sim}{p}(u, v) = \sum_{i=0}^{1} \sum_{j=0}^{1} \left[\phi_i(u)\,\varphi_i(v)\, \underset{\sim i,j}{F}(0,0) + \phi_i(u)\,\Psi_j(v)\, \underset{\sim i,j}{F}(0,1) \right.$$

$$\left. + \psi_i(u)\phi_j(v) F_{i,j}(1,0) + \psi_i(u)\,\psi_j(v)\, F_{i,j}(1,1) \right] \quad , \ (u,v) \in S \ , \tag{4.1}$$

where

$$\phi_0(u) = 1 - 3u^2 + 2u^3 \ , \ \phi_1(u) = u - 2u^2 + u^3 \tag{4.2}$$

and

$\psi_i(u) = (-1)^1 \phi_i(1-u)$, $i = 0, 1$, In this case the boundary data for the pentagonal patch is defined by

$$\underset{\sim}{F}(E_i) = \phi_0(s_{i+2})\,\underset{\sim}{F}(V_{i+2}) + \phi_1(s_{i+2})\,\underset{\sim}{F}n_{i-1}(V_{i+2})$$

$$+ \phi_0(t_{i-2})\,\underset{\sim}{F}(v_{i-2}) + \phi_1(t_{i-2})\,\underset{\sim}{F}n_{i+1}(V_{i+2}) \tag{4.3}$$

$$\underset{\sim}{F}n_i = \phi_0(t_{i-2})\,\underset{\sim}{F}n_i(v_{i+2}) + \phi_1(s_{i+2})\,\underset{\sim}{F}n_i,n_{i-1}(V_{i+2})$$

$$+ \phi_0(t_{i-2})\,\underset{\sim}{F}n_i(V_{i-2}) + \phi_1(t_{i-2})\,\underset{\sim}{F}n_i,n_{i-1}(V_{i-2}) \tag{4.4}$$

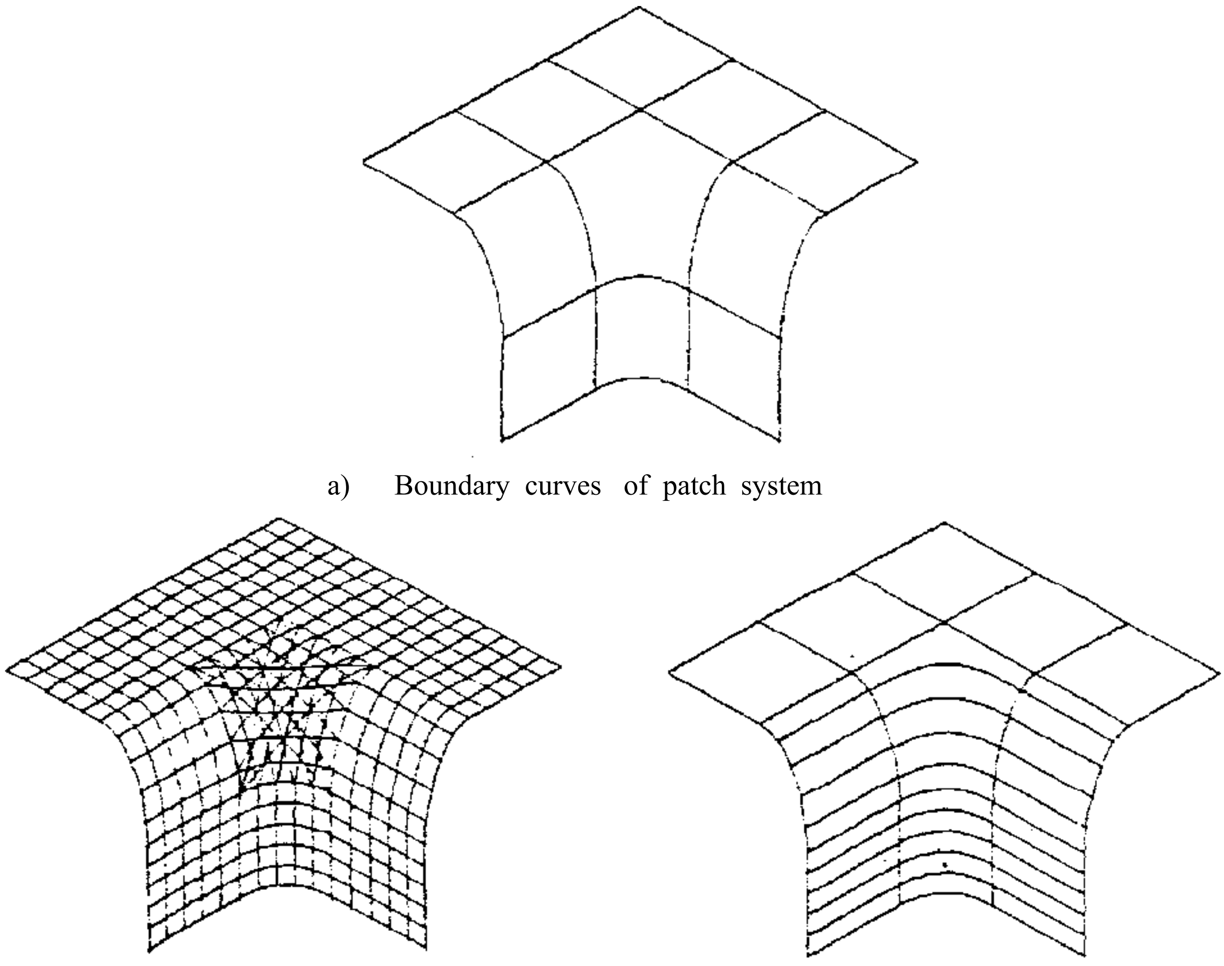
where

$$E_i = t_{i-2} \ V_{i+2} + s_{i+2} \ V_{i-2} \ , \ t_{i-2} + s_{i+2} = 1. \tag{4.5}$$

For the three model problems, the rectangular patches are shown plotted at intervals of 1/5 in the parameter plane, whilst the pentagonal patches are shown with the plotting lines $\lambda_i = 2k \sin\theta/5$; $k = 1,...,5$. The results of an algorithm giving cross sections through the patches are also shown in two cases.

