TR/07/88 September 1988

Non-uniform corner cutting

John A. Gregory and Ruibin Qu.

Z1629668

NON-UNIFORM CORNER CUTTING

by

John A. Gregory and Ruibin Qu
Department of Mathematics and Statistics
Brunei University, Uxbridge, UB8 3PH, England.

Abstract The convergence of a non-uniform corner cutting process is investigated. It is shown that the limit curve will be differentiable provided the proportions of the corner cuts are kept within appropriate constraints.

Keywords Subdivision, corner cutting

1. Introduction

In a recent paper [de Boor '87] it was shown that "cutting corners" of a control polygon "always works", in the sense that the limit curve will be Lipschitz continuous. In this paper we wish to show that the limit curve will be differentiable under some appropriate conditions on the corner cutting process. The key to the analysis is the choice of a parameterization which itself satisfies the corner cutting process (rather than using a uniform diadic point parameterization as in [Micchelli and Prautzsch '87] or [Dyn, Gregory, Levin '88] for uniform subdivision schemes).

2. The corner cutting process

Let $f_i^0 \in R^N$, $i = 0,\ldots,n+1$, denote a given sequence of initial of initial control points in R^N, ≥ 1, which are defined at the parameter values $t_0^0 < t_1^0 < \ldots < t_{n+1}^0$. The corner cutting process is then defined by : For $k = 0,1,2,\ldots$; for i - $0,\ldots,2^k n$;

$$(2.1)\quad \begin{cases} f_{2i}^{h+1} = (1-\alpha_i^k) f_i^k + \alpha_i^k f_{i+1}^k & \text{at} \quad t_{2i}^{k+1} = (1-\alpha_i^k) t_i^k + \alpha_i^k + \alpha_{i+1}^k t_{i+1}^k \,, \\ f_{2i+1}^{k+1} = \beta_i^k f_i^k + (1-\beta_i^k) f_{i+1}^k & \text{at} \quad t_{2i+1}^{k+1} = \beta_i^k t_i^k + (1-\beta_i^k) t_{i+1}^k \,, \end{cases}$$

where

$$(2.2)\qquad \alpha_i^k > 0 \;,\quad \beta_i^k > 0 \quad \text{and} \quad j - \alpha_i^k - \beta_i^k > 0 \;.$$

Denote by f^k the control polygonwith vertices f_i^k, $i = 0,\ldots,2^k n+1$.

Then (2.1) is a process whereby f^{k+1} is created by corner cutting of the polygon f^k . In general, this process is non-uniform since the proportions α_i^k, β_i^k of the corner cuts can depend both on i and k.

For the purposes of the analysis , the control points $\{f_i^k\}$ aer associated with parameter points $\{t_i^k\}$ which also satisfy the corner cutting process, see (2.1). These parametric points always form a strictly monotonic increasing set $t_0^k < t_1^k < \ldots < t_{2^k n+1}^k$ since

$$(2.3)\qquad t_i^k < t_{2i}^{k+1} < t_{2i+1}^{k+1} < t_{i+1}^k \,,\quad i = 0,\ldots,2^k n \;,$$

for α_i^k, β_i^k satisfying (2.2). The control polygon f^k can thus be identified unambiguously as the piecewise linear interpolant

$$(2.4)\quad f^k(t) := \left\{\frac{t_{i+1}^k - t}{t_{i+1}^k - t_i^k}\right\} f_i^k + \left[\frac{t - t_i^k}{t_{i+1}^k - t_i^k}\right] f_{i+1}^k \;,\quad t \in \left\{t_i^k, t_{i+1}^k\right\} \;,\quad i = 0,\ldots,2^k n.$$

We propose to analyse the convergence properties of the component functions of f^k and hence it suffices from now on to consider the scalar case N=1, see Figure 1.

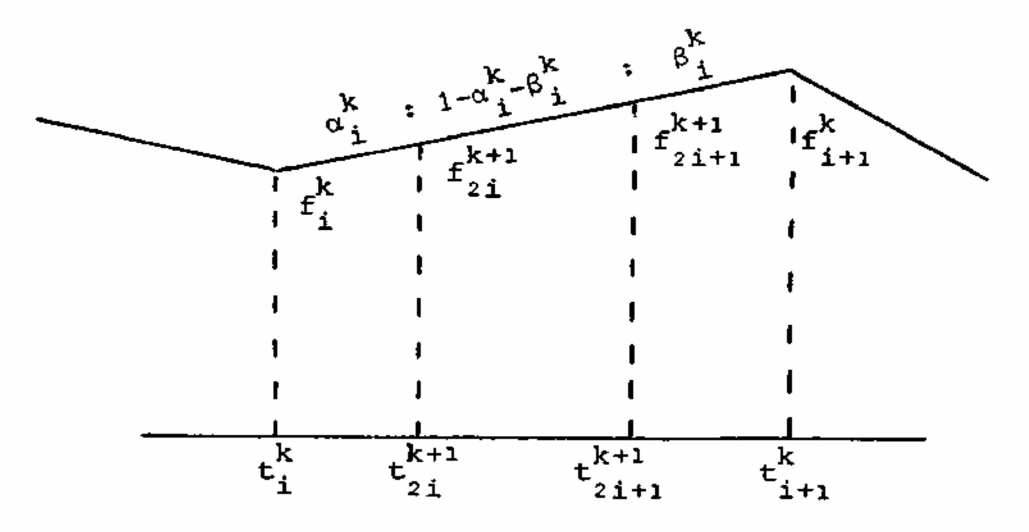

Figure 1

It follows from (2.3) that $\left\{t_0^k\right\}_{k=0}^{\infty}$ and $\left\{t_{2^k n}^k\right\}_{k=0}^{\infty}$ from monotonic increasign and decerasing sequences bounded above and below by t_1^0 and t_n^0 rerspectively.

