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On the comparison of two numerical methods for conformal mapping

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ABSTRACT

Let G be a simply-connected domain in the t—plane ($t = x + iy$), bounded by the three straight lines $x = 0$, $y = 0$, $x = 1$ and a Jordan arc with cartesian equation $y = \tau(x)$. Also, let g be the function which maps conformally a rectangle R onto G, so that the four corners of R are mapped onto those of G. In this paper we show that the method considered recently by Challis and Burley [2], for determining approximations to g, is equivalent to a special case of the well-known method of Garrick [8] for the mapping of doubly-connected domains, Hence, by using results already available in the literature, we provide some theoretical justification for the method of [2].

Keywords : Numerical conformal mapping, method of Garrick.

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1. Introduction

Let Ω be a given doubly-connected domain bounded by two closed Jordan curves, and let f be a function which maps conformally a circular annulus A onto Ω. Also, let G be a simply-connected domain of the form

$$G = \{(x, y) : 0 < x < 1, 0 < y < \tau(x)\}, \tag{1.1}$$

and let g be a function which maps conformally a rectangle R onto G, so that the four corners of R are mapped onto those of G. This paper is concerned with the comparative study of two numerical methods for computing approximations to the conformal maps f and g. The two methods are respectively the well-known method of Garrick [4,8,11-13,15], for the approximation of $f: A \rightarrow \Omega$, and a method proposed recently by Challis and Burley [2], for the approximation of $g: R \rightarrow G$. The motivation for undertaking this study emerges from [7] and [10,p.p.73-74], where it is pointed out that the two methods appear to be closely connected, for the following two reasons:

(i) The problem of determining $g: R \rightarrow G$ is equivalent to that of determining $f: A \rightarrow \Omega$, for a certain symmetric doubly-connected domain Ω.

(ii) Both methods are iterative, and both involve Fourier analysis and Fourier synthesis at each iterative step. Furthermore, both methods can be made computationally efficient by the use of the fast Fourier transform.

In the present paper we investigate further the connection between the two methods, and show that the method of Challis and Burley is, in fact, a special case of the method of Garrick. Hence, by using results already available in the literature, we provide some theoretical justification for the method of [2], We also show that the method of Garrick can be applied directly to the problem of determining $g: R \rightarrow G$, for a wider class of domains than that defined by (1.1).

The details of the presentation are as follows:

Sections 2 and 3 concern the method of Garrick, and are based on the detailed treatment contained in [4,p.p.194-207]. More specifically, in Section 2 we summarize the theory on which the method is based, and in Section 3 we describe the general Garrick algorithm. In this latter section, we also summarize the available theory concerning the convergence of the method.

In Section 4, we describe the simplifications that occur in the Garrick method, in the two cases where the boundary curves of Ω are as follows: (a) Both curves are symmetric with respect to the real axis. (b) The outer curve is the unit circle and the inner curve is symmetric with respect to the real axis.

Section 5 contains the main results of the paper. Here we show that the algorithm of Challis and Burley is equivalent to the simplified algorithm of Garrick, corresponding to a domain of the form (b). We also show that the Garrick algorithm, for a geometry of the form (a), can be applied directly to the problem of determining $g: R \to G$, in the case where G has the more general form,

$$G = \{(x\,,y): 0 < x < 1\,, \tau_1(x) < y < \tau_2(x)\}\,. \tag{1.2}$$

Finally, in Section 6 we present the results of several numerical experiments and make a number of observations concerning the convergence of the method of Garrick, and hence of that of Challis and Burley.

2. Preliminary results

Let A_q be the annulus

$$A_q = \{\, z : q < |\,z\,| < 1 \,\}\,,\quad 0 < q < 1\,, \tag{2.1}$$

and let the function F be regular in A_q and continuous on $\overline{A}_q$. On the boundary of A_q , let

$$F(e^{i\phi}) = u_1(\phi) + iv_1(\phi) \quad , \quad 0 \le \phi \le 2\pi \ , \tag{2.2}$$

and

$$F(qe^{i\phi}) = u_2(\phi) + iv_2(\phi) \quad , \quad 0 \le \phi \le 2\pi \ , \tag{2.3}$$

and observe that

$$\frac{1}{2\pi}\int_0^{2\pi} F(e^{i\phi})d\phi = \frac{1}{2\pi}\int_0^{2\pi} F(qe^{i\phi})d\phi$$

$$= \alpha + i\beta \ , \tag{2.4}$$

say. Also let L_2 be the space of real 2π-periodic functions which are square integrable in [0, 2π] , Then, the real and imaginary parts of the boundary values (2.2)-(2.3) are related by

$$v_1(\phi) = \beta + (K+R_q)\,[u_1(\bullet)]\,(\phi) + S_q\,[u_2(\bullet)]\,(\phi) \tag{2.5}$$

and

$$v_2(\phi) = \beta - S_q\,[u_1(\bullet)]\,(\phi) - (K+R_q)\,[u_2(\bullet)]\,(\phi) \tag{2.6}$$

where K, R_q and S_q are three linear integral operators in the space L_2 . These ooperators are defined as follows.

K is the well-known operator for conjugation on the unit circle. That is, for $u \in L_2$, $K[u(\bullet)](\phi)$ is defined by the Cauchy principal value integral

$$K[u(\bullet)](\phi) := \frac{1}{2\pi} PV\int_0^{2\pi} \cot\left(\frac{\phi - t}{2}\right) u(t)\,dt \tag{2.7}$$

The other two operators, R_q and S_q depend on the real parameter q, $0 < q < 1$, and are defined by

$$R_q\,[u(\cdot)](\phi) := \frac{1}{2\pi}\int_0^{2\pi} g_q(\phi - t)\,u(t)\,dt \tag{2.8}$$

and

$$S_q\,[u(\cdot)]\,(\phi) := \frac{1}{2\pi}\int_0^{2\pi} h_q(\phi - t)\,u(t)\,dt \tag{2.9}$$

where the kernels g_q and h_q are given by the absolutely convergent series

$$g_q(\phi) = 4\sum_{k=1}^{\infty} \frac{q^{2k}}{1-q^{2k}} \sin k\phi \tag{2.8a}$$

and

$$h_q(\phi) = -4\sum_{k=1}^{\infty} \frac{q^{2k}}{1-q^{2k}} \sin k\phi . \tag{2.9a}$$

The relations (2.5) - (2.6) are derived in [4, p. p. 194-197], where also the properties of the operators K, R_q and S_q are studied in detail; see also [16,p.568], [11], [12,p.499] and [13, §17.4]. In particular, we note the following three basic results:

(i) If $u \in L_2$, then also $K[u]$, $R_q[u]$, $S_q[u] \in L_2$.

(ii) For any constant c,

$$K[c] = R_q[c] = S_q[c] = 0 . \tag{2.10}$$

(iii) Let s $(\phi;u)$ denote the Fourier series of a function $u \in L_2$. That is

$$s(\phi;u) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k \cos k\phi + b_k \sin k\phi) , \tag{2.11}$$

where a_k , b_k are the real Fourier coefficients of u . Then,

$$s(\phi;K[u]) \sim \sum_{k=1}^{\infty} (-b_k \cos k\phi + a_k \sin k\phi) , \tag{2.12}$$

$$s(\phi;R_q[u]) \sim \sum_{k=1}^{\infty} \frac{2q^{2k}}{1-q^{2k}}(-b_k \cos k\phi + a_k \sin k\phi), \tag{2.13}$$

$$s(\phi;S_q[u]) \sim \sum_{k=1}^{\infty} \frac{-2q^{k}}{1-q^{2k}}(-b_k \cos k\phi + a_k \sin k\phi) . \tag{2.14}$$

(In fact, the series on the right of (2.13) and (2.14) converge absolute and uniformly. For this reason, the sign "~" in (2.13) and (2.14) can be replaced by "=".)□

We return now to the boundary values (2.2) - (2.3) of F, and let

$$s(\phi;u_j) \sim \frac{1}{2} a_{j,0} + \sum_{k=1}^{\infty}(a_{j,k}\cos k\phi + b_{j,k}\sin k\phi)\ ;\ j=1,2\ , \tag{2.15}$$

