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A CUBIC SPLINE TECHNIQUE FOR THE ONE DIMENSIONAL HEAT CONDUCTION EQUATION.

by

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ABSTRACT

A new method is developed for the numerical solution of the heat conduction equation in one space dimension by replacing the space derivative with a cubic spline approximation and the time derivative with a finite-difference approximation. The method is equivalent to a new finite-difference scheme and produces at each time level an interpolating spline function.

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INTRODUCTION

The use of cubic splines for the numerical solution of linear two-point boundary value problems has been considered by Bickley (1968), Fyfe (1969), and Albasiny and Hoskins (1969). In the present paper a technique similar to that of Albasiny and Hoskins is developed for the heat conduction equation by the use of a cubic spline approximation in the space direction together with a finite-difference approximation in the time direction. This approximate representation, which is shown to be equivalent to a new implicit finite-difference scheme with stability conditions and truncation error similar to those of a well known finite-difference representation, produces at each time level a spline function which may be used to obtain the solution at any point in the range of the space variable.

DESCRIPTION OF PROCEDURE.

Consider the heat conduction problem in which the function u(x,t) satisfies

$$\begin{aligned}
&\frac{\partial u}{\partial t}(x,t) = \frac{\partial^2 u}{\partial x^2}(x,t), && (x,t) \in R, \\
&u(x,0) = f(x), && 0 \leqq x \leqq 1, \\
&u(0,t) = a(t), && t \geqq 0, \\
&u(1,t) = b(t), && t \geqq 0,
\end{aligned} \tag{1}$$

where R = $\{(x,t):\ 0 \leqq x < 1,\ t \geqq 0]$ and f(x), a(t) and b(t) are given functions.

The region R is covered in the usual manner by a rectangular net

$$(x_i t_j) \equiv (ih, jk), \; 0 \leqq i \leqq N, \; j \geqq 0$$

where $Nh = 1$. If U_i^j denotes a discrete approximation to $u(x, t)$ at the point (x_i, t_j) and $S_j(x)$ is the cubic spline interpolating the values U_i^j at the jth time level then the heat conduction equation in (t) is replaced at (x_i, t_j) by

$$\frac{U_i^{j+1} + U_i^j}{k} = \theta M_i^{j+1} + (1-\theta) M_i^j \qquad i = 0,1,...N, \quad j \geqq 0 \qquad (2)$$

where $0 \leqq 0 \leqq 1$ and $M_i^j = S''j\,(x_i)$.

For the jth time level the results of Ahlberg, Nilson and Walsh (1967) show that in the interval $x_{i-1} \leqq x \leqq x_i$

$$S_j(x) = M_{i-1}^j \frac{(x_i - x)^3}{6h} + M_i^j \frac{(x - x_{i-1})^3}{6h} + (U_{1-1}^j - \frac{h^2}{6} M_{i-1}^j) \frac{(x_i - x)}{h}$$

$$+ (U_i^j - \frac{h^2}{6} M_i^j) \frac{(x - x_{i-1})}{h} \qquad i = 1,2 \; \ldots \; N. \qquad (3)$$

Hence

$$S_j'(x_i+) = -\frac{h}{3} M_i^j - \frac{h}{6} M_{i+1}^j + \frac{U_{i+j}^j - U_i^j}{h}, \qquad i = 0,1,2....N-1, \qquad (4)$$

and

$$S_j'(x_i-) = \frac{h}{3} M_i^j + \frac{h}{6} M_{i+1}^j + \frac{U_i^j - U_{i-1}^j}{h}, \qquad i = 1,2,....N, \qquad (5)$$

so that continuity of the first derivatives implies

$$\frac{h}{6}\, M_{i-1}^{j} + \frac{2h}{3}\, M_{i}^{j} + \frac{h}{6}\, M^{j}{}_{i+1} = \frac{U_{i-1}^{j} - 2U_{i}^{j} + U_{i+1}^{j}}{h}\, ,$$

$$i = 1,2,\ldots N - 1\, . \qquad (6)$$

Similarly, for the (j + 1)th time level,

$$\frac{h}{6}\, M_{i-1}^{j+1} + \frac{2h}{3}\, M_{i}^{j+1} + \frac{h}{6}\, M_{i-1}^{j+1} = \frac{U_{i-1}^{j+1} - 2U_{i}^{j+1} + U_{i+1}^{j+1}}{h}\, ,$$