The first model problem is the corner moulding shown in Fig. 1.1 of the introduction. Fig. 4.1a shows the boundary curves of the patch system, whilst Fig. 4.1b shows the patches viewed with the plotting lines described above. Fig. 4.1c shows the patches viewed with cross sections.

The second model problem is the end of a ridge protruding from a plane. This problem is of particular interest since it includes a triangular patch together with the pentagonal patch. The triangular patch illustrated here is that described in [7], The final model problem is the half of a cylindrical T joint shown in Fig. 4.3.

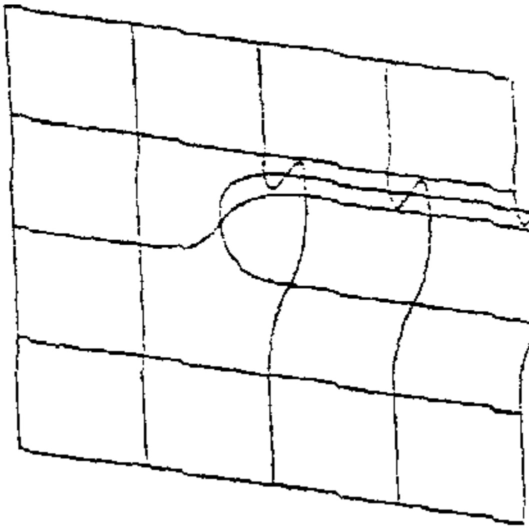
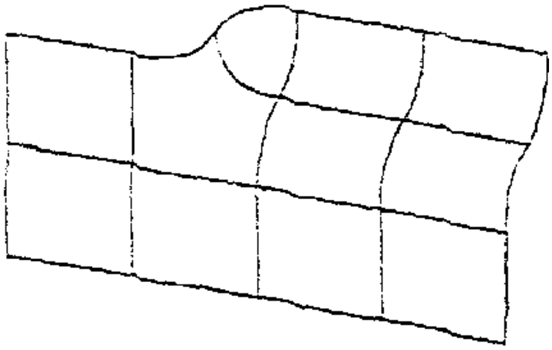


a) Boundary curves of patch system

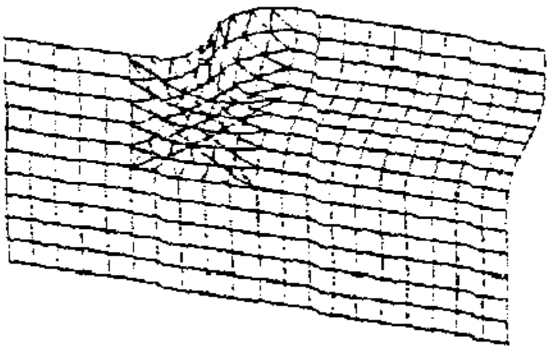
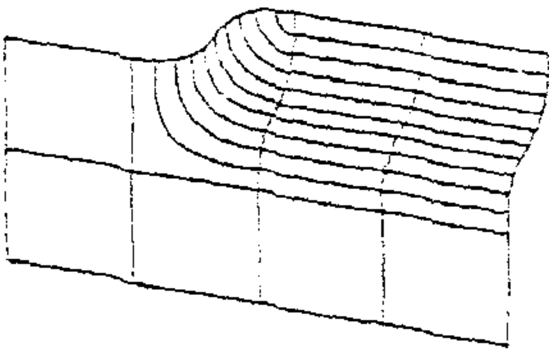
b) Plotting lines λ_i. = constant

c) Cross sections

Fig. 4.1. Model Problem'1 : A corner moulding

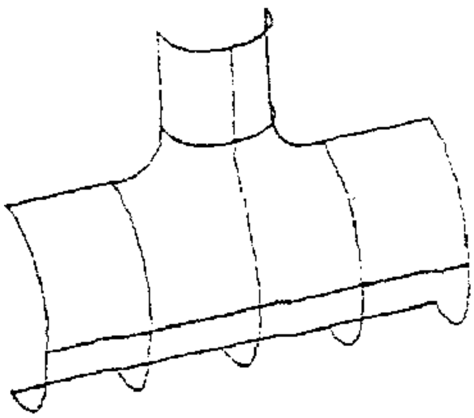
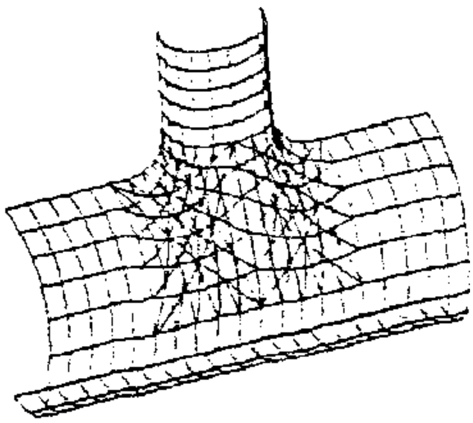



a) Boundary curves of patch system   b) Half of patch system

c) Plotting lines λ_i = constant   d) Cross sections

Fig, 4.2. Model Problem 2 : A ridge

a) Boundary curves of patch system   b) Plotting lines λ_i = constant

Fig. 4.3. Model Problem 3 : A cylindrical T joint.

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