Where, because of (2.4),

$$a_{1,0} = a_{2,0} = 2a. \tag{2.15a}$$

Then it follows from (2.5) - (2.6) and (2.11) - (2 .14) , that

$$s(\phi;v_j) \sim \beta + \sum_{k=1}^{\infty}(A_{j,k}\cos k\phi + B_{j,k}\sin k\phi);\ j=1,2\ , \tag{2.16}$$

Where

$$\left.\begin{aligned} A_{1,k} &= \{-b_{1,k}(1+q^{2k}) + 2b_{2,k}q^{k}\}/(1-q^{2k})\ , \\ B_{1,k} &= \{a_{1,k}(1+q^{2k}) - 2a_{2,k}q^{k}\}/(1-q^{2k})\ , \\ A_{2,k} &= \{-2b_{1,k}q^{k} + b_{2,k}(1+q^{2k})\}/(1-q^{2k})\ , \\ B_{2,k} &= \{2a_{1,k}q^{k} - a_{2,k}(1+q^{2k})\}/(1-q^{2k})\ . \end{aligned}\right\} \tag{2.16a}$$

Furthermore, it can be shown easily that the Laurent series of the function F is

$$F(z) = \sum_{k=-\infty}^{\infty} c_k z^k\ ,\ \ z \in A_q\ , \tag{2.17}$$

where

$$c_0 = \alpha + i\beta, \tag{2.17a}$$

and

$$\left.\begin{aligned} c_k &= \{(a_{1,k} - a_{2,k}q^k) - i(b_{1,k} - b_{2,k}q^k)\}/(1-q^{2k})\ , \\ c_{-k} &= q^k\{(a_{2,k} - a_{1,k}q^k) + i(b_{2,k} - b_{1,k}q^k)\}/(1-q^{2k})\ ; \end{aligned}\right\} \tag{2.17b}$$

$$k = 1,2,\ldots;$$

see [4,p.197] and also [11,p.20] and [16,p.569] .

3. <u>The method of Garrick</u>

Let $\partial\Omega_j$; $j = 1,2$, be two Jordan curves in the w-plane, which are starlike with respect to $w = 0$, and are given in polar coordinates by

$$\partial\Omega_j = \{w : w = \rho_j(\theta) e^{i\theta}, \ 0 \le \theta \le 2\pi\} \ ; \ j = 1,2 , \tag{3.1}$$

where $0 < \rho_2(\theta) < \rho_1(\theta)$, $\theta \in [0,2\pi]$. Also, let Ω be the doubly-connected domain which is bounded externally and internally by $\partial\Omega_1$ and $\partial\Omega_2$ respectively, i.e.

$$\Omega = \mathrm{Int}(\partial\Omega_1) \cap \mathrm{Ext}(\partial\Omega_2). \tag{3.2}$$

Then, for a certain value q, $0 < q < 1$, Ω is conformally equivalent to the annulus

$$A_q = \{z : q < |z| < 1\} , \tag{3.3}$$

and the reciprocal of the inner radius

$$M := 1/q , \tag{3.4}$$

is called the conformal module of Ω.

Let f be the function which maps conformally A_q onto Ω, and recall the following well-known results:

(i) f can be extended continuously to $\overline{A}_q$,

(ii) On the boundaries $|z| = 1$ and $|z| = q$ of A_q, f is given by two continuous boundary correspondence functions θ_1 and θ_2, which are defined by

$$f(e^{i\phi}) = \rho_1(\theta_1(\phi)) e^{i\theta_1(\phi)} , \quad 0 \le \phi \le 2\pi , \tag{3.5}$$

and

$$f(qe^{i\phi}) = \rho_2(\theta_2(\phi)) e^{i\theta_2(\phi)} , \quad 0 \le \phi \le 2\pi , \tag{3.6}$$

(iii) The requirement that $|z| = 1$ is mapped onto $\partial\Omega_1$ defines f uniquely, apart from a rotation in the z-plane. Here, we normalize the mapping by requiring that

$$\int_0^{2\pi}(\theta_1(\phi)-\phi)d\phi=\int_0^{2\pi}(\theta_2(\phi)-\phi)d\phi=0. \square \tag{3.7}$$

The method of Garrick involves the iterative solution of three nonlinear integral equations, for the unknown boundary correspondence functions θ_1 and θ_2, and the unknown inner radius q of A_q. The method is based on applying the results of Section 2 to the auxiliary function

$$\begin{aligned} F(z) &:= \log\{f(z)/z\} \\ &= \log|f(z)/z| + i(\arg f(z) - \arg z) . \end{aligned} \tag{3.8}$$

This function is regular and single-valued in A_q, continuous on $\overline{A}_q$ and has the boundary values

$$F(e^{i\phi})=\log\rho_1(\theta_1(\phi)) + i(\theta_1(\phi)-\phi) \tag{3.9}$$

and

$$F(qe^{i\phi})=\log\rho_2(\theta_2(\phi)-\log q + i(\theta_2(\phi)-\phi) . \tag{3.10}$$

Therefore, the relations (2.5) - (2.6) hold with

$$u_1(\phi) = \log\rho_1(\theta_1(\phi)) , \quad u_2(\phi) = \log\rho_2(\theta_2(\phi)) - \log q \tag{3.11}$$

and

$$v_j(\phi)=\theta_j(\phi)-\phi \ ; \ j=1,2. \tag{3.12}$$

More specifically, because of (3.7), the constant g in (2.5) - (2.6) is zero and the two relations give respectively

$$\theta_1(\phi)=\phi+(K+R_q)[\log\rho_1(\theta_1(\cdot))](\phi)+S_q[\log\rho_2(\theta_2(\cdot))](\phi) \tag{3.13}$$

and

$$\theta_2(\phi)=\phi-S_q[\log\rho_1(\theta_1(\cdot))](\phi)-(K+R_q)[\log\rho_2(\theta_2(\cdot))](\phi) \tag{3.14}$$

(In deriving (3.13) - (3.14), we made use of the fact that $(K+R_q)[\log q] = S_q[\log q] = 0$; see E_q. (2.10).) Also, from (2.4),

$$-\log q = \log M$$

$$= \frac{1}{2\pi}\int_0^{2\pi}\{\log\rho_1(\theta_1(\phi)) - \log\rho_2(\theta_2(\phi))\}d\phi \qquad (3.15)$$

Thus, the functions θ_j ; $j = 1,2$, and the radius q of A_q satisfy the equations (3.13) - (3.15) . These three equations are known as the integral equations of Garrick.

The existence of a solution $(\theta_1, \theta_2; q)$ of (3.13) - (3.15) is guaranteed by the fact that any doubly-connected domain is conformally equivalent to a circular annulus. Regarding uniqueness, we have the following theorem which is proved in [4].

Theorem 3.1 ([4,p.200], The equations (3.13) - (3 .15) always have a unique solution $(\theta_1,\theta_2 ;q)$, such that the functions θ_j ; $j = 1,2$, are continuous and strictly monotonic increasing in $[0,2\pi]$.□

Another uniqueness result, which does not invlove the monotonicity requirement of Theorem 3.1, is also established in [4], under the assumption that the boundary curves $\partial\Omega_j$; $j =1,2$, satisfy an εδ-condition. This condition is defined as follows.

Definition 3.1 ([4.p.200]) . The Jordan curves

$$\partial\Omega_j = \{w : w = \rho_j(\theta)e^{i\theta} , 0 \le \theta \le 2\pi\} ; \quad j=1,2 , \qquad (3.16)$$

are said to satisfy an εδ-condition if:

(i) $\rho_1(\theta) \geq: 1$ and $\rho_2(\theta) \le m < 1$.

(ii) There exist positive constants a,b and e such that

$$a(1+\varepsilon)^{-1} \le \rho_1(\theta) \le a(1+\varepsilon) \text{ and } b(1+\varepsilon)^{-1} \le \rho_2(\theta) \le b(1+\varepsilon) . \qquad (3.17)$$

(iii) The functions $\rho_j(\theta)$; $j = 1,2$, are absolutely continuous in $[0,2\pi]$ and, for almost all $\theta \in [0,2\pi]$,

$$|\rho_j'(\theta)/\rho_j(\theta)| \leq \varepsilon\delta \ ; \ j = 1,2 \ , \tag{3.18}$$

where

$$\delta^{-1} = \frac{1+m}{1-m} + \frac{4m}{(1-m)^2} \ . \square \tag{3.18a}$$

Theorem 3.2 ([4,p.200]). If the boundary curves $\partial\Omega_j$; j = 1,2, satisfy an $\varepsilon\delta$-condition with some $\varepsilon < 1$, then the equations (3.13) - (3.15) have a unique solution $(\theta_1\ \theta_2\ ;q)$, with continuous θ_j ; j=1,2.□

We consider now the following Jacobi type iteration for the analytical solution of Eqs (3.13) - (3.15).