$$i = 1,2,\ldots N - 1. \qquad (7)$$

The addition of (7) multiplied by θ to (6) multiplied by $(1-\theta)$ gives, after the elimination of the M's by means of (2),

$$\frac{U_{i-1}^{j+1} - U_{i-1}^{j} + 4(U_{i}^{j+1} - U_{i}^{j}) + U_{i-1}^{j+1} - U_{i-1}^{j}}{6k}$$

$$= \frac{1}{h^2}\left\{ \theta\delta^2 U_{i}^{j+1} + (1-\theta)\delta^2 U_{i}^{j} \right\}$$

$$i = 1,2,\ldots N -1. \qquad (8)$$

where δ is the usual central difference operator in the x - direction. Thus, for i = 1,2 ... N- 1, the discrete replacement (2) of the heat conduction equation is equivalent to the finite-difference replacement (8).

If U_{i}^{j} is assumed to be known at all mesh points up to the j th time level, then equations (3) written as

$$(1-6r\theta)\, U_{i-1}^{j+1} + 2\,(2 + 6r\theta)\, U_{i}^{j+1} + (1 - 6r\theta)\, U_{i-1}^{j+1}$$

$$= \{1 + 6r(1-\theta)\} U_{i-1}^{j} + 2\{2-6r(1-\theta)] U_{i}^{j} + \{1 + 6r(1-\theta)] U_{i+1}^{j}\, ,$$

$$r = k/h^2\, , \qquad i = 1,2,\ldots N - 1 \qquad (9)$$

together with the boundary conditions of (1) constitute a tri-diagonal set of linear equations which is solved for the unknowns U_i^{j+1}, U_2^{j+1} ... U_{N-1}^{j+1}. Moreover, if the values M_i^j are known at all mesh points up to the jth time level, equation (2), together with the relevant boundary values, gives the values M_i^{j+1}, $i = 0,1,2 \ldots N$, and hence (3) with j replaced by $j + 1$ produces the cubic spline approximating the solution $u(x,t_{j+1})$ at the $(j + 1)$th time level. If $\theta \neq 1$ the determination of the cubic spline for the first time level requires knowledge of the values M_i°, $i = 0,1, \ldots N$ and, provided $f(x) \subset C^2[0,1]$, these may be determined by setting

$$M^\circ = f''(x_i), \qquad i = 0,1,2,\ldots N.$$

If however, f'(x) or f''(x) is discontinuous at some point in (0,1) or if f(x) is given in discrete form, then the values M_i° are taken as the second derivatives of the cubic spline approximating the initial function and are determined in the usual way by solving the tri-diagonal system

$$\frac{h}{6} M_{i-1}^\circ + \frac{2h}{3} M_i^\circ + \frac{h}{6} M_{i-1}^\circ = \frac{f(x_{i-1}) - 2f(x_i) + f(x_{i-1})}{h}$$

$$i = 1,2, \ldots N - 1, \tag{10}$$

$$M_\circ^\circ = \frac{a(k) - a(0)}{k}, \qquad M_N^\circ = \frac{b(k) - b(0)}{k}.$$

STABILITY ANALYSIS

Equations (9) may be written in matrix form as

$$A\underline{U}^{j+1} = B\underline{U}^j + \underline{c} \tag{11}$$

where A and B are tri-diagonal symmetric matrices of order (N-1) with respectively diagonal elements

$$2(2 + 6r\theta) \quad \text{and} \quad 2[2 - 6r(1 - \theta)\}$$

and off diagonal elements

$$1 - 6r\theta \quad \text{and} \quad 1 + 6r(1 - \theta) \quad ,$$

$\underline{U}^s$ (s = j,j+l) is the column vector

$$[U_1^s, U_2^s \ldots\ldots\ldots U_{N-1}^s]^T$$

and the components of the column vector $\underline{c}$ are known boundary values.

Equation (11) can "be rewritten as

$$\underline{U}^{j+1} = A^{-1} B\underline{U}^j + A^{-1}\underline{c}$$

and since $A^{-i}B$ is symmetric for stability we require that

$$\rho(A^{-1} B) \leqq 1 \quad ,$$

where $p(A^{-1}B)$ is the spectral radius of $A^{-1} B$. The eigenvalues of A and B are given respectively by

$$\lambda_i = 6 + 4(6r\theta - 1)\sin^2 a$$

and $\qquad\qquad$ i= 1, 2 .. .. N-1 ,

$$\mu_\iota = 6 - 4\{6r(1-\theta) + 1\}\sin^2 a$$

where $a = \frac{i\pi}{2N}$ and since the matrices A and B commute, the eigenvalues of A^{-1} B are

$$v_i = \frac{\mu_i}{\lambda_i} = \frac{3 - 2\sin^2\alpha - 12r(1-\theta)\sin^2\alpha}{3 - 2\sin^2\alpha - 12r\theta\sin^2\alpha}$$

Hence,

(i) if $\frac{1}{2} \leqq \theta \leqq 1$, $|v_i| \leqq 1$ for all $r > 0$,

so that the scheme (8) is unconditionally stable, and

(ii) if $0 \leqq \theta < \frac{1}{2}$, $|v_i| \leqq 1$ i.e., the scheme (8) is

stable when

$$r \leqq \frac{1}{6(1-2\theta)} .$$

We note that the stability conditions of the finite-difference scheme (8) are similar to those of the well known finite-difference replacement

$$\frac{U_i^{j-1} - U_i^j}{k} = \frac{1}{h^2} \{\theta\delta^2 U_i^{j+1} + (1-\theta)\delta^2 U_i^j\} . \tag{12}$$

Thus scheme (12), which includes as special cases the explicit ($\theta = 0$), fully implicit ($\theta = 1$) and Crank-Nicolson ($\theta = 1/2$) schemes, (see e.g. Richtmyer and Morton (1967),p. 189), is:

(i) for $\frac{1}{2} \leqq \theta \leqq 1$ unconditionally stable

and

(ii) for $0 < 0 < \frac{1}{2}$ stable if $r \leqq \frac{1}{2(1-2\theta)}$.

For $0 \leqq \theta < \frac{1}{2}$ the stability condition of scheme (12) is of course less restrictive than that of scheme (8).

TRUNCATION ERROR

If $u = u(x,t)$ is an exact solution of the equation in (1) then by the definition of Richtmyer and Morton (1967), the truncation error e[u] of the finite-difference scheme (8) is given by

$$e[u] = \frac{u_i^{j+1} - u_i^j + 4(u_i^{j+1} - u_i^j) + u_{i+1}^{j+1} - u_{i-1}^j}{6k}$$

$$- \frac{1}{h^2}\left\{ \theta\delta^2 u_i^{j+1} + (1-\theta)\delta^2 u_i^j \right\} .$$

By Taylor's series expansion about the point $(x_i, ,t_j)$ we find that

$$e[u] = 0(k) + 0(h^2)$$

and since this tends to zero as h, k $\longrightarrow$ 0 the scheme (8) is consistent with the differential equation of (1).

If u (x , t) is sufficiently differentialle the truncation error can be written as

$$e\ [u] = \frac{\partial^4 u}{\partial x^4}\left\{k(\frac{1}{2}-\theta) + \frac{1}{12}h^2\right\} + \frac{k^2}{6}\left\{\frac{\partial^3 u}{\partial t^3} - 3\theta\frac{\partial^4 u}{\partial x^2 \partial t^2}\right\}$$

$$+ 0(k^3) + 0(h^4) .$$

The truncation error of scheme (12) is

$$e\ [u] = \frac{\partial^4 u}{\partial x^4}\left\{k(\frac{1}{2}-\theta) + \frac{1}{12}h^2\right\} + \frac{k^2}{6}\left\{\frac{\partial^3 u}{\partial t^3} - 3\theta\frac{\partial^4 u}{\partial x^2 \partial t^2}\right\}$$

$$+ 0(k^3) + 0(h^4)$$

and thus, for both schemes (8) and (12) , $e[u] = 0(k) + 0(h^2)$.
For the special ease $\theta = \frac{1}{2}$ this reduces to $0(k^2) + 0(h^2)$.

It is interesting to observe that when $\theta = \frac{1}{2}$ the addition of equations (8) and (12) leads to the finite-difference scheme

$$\frac{U_{i-1}^{j+1} - U_{i-1}^{j} + 10(U_{i}^{j+1} - U_{i}^{j}) + U_{i-1}^{j+1} - U_{i-1}^{j}}{12k} = \frac{1}{2h^2}(\delta^2 U_i^{j+1} + \delta^2 U_i^{j}) \qquad (13)$$

whose truncation error is $0(k^2) + 0(h^4)$. Scheme (13), which is unconditionally stable, does in fact represent the six-point implicit scheme of maximum accuracy considered by Douglas (1956).