Iteration 3.1

(I) Set

$$\theta_j^{(0)}(\phi) = \phi \ ; \ j = 1,2 \ .$$

(II) Do steps (a) and (b) , with n = 0,1,2,... , until convergence:

(a) Determine q_n by means of

$$-\log q_n = \frac{1}{2\pi}\int_0^{2\pi}\left\{\log\rho_1\ (\theta_2^{(n)}(\phi) - \log\rho_2\ (\theta_2^{(n)}(\phi))\right\}d\phi \ .$$

(b) Determine

$$\theta_1^{(n)}(\phi) = \phi + (K + R_{q_n.})\left[\log\ \rho_1(\theta_1^{(n)}(\cdot))\right](\phi) + S_{q_n}\left[\log\ \rho_2(\theta_2^{(n)}(\cdot))\right](\phi)$$

and

$$\theta_2^{(n+1)}(\phi) = \phi - S_{q_n}\left[\log\ \rho_1(\theta_1^{(n)}(\cdot))\right](\phi) - (K + R_{q_n.})\left[\log\ \rho_2(\theta_2^{(n)}(\cdot))\right](\phi) \ . \square$$

The following convergence theorem is proved in [4].

Theorem 3.3 ([4,p.202]). Let the boundary curves $\partial\Omega_j$; j = 1,2, satisfy an $\varepsilon\delta$-condition, with some $\varepsilon < 1$, and let $(\theta_1, \theta_2;q)$ be the unique solution of Eqs (3.13) - (3.15). Also, let $(\theta_1^{(n)},\theta_2^{(n)};q_n)$ be the nth iterative values of Iteration 3.1. Then,

$$|\theta_j^{(n)}(\varphi)-\theta_j(\varphi)| \le \sqrt{2\pi A}\ \varepsilon^{n/2} \quad ; \quad j = 1,2 \ , \tag{3.19}$$

and

$$|q_n - q| \le \frac{1}{4} A\ \varepsilon^{n+1} \ , \tag{3.20}$$

where

$$A = 2\left(\frac{1+m}{1-m}\right)\varepsilon \quad \text{and} \quad B = \frac{4\varepsilon}{1-\varepsilon^2} \ . \square \tag{3.21}$$

In practice, Iteration 2.1 is performed in discretized form where, in every iterative step, each of the functions log $\rho_j(\theta_j(\phi))$; $j = 1,2$, is replaced by its interpolating trigonometric polynomial of degree N corresponding to the nodes $r\pi/N$; $r = 0\ (1)\ 2N-1$. The resulting algorithm is based on the results (2.15) - (2.16), and may be stated as follows.

Algorithm 3.1.

(I) Set

$$\theta_j^{(0)} = \phi \ ; \ j = 1,2 \ .$$

(II) Do steps (a), (b) and (c) with n = 0,1,2,..., until convergence:

(a) Compute the coefficients $a_{j,k}^{(n)}$ and $b_{j,k}^{(n)}$ of the trigonometric Polynomials

$$T_j^{(n)}(\phi) = \frac{1}{2} a_{j,0}^{(n)} + \sum_{k=0}^{N-1} (a_{j,k}^{(n)} \cos k\phi + b_{j,k}^{(n)} \sin k\phi) + \frac{1}{2} a_{j,N}^{(n)} \cos N\phi \ ; \ j = 1,2 \ , \tag{3.22}$$

Which interpolate the function $\log\rho_j\,(\theta_j^{(n)}(\phi)$; $j = 1,2$, at the points

$$\phi_r = r\pi/N \ ; \ r = 0(1)2N-1 \ .$$

That is,

$$\left.\begin{aligned} a_{j,k}^{(n)} &= \frac{1}{N} \sum_{r=0}^{2N-1} \log \rho_j\,(\theta_j^{(n)}.(\phi_r)) \cos k\phi_r \ , \quad k = 0(1)N \ , \\ &\text{and} \\ b_{j,k}^{(n)} &= \frac{1}{N} \sum_{r=1}^{2N-1} \log \rho_j\,(\theta_j^{(n)}(\phi_r)) \sin \phi_r \ ; \quad k = 1(1)N-1 \ . \end{aligned}\right\} \tag{3.22a}$$

(b) Compute q_n by means of

$$-\log q_n = \frac{1}{2}\ (a_{1,0}^{(n)} - a_{2,0}^{(n)})$$

$$= \frac{1}{2N}\sum_{r=0}^{2N-1}\{\log\rho_1\ (\theta_1^{(n)}(\phi_r)) - \log\rho_2\ (\theta_1^{(n)}(\phi_r))\} \tag{3.23}$$

(c) With r = 0(1)2N-1, compute the values,

$$\theta_1^{(n+1)}(\phi_r) = \phi_r + (K+R_{q_n})[T_1^{(n)}(\cdot)](\phi_r) + S_{q_n}[T_2^{(n)}(\cdot)](\phi_r)$$

and

$$\theta_2^{(n+1)}(\phi_r) = \phi_r - S_{q_n}[T_1^{(n)}(\cdot)](\phi_r) - (K+R_{q_n})[T_1^{(n)}(\cdot)](\phi_r)$$

That is, from (2.15)-(2.16)

$$\theta_j^{(n+1)}(\phi_r) = \phi_r + \sum_{k=1}^{N-1}\{A_{j,k}^{(n)}\cos k\phi_r + B_{j,k}^{(n)}\sin k\phi_r\}\ ;\ r = 0(1)2N-1\ , \tag{3.24}$$

Where, for k = 1(1)N,

And

$$\left.\begin{aligned}
A_{1,k}^{(n)} &= \{-b_{1,k}^{(n)}(1+q_n^{2k}) + 2b_{2,k}^{(n)}q_n^k\}/(1-q_n^{2k})\ ,\\
B_{1,k}^{(n)} &= \{a_{1,k}^{(n)}(1+q_n^{2k}) - 2a_{2,k}^{(n)}q_n^k\}/(1-q_n^{2k})\ ,\\
A_{2,k}^{(n)} &= \{-2b_{1,k}^{(n)}q_n^k + b_{2,k}^{(n)}(1+q_n^{2k})\}/(1-q_n^{2k})\ ,\\
B_{2,k}^{(n)} &= \{2a_{1,k}^{(n)}q_n^k - a_{2,k}^{(n)}(1+q_n^{2k})\}/(1-q_n^{2k})\ ,
\end{aligned}\right\} \tag{3.24a}$$

with

$$b_{1,N}^{(n)} - b_{2,N}^{(n)} = 0\ .\ \square$$

The coefficients $a_{j,k}^{(n)}, b_{j,k}^{(n)}$; j =1,2, in Step II(a) of the algorithm can be computed efficiently, in 0(NlogN) operations by the use of the fast Fourier transform (FFT). Similarly, in Step II(b), the computation of the values $\theta_j\ (\phi_j)$; j = 1,2, can be performed by the use of the FFT. That is, the algorithm requires the application of four FFT s in each iterative step; see [12,13].

Let $\tilde{a}_{j,k}, \tilde{b}_{j,k}$ and $\tilde{A}_{j,k}, \tilde{B}_{j,k}$ be the final computed values of the coefficients (3.22a) and (3.24a), and let $\tilde{q}$ be the final approximation to the inner radius q. Then, at points on the boundaries $|z| = 1$ and $|z| = q$ of A_q, the mapping function is approximated by

$$\left.\begin{aligned} &\tilde{f}(e^{i\phi}) = \exp\{\tilde{T}_1(\phi) + i\tilde{\theta}_1(\phi)\} \\ \text{and} \quad & \\ &\tilde{f}(e^{i\phi}) = \exp\{\tilde{T}_1(\phi) + i\tilde{\theta}_1(\phi)\} \;, \end{aligned}\right\} \qquad (3.25)$$

where $\tilde{T}_j$; j =1,2, are the final interpolating polynomials, and $\tilde{\theta}_j$; j = 1,2, are the final approximations to the boundary correspondence functions θ_j ; j=1,2. That is,

$$\left.\begin{aligned} &\tilde{T}_j(\phi) = \frac{1}{2}\tilde{a}_{0,j} + \sum_{k=1}^{N-1}(\tilde{a}_{j,k}\cos k\phi + \tilde{b}_{j,k}\sin k\phi) + \frac{1}{2}\tilde{a}_{N,j}\cos N\phi \\ \text{and} \quad & \\ &\tilde{\theta}_j(\phi) = \phi + \sum_{k=1}^{N-1}(\tilde{A}_{j,k}\cos k\phi + \tilde{B}_{j,k}\sin k\phi) + \frac{1}{2}\tilde{B}_{j,N}\sin N\phi \;;\; j = 1,2\,. \end{aligned}\right\} \qquad (3.25a)$$

At interior points $z \in A_q$, f(z) may be approximated by making use of the Laurent expansion (2.17) of the auxiliary function F. That is,

$$f(z) = z\exp\left\{\sum_{k=-N}^{N}\tilde{c}_k z^k\right\} \qquad (3.26)$$

Where, with $b_{1,N} = b_{2,N} = 0$,

$$\tilde{c}_0 = \tilde{a}_{1,0}/2 \qquad (3.26a)$$

and

$$\tilde{c}_k = \{(\tilde{a}_{1,k} - \tilde{a}_{2,k}\tilde{q}^k) - i(\tilde{b}_{1,k} - \tilde{b}_{2,k}\tilde{q}^k)\}/(1-\tilde{q}^{2k}) \qquad (3.26b)$$

$$\tilde{c}_{-k} = \tilde{q}^k\{\tilde{a}_{2,k} - \tilde{a}_{1,k}\tilde{q}^k) + i(\tilde{b}_{2,k} - \tilde{b}_{1,k}\tilde{q}^k)\}/(1-\tilde{q}^{2k}) \;;\; k = 1(1)N\,.$$

Naturally, when $z = e^{i\phi}$ and $z = qe^{i\phi}$, (3.26) simplifies to (3.25). Therefore, E_q. (3.26) can be used to represent the approximate conformal map $\tilde{f}(z)$ at any point $z \in \bar{A}_q$.

Let $\underline{x}_j^{(n)}$; j = 1,2, denote respectively the 2N—dimensional column vectors

$$\underline{x}_j^{(n)} = \left\{ \theta_j^{(n)} (\phi_r) \right\}_{r=0}^{2N-1} \; ; \; j = 1,2 \; , \tag{3.27}$$

where $\theta_j^{(n)}(\phi_r)$; j = 1, 2, are the nth iterative values generated by Algorithm 3.1. Then, the results of the following theorem are established in [11].

Theorem 3.4. ([11, Theorems 4.6,6.1 and 12.1), Let the boundary curves $\partial\Omega_j$; j = 1,2, satisfy an εδ-condition, with some ε < 1, and let $(\theta_1, \theta_2; q)$ be the unique solution of Eqs (3.13) - (3.15). Also, let

$$\underline{x}_j = \left\{ \theta_j (\phi_r) \right\}_{r=0}^{2N-1} \; ; \; j = 1,2 \; . \tag{3.28}$$

Then, the sequence of iterates $\{(\underline{x}_1^{(n)}, \underline{x}_2^{(n)}; q_n)\}$, generated by Algorithm 3.1, converge linearly to a unique solution $(\underline{x}_1^*, \underline{x}_2^*; q^*)$ where

$$\| \underline{x}_1^* - \underline{x}_1 \| + \| \underline{x}_2^* - \underline{x}_2 \| = 0(1/N) \tag{3.29}$$

and

$$|q^* - q| = 0(1/N) \; . \tag{3.30}$$

Furthermore, if the functions ρ_j ; j = 1, 2, satisfy the additional condition

$$\left| \frac{\rho_j'}{\rho_j}(\theta_1) - \frac{\rho_j'}{\rho_j}(\theta_2) \right| \le M \left| \theta_1 - \theta_2 \right| , \tag{3.31}$$

for some M > 0, then

$$\| \underline{x}_1^* - \underline{x}_1 \| + \| \underline{x}_2^* - \underline{x}_2 \| = 0(1/N^2) \tag{3.32}$$

and

$$|q^* - q| = 0(1/N^2) \; .$$

(Given a vector $\underline{x} = \left\{ x_r \right\}_{r=0}^{2N-1}$, the norm $\|\,.\|$ in (3.29) and (3.32) is defined by

$$\|\underline{x}\| := \left\{\frac{1}{2N}\sum_{r=0}^{2N-1} x_r^2\right\}^{\frac{1}{2}} .) \square$$

4. Special geometries

4.1 Symmetric boundary curves

If the two boundary curves $\partial\Omega_j$; j = 1,2, are both symmetric with respect to the real axis, then

$$\theta_j(-\phi) = -\theta_j(\phi) \; ; \; j=1,2 , \tag{4.1}$$

and

$$\log\rho_j(\theta_j(-\phi) = \log\rho_j(\theta_j(\phi)) \; ; \; j = 1,2 , \tag{4.2}$$

Therefore, if u_j, and v_j; j = 1,2, denote the functions (3.11) and (3.12), then $u_j(-\phi) = u_j(-\phi)$, $v_j(-\phi) = v_j(-\phi)$; j = 1,2, and the results (2.15)- (2.17) simplify as follows:

R4.1.1. The Fourier series of the functions u_j; j = 1,2, are of the form

$$s(\phi;u_j) \sim \frac{1}{2}a_{j,0} + \sum_{k=1}^{\infty} a_{j,k}\cos k\phi \; ; \quad j = 1,2 . \tag{4.3}$$

That is, the sine coefficients b. , in (2.15) are all zero.□

R4 .1 .2. The Fourier series of the conjugate functions v_j ; j = 1,2, are

$$s(\phi;v_j) \sim \sum_{k=1}^{\infty} B_{j,k}\sin k\phi \; ; \; j = 1,2 , \tag{4.4}$$

where the coefficients $B_{1,k}$ and $B_{2,k}$ are given by the second and fourth formulae in (2.16a). □

R4. 1 .3. The Laurent series expansion of the auxiliary function F is

$$F(z) = \sum_{k=-\infty}^{\infty} c_k z^k \; , \tag{4.5}$$

where the c_k are all real and are given by

$$c_0 = a_{1,0}/2 , \tag{4.5a}$$

and

$$c_k = \{a_{1,k} - a_{2,k}q^k\}/(1-q^{2k}) , \; c_{-k} = \{a_{2,k}q^k - a_{1,k}q^{2k}\}/(1-q^{2k}) ;$$

$$k = 1,2,\ldots . \square \tag{4.5b}$$

Iteration 2.1 and the results of Theorems 3.1-3.4 remain unchanged, but because of R4.1.1, formulae (3.22) and (3.23) of Algorithm 3.1 simplify respectively to,

$$T_j^{(n)}(\phi) = \sum_{k=0}^{N} a_{j,k}^{(n)} \cos k\phi \; ; \; j = 1,2 , \tag{4.6}$$

with

$$a_{j,k}^{(n)} = \frac{2}{N} \sum_{r=0}^{N}{}' \log\rho_j\,(\theta_j^{(n)}(\phi_r))\cos k\phi_r \; ; \; k = 0(1)N , \tag{4.6a}$$

and

$$-\log q_n = \frac{1}{N} \sum_{r=0}^{N} \{\log\rho_1(\theta_1^{(n)}(\phi_r)) - \log\rho_2(\theta_2^{(n)}(\phi_r))\} . \tag{4.7}$$

(In the above the prime indicates that the first and last terms of the series are to be taken with weight 1/2). Also, because of R4.1.2, equation (3.24) simplifies to

$$\theta_j^{(n+1)}(\phi_r) = \phi_r + \sum_{k=1}^{N-1} B_{j,k}^{(n)} \sin k\phi_r \; ; \; r = 1(1)N-1 , \tag{4.8}$$

where the coefficients $B_{1,k}^{(n)}$ and $B_{2,k}^{(n)}$ are given by the second and fourth formulae in (3.24a). Finally, because of R4.1.3, the series (3.26) for the approximate conformal map $\tilde{f}$ simplifies as indicated by (4.5).