NUMERICAL RESULTS

The cubic spline method, defined by equation (2), is applied to the following two problems:

(i) Problem (l) with,

$$a(t) = b(t) = 0, \qquad f(x) = \sin\pi x, \qquad (14)$$

and

(ii) Problem (1) with,

$$a(t) = b(t) = 0, \qquad f(x) = \begin{cases} 2x, & 0 \leqq x \leqq \frac{1}{2}, \\ 2(1-x), & \frac{1}{2} \leqq x \leqq 1. \end{cases} \qquad (15)$$

Results, computed with $\theta = \frac{1}{2}$, for problems (14) and (I5) are given in Tables 1 and 2 respectively. In both tables the values computed at mesh points from the resulting finite-difference scheme (9), are given together with values calculated from the analytic solution and values computed by the application of the Crank-Nicolson scheme, At non-mash points the values computed, from the resulting spline function (3), are given together with values calculated from the analytic solution. In all computations $h = 0.1$ and $k = 0.01$ 30 that $r=1$ and, at each time level, the mesh points correspond to $x = 0.1(0.1)1.0$. The points corresponding to $x = 0.05(0.1)0.95$ are the intermediate points where results are computed by cubic spline interpolation. Since both problems (14) and (15) are symmetric with respect to $x = 0.5$ results are exhibited only for $0 < x \leqq 0.5$. In both tables the significance of the recorded values at each time level is :

(i) Upper entry: Value computed by the cubic spline method,

$$\theta = 1/2, \quad r = \frac{0.01}{(0.1)^2} = 1.$$

(ii) Middle entry : Value computed from the analytic solution.

(iii) Lower entry : Value computed by the Crank-Nicolson method,

$$r = \frac{0.01}{(0.1)^2} = 1.$$

It is important to observe that whereas in problem (14) the initial function $f(x)$ is $C^2[0,1]$ and the values M_i° are determined by setting

$$M_i^\circ = f''(x_i) = -\pi^2 \sin \pi x_i$$

in problem (15) $f'(x)$ has a discontinuity at $x = 0.5$ and so the values M_i° are computed from the tri-diagonal system (10).

t \ x	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50
0.05	0.09509 0.09550	0.18781 0.18865 0.18934	0.27594 0,27716	0.35724 0.35884 0.36016	0.42982 0.43169	0.49169 0.49390 0.49571	0.54160 0.54396	0.57802 0.58062 0.58274	0.60037 0.60298	0.60777 0.61050 0.61073
0.10	0.05777 0.05330	0.11415 0.11517 0.11602	0.16765 0.16921	0.21712 0.21907 0.22068	0.26111 0.26354	0.29884 0.30153 0.30374	0.32902 0.33209	0.35130 0.35447 0.35707	0.36 72 0.368 2	0.36938 0.3727 0.37545
0.15	0.03513 0.03559	0.06937 0.07031 0.07109	0.10196 0.10330	0.13196 0.13374 0.13522	0.15881 0.16089	0.18162 0.18408 0.18611	0.20011 0.20274	0.21351 0.21640 0.21879	0.22183 0.22474	0.22450 0.22754 0.23057
0.20	0.02133 0.02173	0.04216 0.04293 0.04356	0.06190 0.06306	0.08020 0.08165 0.08285	0.09641 0.09823	0.11038 0.11238 0.11404	0.12148 0.12377	0.12976 0.13211 0.13406	0.13466 0.13720	0.13644 0.13891 0.14096
0.25	0.01299 0.01327	0.02563 0.02621 0.02669	0.03769 0.03850	0.04874 0.04985 0.05077	0.05871 0.05997	0.06709 0.06861 0.06987	0.07397 0.07556	0.07887 0.08065 0.08214	0.08200 0.08376	0.08293 0.08480 0.08637
0.30	0.00787 0.00810	0.01557 0.01600 0.01635	0.02283 0.02350	0.02962 0.03043 0.03111	0.03557 0.03661	0.04077 0.04189 0.04281	0.04482 0.04613	0.04793 0.04924 0.05033	0.04968 0.05114	0.05040 0.05177 0.05292
0.35	0.00481 0.00494	0.00947 0.00977 0.01002	0.01395 0.01435	0.01800 0.01858 0.01906	0.02173 0.02235	0.02478 0.02557 0.02623	0.02738 0.02816	0.02913 0.03006 0.03084	0.03035 0.03122	0.03063 0.03161 0.03243
0.40	0.00290 0.00302	0.00575 0.00596 0.00614	0.00841 0.00876	0.01094 0.01134 0.01168	0.01309 0.01364	0.01506 0.01561 0.01607	0.01650 0.01719	0.01771 0.01835 0.01890	0.01829 0.01906	0.01862 0.01930 0.01987
0.45	0.00179 0.00184	0.00350 0.00364 0.00376	0.00518 0.00535	0.00665 0.00692 0.00716	0.00807 0.00833	0.00915 0.00953 0.00985	0.01017 0.01050	0.01076 0.01120 0.01158	0.01127 0.01164	0.01131 0.01178 0.01217

TABLE 1

t \ x	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50
0.05	0,07709 0.07693	0.15297 0.15206 0.15376	0.22592 0.22361	0.29229 0.28986 0.29322	0.34926 0.34916	0.39935 0.40001 0.40470	0.44420 0.44108	0.47660 0.47125 0.47705	0.49040 0.48970	0.49270 0.49591 0.50159
0.10	0.04738 0.04725	0.09341 0.09335 0.09481	0.13687 0.13714	0.17717 0.17756 0.18034	0.21363 0.21361	0.24465 0.24440 0.24823	0.26883 0.26918	0.28656 0.28733 0.29183	0.2998 0.2084	0.30249 0.30212 0.30685
0.15	0.02867 0.02885	0.05667 0.05699 0.05810	0.08332 0.08373	0.10789 0.10841 0.11051	0.12971 0.13042	0.14836 0.14921 0.15210	0.16349 0.16433	0.17458 0.17541 0.17881	0.18121 0.18216	0.18338 0.184 0.13
0.20	0.01745 0.01761	0.03446 0.03479 0.03560	0.05062 0.05112	0.06554 0.06618 0.06771	0.07885 0.07962	0.09023 0.09109 0.09320	0.09935 0.10032	0.10604 0.10709 0.10956	0.11014 0.11121	0.11152 0.11260 0.11520
0.25	0.01060 0.01075	0.02094 0.02124 0.02181	0.03077 0.03121	0.03984 0.04040 0.04149	0.04792 0.04861	0.05483 0.05561 0.05711	0.06038 0.06125	0.06446 0.06538 0.06713	0.06694 0.06789	0.06777 0.06874 0.07059
0.30	0.00644 0.00656	0.01273 0.01297 0.01337	0.01870 0.01905	0.02421 0.02467 0.02542	0.02912 0.02967	0.03332 0.03395 0.03499	0.03670 0.03739	0.03917 0.03991 0.04113	0.04068 0.04145	0.04119 0.04297 0.04325
0.35	0.00392 0.00401	0.00774 0.00792 0.00819	0.01136 0.01163	0.01471 0.01506 0.01558	0.01770 0.01812	0.02025 0.02073 0.02144	0.02230 0.02283	0.02381 0.02437 0.02520	0.02472 0.02530	0.02503 0.02562 0.02650
0.40	0.00238 0.00245	0.00470 0.00483 0.00502	0.00691 0.00710	0.00894 0.00919 0.00954	0.01076 0.01106	0.01231 0.01265 0.01314	0.01356 0.01394	0.01447 0,01488 0.01544	0.02503 0.01545	0.01521 0.01564 0.01624
0.45	0.00145 0.00149	0.00286 0.00295 0.00307	0.00420 0.00434	0.00544 0.00561 0.00585	0.00654 0.00675	0.00748 0.00773 0.00805	0.00824 0.00851	0.00879 0.00908 0.00946	0.00913 0.00943	0.00925 0.00955 0.00995

TABLE 2.

DISCUSSION

As expected the results obtained at mesh points indicate that when $\theta = \frac{1}{2}$ the accuracy of scheme (9) is comparable to that of the Crank-Nioolson scheme. Examination of the results in Tables 1 and 2 shows that the Crank-Nicolson scheme gives the more accurate solution for problem (14) and scheme (9) the more accurate solution for problem (15). We note however, that for the latter problem the effect of the discontinuity in $f'(x)$ is initially more damaging to the results of scheme (9) and, although this effect dies away quite- rapidly, for the first few time steps ($j < 4$) the Crank-Nicolson results are more accurate. From the above we conclude that when $\theta = \frac{1}{2}$ the scheme (9), considered as a finite-difference scheme for determining the solution at mesh points, is not in general superior to the Crank-Nicolson scheme. The important advantage of the method described in this report, over other existing methods, is that at each time level it produces a spline function from which the solution can be determined with ease at any intermediate point in the space direction as accurately as the solution at the mesh points. Furthermore, equations (4) or (5) give approximations to $\frac{\partial u}{\partial x}$ at the mesh points.