4.2 Circular outer boundary and symmetric inner boundary

If the boundary $\partial\Omega_1$ of Ω is the unit circle, then $\rho_1(\theta) = 1$ and (3.13) - (3.15) simplify respectively to

$$\theta_1(\phi) = \phi + S_q[\log\rho_2(\theta_2(.))](\phi) \quad , \tag{4.9}$$

$$\theta_2(\phi) = \phi - (K + R_q)[\log\rho_2(\theta_2(.))](\phi) \tag{4.10}$$

and

$$\log q = \frac{1}{2\pi}\int_0^{2\pi} \log\rho_2(\theta_2(\phi))d\phi \quad . \tag{4.11}$$

Therefore, in this case, only the last two equations (4.10) - (4.11) need to be solved iteratively for the unknowns θ_2 and q. Because of this, Iteration 3.1 simplifies considerably as indicated below.

Iteration 4.1

(I) Set

$$\theta_2^{(0)} = \phi \ .$$

(II) Do steps (a) and (b), with n = 0,1,2,..., until convergence:

(a) Determine q_n by means of

$$\log q_n = \frac{1}{2\pi}\int_0^{2\pi} \log\rho_2(\theta_2^{(n)}(\phi))\,d\phi \ .$$

(b) Determine

$$\theta_2^{(n+1)}(\phi) = \phi - (K + R_{q_n})[\log\rho_2(\theta_2^{(n)}(.))](\phi) \ . \square$$

Further substantial simplifications occur if, in addition to $\partial\Omega_1$ being the unit circle, the inner boundary $\partial\Omega_2$ is symmetric with respect to the real axis. In this case, corresponding to R4.1.1 - R4.1.3, we have the following results.

R4. 2.1 . The Fourier series of the function $u_2(\phi) = \log\rho_2(\theta_2(\phi)) - \log q$

is of the form,

$$s(\phi;u_2) \sim \sum_{k=1}^{\infty} a_{2,k} \cos k\phi \,. \tag{4.12}$$

(The coefficient $a_{2,0}$ is zero, because $\int_0^{2\pi} u_2(\phi)d\phi = 0$.) □

R4. 2.2 . The Fourier series of the conjugate functions $v_j(\phi) = \theta_j(\phi) - \phi$; $j = 1,2$, are

$$s(\phi;v_j) \sim \sum_{k=1}^{N-1} B_{j,k} \sin k\phi \; ; \; j = 1,2 \,, \tag{4.13}$$

where

$$B_{1,k} = -\, 2a_{2,k}q^k / \, (1-q^{2k}) \,, \tag{4.13a}$$

and

$$B_{2,k} = -\, a_{2,k} \, (1-q^{2k}) / \, (1-q^{2k}) \,. \; □$$

R4.2.3. The Laurent series expansion of the auxilary function F is

$$F(z) = \sum_{k=1}^{\infty} c_k (z^k - 1/z^k) \,, \tag{4.14}$$

where the c_k are all real and are given by.

$$C_k = -\, q^k a_{2,\,k} \, / \, (1-q^{2k}) \; ; \; k = 1,2,\ldots \,. \; □ \tag{4.14a}$$

The resulting simplified algorithm can be stated as follows:

Algorithm 4.1

(I) Set

$$\theta_2^{(0)} = \phi \,.$$

(II) Do steps (a), (b) and (c), with n = 0,1,2,..., until convergence:

(a) Compute the coefficients $a_{2,k}^{(n)}$ of the trigonometric polynomial

$$T_2^{(n)}(\phi) = \sum_{k=0}^{N} a_{2,k}^{(n)} \cos k\phi \,, \tag{4.15}$$

which interpolates the function $\log\rho_2(\theta_2^{(n)}(\phi))$ at the points

$$\phi_r = r\pi/N \,;\, r = 0(1)N \;.$$

That is,

$$a_{2,k}^{(n)} = \frac{2}{N}\sum_{r=0}^{N}{}' \log\rho_2(\theta_2^{(n)}(\phi_r))\cos k\phi_r \,. \tag{4.15a}$$

(b) Compute q_n by means of

$$\log q_n = \frac{1}{2}\, a_{2,0}^{(n)}$$

$$= \frac{1}{N}\sum_{r=0}^{N}{}' \log\rho_2(\theta_2^{(n)}(\phi_r) \;. \tag{4.16}$$

(c) Compute the values

$$\theta_2^{(n+1)}(\phi_r) = \phi_r + \sum_{k=1}^{N-1} B_{2,k}^{(n)} \sin k\phi_r \;;\; r = 0(1)N \,, \tag{4.17}$$

where

$$B_{2,k}^{(n)} = -\, a_{2,k}^{(n)}(1+q_n^{2k})/(1-q_n^{2k}) \,. \tag{4.17a}$$

Let $\widetilde{a}_{2,k}$ and $\widetilde{q}$ be respectively the final computed values of the coefficients (4.15a) and the final approximation to q. Then, because of R4.2.3, formula (3.26) for the approximate conformal map simplifies to

$$\widetilde{f}(z) = z\exp\left\{ \sum_{k=1}^{N} \widetilde{c}_k\,(z^k - 1/z^k) \right\} \tag{4.18}$$

where

$$\widetilde{c}_k = -\,\widetilde{q}^{\,k}\,\widetilde{a}_{2,k}/(1-\widetilde{q}^{\,2k}) \,. \tag{4.18a}$$

The results of Theorems 3.1-3.4 remain unchanged but, because $\rho_1(\theta) \equiv 1$, Definition 3.1 of the $\varepsilon\delta$-condition simplifies in an obvious manner. In particular, the requirements (3.16) and (3.18) simplify respectively to

$$\rho_2(\theta) \leq m < 1 \tag{4.19}$$

and

$$|\rho_2'(\theta)/\rho_2(\theta)| \leq \varepsilon\delta . \tag{4.20}$$

Furthermore, it follows easily from the analysis in [4,p.p.203-206] that, in this special case, the value of δ^{-1} can be taken as

$$\delta^{-1} = \frac{1}{1-m^2}\left\{1+m^2 + \frac{4m^2}{1-m^2}\right\} . \tag{4.20a}$$

5. <u>The method of Challis and Burley [2] and its connection to the method of Garrick</u>

Let G be a simply-connected domain in the t-plane (t=x+iy), bounded by the three straight lines x = 0, y = 0, x = 1 , and a Jordan arc with cartesian equation y = τ (x), where τ is positive in [0,1], i.e.

$$G = \{(x,y) : 0 < x < 1 , 0 < y < \tau(x)\} . \tag{5.1}$$

Also, let A_j ; j = 1(1)4, be the four corners of G, i.e.

$$A_1 = (0,0) , \ A_2 = (1,0) , \ A_3 = (1,\tau(1)) , \ A_4 = (0,\tau(0)) , \tag{5.2}$$

and let R_H denote the rectangle

$$R_H = \{(\xi,\eta) : 0 < \xi < , 0 < \eta < H\} , \tag{5.3}$$

in the ζ-plane ($\zeta = \xi + i\eta$) . Then, it follows from the Riemann mapping theorem that, for a certain H, there exists a unique conformal map g: $R^H \rightarrow$ G, which takes the four vertices (0,0), (1,0), (1,H) and (0,H) of R^H respectively onto the four corners A_j ; j = 1(1)4, of G. The unique value of H for which the above conformal map is possible is called the conformal module of the quadrilateral defined by the four boundary points A_j ; $j = 1\ (1)4$; see [5,6,14].