The method may of course be used with values of θ other than $\frac{1}{2}$. However, when $\theta < \frac{1}{2}$ very small time steps are required for stability and the method is thus quite impractical. The case $\theta = 1$ is that for which the least computational effort is required for determining the spline function at each time level. For this value of θ scheme (9) is unconditionally stable and its accuracy is comparable to that of the fully implicit scheme.

APPLICATION TO OTHER PARABOLIC PROBLEMS

The cubic spline method may be used for the solution of more general parabolic problems. Its application to two such problems is briefly outlined below.

(i) If the boundary conditions in problem (1) are

$$\frac{\partial u}{\partial x} + a_o U = b_o(t), \qquad x = 0, \ t \geqq 0,$$

and

$$\frac{\partial u}{\partial x} + a_1 u = b_t(t), \qquad x = 1, \ t \geqq 0,$$

then the differential equation is approximated by equation (2), and the boundary conditions at the points (x_0,tj) and (x_N, t_j) respectively by

$$Sj(xo) + aoU_\circ^j = b_0(tj) \tag{16}$$

and

$$Sj(x_N) + a_1 U_i^j - b_i (t_j) . \tag{17}$$

The approximation (2) of the heat conduction equation leads as before to the finite-difference replacement (9) at internal mesh points and (16) and (17) when used in conjunction with equations (4), (5) and (2) give the relations

$$(2 + 6r\theta - 6rh\theta a_o)U_\circ^{j+i} + (1 - 6r\theta) U_1^{j+1} =$$

$$= \{2 - 6r(1-\theta) + 6rhao(1-\theta)\} U_\circ^j + \{1 + 6r(1 - \theta)\} U_1^j - 6rhc_\circ^j \tag{18}$$

and

$$(1 - 6r\theta) U_{N-1}^{j+1} \quad (2 + 6r\theta + 6rh\theta a_1) U_N^{j+i} \quad [1 + 6r(1 - \theta)\} U_{N-1}^j$$

$$+ \{2 - 6r(1-\theta)-6rha_1(1-\theta)] U_N^j + 6rhc_1^j \tag{19}$$

where

$$c^j_i = \theta b_i(t_{j+1}) + (1-\theta) b_i(t_j), \quad i = 0,1.$$

Thus, the equations (18) and (19), which hold on the boundary lines x = 0 and x = 1 respectively, together with equations (9) produce a tri-diagonal system of (N+ 1) equations which is solved for the unknowns U_0^{j+1}, U_1^{J+1} .... U_N^{j+1}

(ii) We consider a parabolic equation of the form

$$\frac{\partial \phi}{\partial t} = \frac{\partial^2 \phi}{\partial x^2} + a \frac{\partial \phi}{\partial x} + b\phi \tag{20}$$

where a and b are constants, and assume boundary and initial conditions similar to those of problem (1).

The substitution

$$\phi = \theta^{\frac{1}{2}ax} u$$

transforms (20) {see Todd (1956)} into the equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + (b - \frac{1}{4}a^2)u$$

and this is approximated at the point (x_t, t_j) by

$$\frac{U_i^{j+1} + U_i^j}{k} = \theta M_i^{j+1} + (1-\theta) M_i^j + (b - \frac{1}{4}a^2) U_i^j \quad . \tag{21}$$

Equation (21) may be written as

$$\frac{U_i^{j+1} - (1 + b'k) U_i^j}{k} = \theta M_i^{j+1} + (1-\theta) M_i^j \quad ,$$

where $b' = b - \frac{1}{4}a^2$, and with $\theta = \frac{1}{2}$ for example it leads to one finite-difference equation

$$(1-3r)U_{i-1}^{j+1} + 2(2+3r)\,U_i^{j+1} + (1-3r)\,U_{i-1}^{j+1}$$
$$= \{(1+b'k) + 3r\}\,U_{i-1}^{j} + 2\{2(1+b'k) - 3r\}\,U_i^{j}$$
$$+ \{(1+b'k) + 3r\}\,U_{i-1}^{j} \quad .$$

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