The purpose of this section is to show that the algorithm proposed recently by Challis and Burley [2], for computing approximations to the conformal module H and the mapping function g, is equivalent to applying the simplified Garrick Algorithm 4.1 to a certain doubly-connected domain Ω. Our method of analysis is based on the following two observations; see [7] and [10,p.p.73-74]:

(i) By using the Schwarz reflection principle, the conformal map g can be extended to map the infinite strip $\{(\xi,\eta) : -\infty < \xi < \infty,\ 0 < \eta <(H\}$ onto the infinite domain bounded by the x-axis and the curve $y = \hat{\tau}(x)$ where $\hat{\tau}$ is the periodic function defined by

$$\hat{\tau}(\pm x) = \tau(x)\ ,\ \ 0 \le x \le 1\ ,$$

$$\hat{\tau}\ (2+x) = \hat{\tau}(x)\ . \tag{5.4}$$

This also shows that the function $g(\zeta) - \zeta$ is periodic with period 2. □

(ii) The exponential function

$$w = \exp(i\pi t) \tag{5.5}$$

maps G conformally onto the upper half of a symmetric doubly-connected domain Ω, bounded externally by the unit circle

$$\partial\Omega_2 = \{w : w = p_2(\theta)e^{i\theta}\ ,\ \ 0 \le \theta \le 2\pi\}\ , \tag{5.7}$$

where

$$\rho_2(\theta) = \exp\{-\pi\tau\ (\theta/\pi)\}\ ,\ \ 0 \le \theta \le \pi$$

and (5.7a)

$$\rho_2\ (2\pi-\theta) = \rho_2(\theta)\ ,\qquad \pi < \theta \le 2\pi\ .$$

Similarly

$$z = \exp\ (i\pi\zeta) \tag{5.8}$$

maps the rectangle R_H conformally onto the upper half of the annulus

$$A_q = \{z : q < |z| < 1\} , \tag{5.9}$$

where

$$q = \exp(-\pi H) . \square \tag{5.9a}$$

It follows from the above that the problem of determining $g : R_H \to G$ is equivalent to that of determining $f : A_q \to \Omega$, where $\Omega = \text{Int}(\partial\Omega_1) \cap \text{Ext}(\partial\Omega_2)$ is the doubly-connected domain bounded by the unit circle (5.6) and the symmetric curve (5.4). That is, the problem considered by Challis and Burley [2] is equivalent to a problem of the type studied in Section 4.2, Because of this, the equations on which the method of [2] is based emerge easily from the results R4.2 .1 – R4 .2.3, as follows.

Let

$$\hat{x}(\xi) := \text{Re}\{g(\xi + iH)\} , \tag{5.10}$$

so that

$$g(\xi + iH) = \hat{x}(\xi) + i\ \tau(\hat{x}(\xi)) \quad , \ 0 \le \xi \le 1,. \tag{5.11}$$

Then, since

$$f(z) = \exp\left\{i\pi g\left(\frac{1}{i\pi} \log z\right)\right\} \quad , \tag{5.12}$$

the functions $\hat{x}(\xi)$ and $\tau(\hat{x}(\xi))$ are related to the inner boundary correspondence function θ_2 of the conformal map f by

$$\hat{x}(\xi) = \frac{1}{\pi}\,\theta_2(\pi\xi) \tag{5.13}$$

and

$$\tau(\hat{x}(\xi)) = -\frac{1}{\pi} \log\rho_2(\theta_2(\pi\xi)) \ , \ 0 \le \xi \le 1 . \tag{5.14}$$

Therefore, the results R4,2.1 - R4.2.3 imply the following:

<u>R5.1.</u> Let

$$\hat{u}(\pi\xi) := -\frac{1}{\pi} \log\rho_2(\theta_2(\pi\xi) + \frac{1}{\pi} \log q \tag{5.15}$$

$$= \hat{\tau}(\hat{x}(\xi)) - H .$$

Then,

$$s(\pi\xi;\hat{u}) \sim \sum_{k=1}^{\infty} \hat{a}_k \cos k\pi\xi \ , \tag{5.16}$$

where the coefficients $\hat{a}_k$ are related to those of (4.12) by

$$\hat{a}_k = -\frac{1}{\pi}\, a_k \ . \square \tag{5.16a}$$

<u>R5.2.</u> Let

$$\hat{v}(\pi\xi) := \frac{1}{\pi}\,(\theta_2(\pi\xi) - \pi\xi)$$

$$= \hat{x}(\xi) - \xi \ . \tag{5.17}$$

Then,

$$s(\pi\xi;\hat{v}) \sim \sum_{k=1}^{\infty} \hat{B}_k \sin k\pi\xi \ , \tag{5.18}$$

where

$$\hat{B}_k = \hat{a}_k (1+q^{2k})/(1-q^{2k}) \tag{5.18a}$$

or, since $q = \exp(-\pi H)$,

$$\hat{B}_k = \hat{a}_k \coth \pi k H \ . \square \tag{5.18b}$$

<u>R5.3.</u> Because of (4.14) and (5.12), the mapping function g has a series expansion of the form

$$g(\varsigma) = \varsigma + \sum_{k=1}^{\infty} \hat{c}_k \sin k\pi\varsigma \ , \tag{5.19}$$

where the coefficients $\hat{c}_k$ are related to those of (4.14) by

$$\hat{c}_k = \frac{2}{\pi}\, c_k \ . \tag{5.19.a}$$

Furthermore, because of (4.14a) and (5.16a), the coefficients $\hat{c}_k$ are related to the Fourier coefficients of the function $\hat{u}$ by

$$\hat{a}_k = \hat{c}_k \sin k\pi H \ . \square \tag{5.19b}$$

The above results contain all the relations on which the method of Challis and Burley [2] is based. (These relations are derived in [2] by solving, by the method of separation of variables, two harmonic mixed boundary value problems in the rectangle R_H.) The results also show that Algorithm 4.1 can be expressed, in terms of the functions $\hat{x}(\xi)$, $\tau(\hat{x}(\xi))$ and the conformal module H, as indicated below.

<u>Algorithm 5.1</u>

(I) Set

$$\hat{x}^{(0)}(\xi) = \xi .$$

(II) Do steps (a), (b) and (c), with n=0,1,2,..., until convergence:

(a) Compute the coefficients $\hat{a}_k^{(n)}$ of the trigonometric polynomial

$$\hat{T}^{(n)}(\xi) = \sum_{k=0}^{N} \hat{a}_k^{(n)} \cos k\pi\xi \tag{5.20}$$

which interpolates the function $\tau(\hat{x}^{(n)}(\xi))$ at the points

$$\xi_r = r/N ; \quad r = 0(1)N .$$

That is,

$$\hat{a}_k^{(n)} = \frac{2}{N} \sum_{r=0}^{N}{}' \tau(\hat{x}^{(n)}(\xi_r)) \cos k\pi \xi_r . \tag{5.20a}$$

(b) Compute

$$H_n = \frac{1}{2} \hat{a}^{(n)}$$

$$= \frac{1}{N} \sum_{r=0}^{N}{}' \tau(\hat{x}^{(n)}(\xi_r)) . \tag{5.21}$$

(c) Compute the values

$$\hat{x}^{(n+1)}(\xi_r) = \xi_r + \sum_{k=1}^{N-1} \hat{B}_k^{(n)} \sin k\pi\xi_r ; r = 0(1)N , \tag{5.22}$$

where

$$\hat{B}_k^{(n)} = \hat{a}_k^{(n)} \coth \pi k H_n . \ \square \tag{5.22a}$$

Let $\widetilde{a}_k$ and $\widetilde{H}$ be respectively the final computed values of the coefficients (5.20a), and the final approximation to the conformal module H. Then, because of (5.19), the approximation to the conformal map g is given by

$$\widetilde{g}(\varsigma) = \varsigma + \sum_{k=1}^{N} \widetilde{c}_k \sin k\pi\varsigma \, , \tag{5.23}$$

where

$$\widetilde{c}_k = \widetilde{a}_k \operatorname{cosec} \ k\pi H \tag{5.23a}$$

It turns out that Algorithm 5.1 is, in fact, precisely the algorithm proposed by Challis and Burley [2], This follows at once from their paper, by observing that their notations f, α, γ, a and b_k are related to ours by

$$f(x) \ := \ \tau(x) \, , \ \alpha(\xi) \ := \hat{x}(\xi) - \xi \, , \ \gamma(\xi) \ := \tau(\hat{x}(\xi)) \quad ,$$

$$a := 1/H \text{ and } b_k \ := \ \hat{B}_k \quad .$$

In other words, the algorithm of [2] is just the special case of the simplified Garrick Algorithm 4.1, corresponding to a function ρ_2 of the form (5.7). Therefore, the results of Theorems 3.1 -3.4 hold and, in this case, the requirements (3.16) and (3.18) of the $\varepsilon\delta$-condition can be replaced respectively by,

$$\exp\{-\pi\tau(x)\} \le m < 1 \, , \quad x \in [0,1] \, , \tag{5.24}$$

and

$$|\tau'(x)| \le \varepsilon\delta \, , \quad x \in [0,1] \, , \tag{5.25}$$

where δ is given by (4.20a). If the above two requirements are satisfied, with some $\varepsilon < 1$, then the Jacobi iterations of Algorithm 5.1 converge linearly to a unique solution $(\underline{\hat{x}}^*, H^*)$; $\underline{\hat{x}}^* = \{\hat{x}^* \ (\xi_r)\}_{r=0}^{N}$, where

$$\| \underline{\hat{x}}^* - \underline{\hat{x}^*} \| = 0(1/N) \text{ and } |H^* - H| = 0(1/N) \ ; \tag{5.26}$$

see Theorem 3.4. In fact, our numerical results suggest that the above theoretical predictions are somewhat pessimistic; see Section 6.

In their paper Challis and Burley do not present any theoretical results concerning the convergence of the Jacobi iterations or the quality of the computed approximations. With reference to the iterations they simply state the following; [2, p.173]:

"Some under-relaxation of the α_r values (i.e. the values $\hat{x}(\xi_r)-\xi_r$, in our notation) is necessary in some cases, and helps to speed convergence in other cases. Typically a relaxation factor of 0.5 must be used,"

However, our numerical experiments indicate that under-relaxation is essential for the convergence of the iteration in four out of the five examples considered in [2]. Furthermore, in three of these examples a relaxation factor considerably less than 0.5 is needed to give as reasonable rate of convergence; see Section 6. This is not surprising, since all the curves considered in [2] do not satisfy an $\varepsilon\delta$-condition, with $\varepsilon < 1$.

We end this section by observing that the equivalence of the conformal maps $g : R_H \to G$ and $f : A_q \to \Omega$ persists in the case where G has the more general form

$$G -\{(x,y) : 0 < x < 1 , \tau_1(x) < y < \tau_2(x)\} , \tag{5.27}$$

with $\tau_1(x) < \tau_2(x)$, $x \in [0,1]$; see [10, p. 74]. This follows easily from the discussion at the beginning of the section. It also follows that, in this case, the doubly-connected domain ft is bounded externally and internally by the two symmetric curves

$$\partial\Omega_j = \{w : w = \rho_j(\theta)e^{i\theta} , 0 \le \theta \le 2\pi ; j = 1,2 \tag{5.28}$$

where

$$\rho_j(\theta) = \exp\{-\pi\tau_j(\theta/\pi)\} \ , \ 0 \le \theta \le \pi$$

and (5.28a)

$$\rho_j(2\pi-\theta) = \rho_1(\theta) \quad \pi < \theta \le 2\pi \, .$$

Therefore, the Garrick Algorithm 3.1, modified as indicated in Section 4.1, can be applied directly to the problem of determining the conformal map $g : R_H \to G$, in the case where G has the more general form (5.27).

6. Numerical examples

The examples of this section involve the use of Algorithm 4.1, or the equivalent Algorithm 5.1, for computing conformal modules of symmetric doubly-connected domains of the form (5.1). However, we also present one example involving the use of the general Garrick Algorithm 3.1.

The numerical results were computed on a Honeywell level 68 computer by H. Freter [3] and M. Modi [17], two students at Brunei University. They used programs written in double-precision Fortran, and performed the trigonometric summations by means of the NAG Library FFT subroutine CO6FAF.

In presenting the results we use the following notations:

N: Degree of interpolating trigonometric polynomials.

M_N, H_N : Corresponding approximations to the conformal modules M, H,

n_w : Number of iterations needed for the convergence of the JOR with relaxation parameter w, where $0 < w \le 1$. (The abbreviation JOR is used here to denote the Jacobi method with under-relaxation, rather than over-relaxation. Of course, w = 1 gives the basic Jacobi method described in the algorithms) . Unless otherwise stated, the iteration is terminated when two successive iterates to M or H differe by less than 10^{-12}

w_b : "Best" under-relaxation parameter for N=16, obtained by a simple search procedure written by Modi [17].

R_N : Used for estimating the rate of convergence of the sequences of approximations $\{M_N\}$ or $\{H_N\}$. If the exact values of M, H are known then we take

$R_{2N} = |M_N - M| / |M_{2N} - M|$ or $R_{2N} = |H_N - H| / |H_{2N} - H|$.

Otherwise, we replace M, H by $\hat{M}, \hat{H}$ were:

(i) $\hat{M}$ denote accurate approximations to M, obtained by an orthonormalization method in [18,19],

(ii) $\hat{H}$ is our most accurate computed approximation to H, i.e. $\hat{H} = H_{\hat{N}}$ where $\hat{N}$ is the largest value of N used.

<u>Example 1.</u> Algorithm 5.1 for the five domains considered by Challis and Burley [2]. That is

$$G = \{(x,y) : 0 < x < 1 , 0 < y < \tau(x)\}, \quad (6.1)$$

where

(i) $\tau(x) = 0.25 + 0.2\text{sech}^2 (2.5x)$.

(ii) $\tau(x) = 1.5 + x$.

(iii) $\tau(x) = 1 - 0.25 \cos 27\pi x$.

(iv) $\tau(x) = 1 + 0.25 \cos 2\pi x$.

(v) $\tau(x) = 1 .25 - (x- 0.25)^2$.

The numerical results corresponding to (i) - (iii) and (v) are listed in Tables 1 (i) - (iii) and (v) respectively. (As might be expected the results for (iv) are exactly the same as those for (iii).) □

Before presenting the results of our other examples we make the following remarks concerning the convergence of the Jacobi iterations, the choice of under—relaxation parameter w, and the convergence of the sequence of approximations $\{H_N\}$.

Remark 1. Let G be a simply—connected domain of the form (5.1), and let

$$\hat{\varepsilon} := \sup_{0 \le x \le 1} |\tau'(x)|. \tag{6.2}$$

Then, Theor. 3,4 guarantees the convergence of the Jacobi iterations provided that

$$\hat{\varepsilon}\delta^{-1} < 1, \tag{6.3}$$

where δ^{-1} is given by (4.20a) with $m = \max_{0 \le x \le 1} \{\exp(-\pi\tau(x))\}$. For the curves of Ex. 1 the values of $\hat{\varepsilon}$ and δ^{-1} are as follows:

(i) $\hat{\varepsilon} = 0.3849$, $\delta^{-1} = 2.7631$. (ii) $\hat{\varepsilon} = 1.0$, $\delta^{-1} = 1.0005$. (iii),(iv) : $\hat{\varepsilon} = 1.5708$, $\delta^{-1} = 1.0547$. (v) $\hat{\varepsilon} = 1.5$, $\delta^{-1} = 1.0816$. That is, all the curves of Ex.1 violate the convergence criterion (6.3). Thus, with reference to the numerical results, it is not surprising that the Jacobi iterations do not converge in the four cases (ii) - (v), for which $\hat{\varepsilon} \ge 1$. For the curve (i) however, $\hat{\varepsilon}$ is appreciably less than one and the iterations converge reasonably fast. This indicates that the condition (6.3) can be rather pessimistic, because of the value of δ^{-1}. That is, if $\hat{\varepsilon} < 1$ then the Jacobi iterations may converge rapidly, even if the $\varepsilon\delta$-condition of Theor. 3.4 is not fulfilled.□

Remark 2. The results of Ex.1, and those of several other experiments not presented here, suggest

$$w = 1/(1 + \hat{\varepsilon}^2) \tag{6,4}$$

as a suitable relaxation parameter for use with the JOR. For example, for the domains (6.1), (i) - (v), this formula gives respectively the values w = 0.871, 0.5, 0.288, 0.288 and 0.296, which agree closely with the experimentally determined "best" under-relaxation parameters w_b . (Our motivation for experimenting with (6.4) emerged from the theoretical results of Gutknecht [9] concerning the convergence of the Theodorsen iteration for the mapping of simply-connected domains).□

Remark 3. The function $\rho_2(\theta) = \exp\{-\pi\tau(x)\}$ corresponding to the curves (i), (ii) and (v) of Ex. 1 are only piecewise differentiable. Thus, although the conditions of Theor. 3.4 are not fulfilled, the theorem indicates that

$$|H_N - H| = 0(1/N) \tag{6.5}$$

might be true. However, the values R_N listed in Tables 1(i) , 1 (ii) and 1(v) suggest strongly that

$$|H_N - H| = 0(1/N^2) \ . \tag{6.6}$$

This experimental observation is also supported by the results of the examples given below, and those of several other experiments contained in [3]. Unfortunately, we have not been able to prove (6.6).

For the domain (6.1 iii) , the curve $\partial\Omega_2 : w = \rho_2(\theta)e^{i\theta}$ is analytic and, in this case, the values $\hat{R}_N = R_N^{-1/N}$ listed in Table 1 (iii) suggest that

$$|H_N - H| = 0(\alpha^N) \ , \tag{6.7}$$

where $0.8 < a < 1$. Of course, exactly the same remark applies to the domain (6.1 iv).□

Example 2. Algorithm 4.1 for the following three doubly-connected domains:

(i) $$\Omega = \mathrm{Int}(\partial\Omega_1) \cap \mathrm{Ext}(\partial\Omega_1) \ , \tag{6.8}$$

where

$$\partial\Omega_1 = \{w : |w| = 1\} \quad \text{and} \quad \partial\Omega_2 = \{w : w = \rho_2(\theta)e^{i\theta} \ , \ 0 \le \theta \le 2\pi\} \ , \tag{6.8a}$$

with

$$\rho_2(\theta) = \frac{5}{8}\left\{0.5 + \cos\frac{(\theta-\pi)}{2}\right\} \ . \tag{6.8b}$$

That is Ω is a unit disc with a cardioid shaped hole; see [19,p. 100] .

(ii)

$$\Omega = \{(X,Y) : |X| < 2.5,\ |Y| < 2.5\} \cap \{w : |w| > 1.0\}\ . \qquad (6.9)$$

That is Ω is a square of side length 5.0, with a circular hole of unit radius; see [18,p.691]. (In this case, Algorithm 4.1 must be modified in an obvious manner to take care of the fact that the inner, and not the outer, boundary is circular.)

(iii)

$$\Omega = \{w: |w| < 1.0\} \cap \{(X,Y) : |x| > 0.5,\ |Y| > 0.5\}\ . \qquad (6.10)$$

That is Ω is a unit disc with a square hole of side length 1.0; see [18,p.694] .

For each of the above domains $\hat{\varepsilon} = 1.0$, where now

$$\hat{\varepsilon} := \sup_{0 \le \theta \le \pi} \left|\rho_2'(\theta)/\rho_2(\theta)\right|\ . \qquad (6.11)$$

Thus, following the observation of Remark 2, we perform the iterations by applying under-relaxation with w = 0.5. The numerical results obtained are listed in Tables 2(i)-2(iii).□

<u>Example 3.</u> Algorithm 3.1 simplified as described in Section 3.1 for the square frame

$$\Omega = \{(X,Y) : |x| < 1,\ |Y| <1\} \cap \{(X,Y): |x| > a,\ |Y| > a\ ;\ a < 1\}\ . \qquad (6.12)$$

Here the iterations are performed by applying under-relaxation with w = 0.5 and using 10^{-8}, instead of 10^{-12}, as termination criterion. The numerical results corresponding to the values a = 0.2, 0.5 and 0.8 are listed in Tables 3(i) - (iii) respectively.□

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TABLE 1

(i) $\tau(x) = 0.25 + 0.2 \operatorname{sech}^2 (2.5x)$

$w_b = 0.894$

N	$n_{1.0}$	$n_{0.5}$	n_{wb}	H_N	R_N
8	20	32	19	0.312 412 411	-
16	20	31	17	0.312 436 037	3.8
32	20	31	17	0.312 442 327	4.0
64	20	31	17	0.312 443 901	4.0
128	20	31	17	0.312 444 294	4.0
256	20	31	17	0.312 444 393	4.2
512	20	31	17	0.312 444 417	4.4
1024	20	31	17	0.312 444 424	-

(ii) $\tau(x) = 1.5 + x$

Exact H = 1.779 359 959; [1].

$w_b = 0.450$

N	$n_{1.0}$	$n_{0.5}$	n_{wb}	H_N	R_N
8	d	56	60	1.777 332 776	-
16	d	62	57	1.777 849 844	4.0
32	d	62	58	1.779 232 217	4.0
64	d	58	62	1.779 328 010	4.0
128	d	62	62	1.779 351 971	4.0
256	d	62	62	1.779 357 962	4.0
512	d	62	53	1.779 359 460	4.0
1024	d	62	62	1.779 359 834	4.0

d : The iteration does not converge.

(iii) (x) = 1 - 0.25 cos (2πx)

$w_b = 0.293$

N	$n_{1.0}$	$n_{0.5}$	n_{wb}	H_N	$\hat{R}_N = R_N^{-2/N}$
8	104	54	81	0.873 139 197 963	-
16	d	105	85	0.866 080 847 503	0.83
32	d	203	109	0.869 396 169 077	0.89
64	d	216	103	0.864 113 335 178	0.93
128	d	194	103	0.864 087 604 097	0.95
256	d	194	103	0.864 086 767 469	0.96
512	d	194	103	0.864 086 763 146	-
1024	d	194	103	0.864 086 763 146	-

(v) $\tau(x) = 1.25 - (x-0.25)^2$

$w_b = 0.298$

N	$n_{1.0}$	$n_{0.5}$	n_{wb}	H_N	R_N
8	d	97	93	0.969 915 579	-
16	d	117	80	0.970 314 993	3.0
32	d	103	85	0.970 459 144	3.7
64	d	125	79	0.970 499 045	3.9
128	d	125	84	0.970 509 396	4.0
256	d	125	84	0.970 512 022	4.2
512	d	125	84	0.970 512 682	5.0
1024	d	125	84	0.970 512 847	-

□

TABLE 2

(i) Disc with cardioid shaped hole; Eq. (6.8) .

Comparison value: $\hat{M}$ = 1.196 339 075; [19]

N	$n_{0.5}$	M_N	R_N
8	37	1.205 112 270	-
16	43	1.198 257 181	4.6
32	41	1.196 809 217	4.1
64	39	1.196 457 208	4.0
128	40	1.196 368 849	4.0
256	40	1.196 346 572	4.0
512	40	1.196 340 959	4.0
1024	40	1.196 339 548	4.0

(ii)square with circular hole; Eq. (6.9)

Comparison value: $\hat{M}$=2.696 724 431; [18].

N	$n_{0.5}$	M_N	R_N
8	36	2.700 726 343	-
16	35	2.697 743 864	3.9
32	40	2.696 982 789	3.9
64	39	2.696 789 629	4.0
128	38	2.696 740 835	4.0
256	38	2.696 728 550	4.0
512	38	2.696 725 464	4.0
1024	38	2.696 724 690	4.0

(iii) Disc with square hole; Eq. (6.10).

Comparison value: M = 1.691 564 903; [18].

N	$n_{0.5}$	M_N	R_N
8	41	1.689 012 511	-
16	47	1.690 899 766	3.9
32	48	1.691 396 274	4.0
64	49	1.691 522 547	4.0
128	44	1.691 554 297	4.0
256	51	1.691 562 250	4.0
512	46	1.691 564 239	4.0
1024	46	1.691 564 737	4.0

□

TABLE 3

Square frame; Eq. (6.11).

(i) a = 0.2. Exact M = 4.570 860* (ii) a = 0.5. Exact M = 1 .847 709

N	$n_{0.5}$	M_N	R_N	N	$n_{0.5}$	M_N	R_N
36	27	4.574 809	-	-	23	1.849 281	-
72	30	4.571 867	3.9	72	30	1.848 110	3.9
144	33	4.571 115	3.9	144	24	1.847 811	3.9
288	27	4.571924	4	288	22	1.847 734	4.1

(iii) a=0.8. Exaxt M =1.201 453*

N	$n_{0.5}$	M_N	R_N
36	27	1.202 399	-
72	26	1.201 691	4.0
144	20	1.201 513	4.0
288	28	1.201 468	4.0

See [1]. □

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