Hence there exist

(2.5) $$a := \lim_k t_0^k \le t_1^0 \text{ and } b := \lim_k t_{2^k n}^k \ge t_n^0 .$$

We than make use of the uniform norm

(2.7) $$\|f\| = \max_{a \le t \le b} |f(t)| \quad , \quad f \in C[a,b] \quad ,$$

on the interval $[a,b]$.

3. Cutting corners is C^0

Although our main purpose is to find conditions under which the corner cutting process has a C^1 limit, we begin by considering a C^0 analysis. We will show that $\left\{f^k\right\}_{k=0}^{\infty}$ defines a Cauchy sequence no $C[a,b]$ and for thsi we require the following Lemma:

Lemma 3.1

(3.1) $$|f^{k+p} - f^k| \le 2 \max_i |\Delta f_i^k| \qquad \forall \ k,p \ge 0 \ ,$$

where

(3.2) $$\Delta f_i^k := f_{i+1}^{k+p} - f_i^k .$$

Proof Consider $f^{k+p}(t)$ and $f^k(t)$ on $\left[t^{k+p}_{2^p i}, t^{k+p}_{2^p(i+1)}\right]$. From (2.3) we haev

(3.3) $$t_i^k < t^{k+p}_{2^p i} < t^{k+p}_{2^p(i+1)} < t^k_{i+2}$$

and since the process (2.1) defines a convex combination we can obtain

(3.4) $$m_i \le f_j^{k+p} \le M_i , \quad \forall \ j = 2^P i, \ldots, 2^P(i+1) ,$$

where

(3.5) $$m_i = \min\left\{f_i^k, f_{i+1}^k, f_{i+2}^k\right\} , \quad M_i = \max\left\{f_i^k, f_{i+1}^k, f_{i+2}^k\right\} .$$

Hence

$$m_i \le f^{k+p}(t) \le M_i - m_i \le f^k(t) \le M_i$$

which gives

(3.6) $$|f^{k+p}(t) - f^k(t)| \le M_i - m_i \le |f_{i+1}^k - f_i^k| + |f_{i+2}^k - f_{i+1}^k|$$

$\forall \ t \in \left[t^{k+p}_{2^P i}, t^{k+p}_{2^P(i+1)}\right]$ and the Lemma follows.

Lemma 3.1 suggests an analysis of the difference process which is obtained from (2.1) as

(3.7) $$\begin{cases} \Delta f_{2i}^{k+1} = (1-\alpha_i^k - \beta_i^k)\Delta f_i^k , \\ \Delta f_{2i+1}^{k+1} = \beta_i^k \Delta f_i^k + \alpha_{i+1}^k \Delta f_{i+1}^k . \end{cases}$$

Let

(3.8) $$\begin{cases} \overline{\alpha} = \overline{\lim_k} \ \max_i \ \alpha_i^k , \quad \underline{\alpha} = \underline{\lim} \ \min_i \ \alpha_i^k , \\ \overline{\beta} = \overline{\lim_k} \ \max_i \ \beta_i^k , \quad \underline{\beta} = \underline{\lim} \ \max_i \ \beta_i^k . \end{cases}$$

Then we have the following:

Theorem 3.2 (C° convergence) The corner cutting process defined by (2.1) and (2.2) converges to a C° limit if

$$\underline{\alpha} > 0 \ , \ \underline{\beta} > 0 \quad \text{and} \quad 1 - \bar{\alpha} - \bar{\beta} > 0 \ . \tag{3.9}$$

Proof It follows from the definition of the difference process (3.7) that

$$\max_i |\Delta f_i^{k+1}| \leq B_k \max_i |\Delta f_i^k| \ , \tag{3.10}$$

where

$$B_k = \max_i \left\{ 1 - \alpha_i^k - \beta_i^k , \beta_i^k + \alpha_{i+1}^k \right\} \ . \tag{3.11}$$

Moreover, it can be shown that

$$B_k \leq B < 1 \tag{3.12}$$

for some constant B, independent of k, if (3.9) holds. Hence the differences are contracting and from Lemma 3.1 it follows that $\left\{ f^k \right\}_{k=0}^{\infty}$ defines a Cauchy sequenc e on C[a,b] which completes the proof.

Conditions (3.9) require $(\underline{\alpha}, \underline{\beta})$ and $(\bar{\alpha}, \bar{\beta})$ to lie strictly within the region Ω_0 depicted in Figure 3.1. In particular $(0,0) < (\underline{\alpha}, \underline{\beta}) \leq (\bar{\alpha}, \bar{\beta}) < (\frac{1}{2}, \frac{1}{2})$

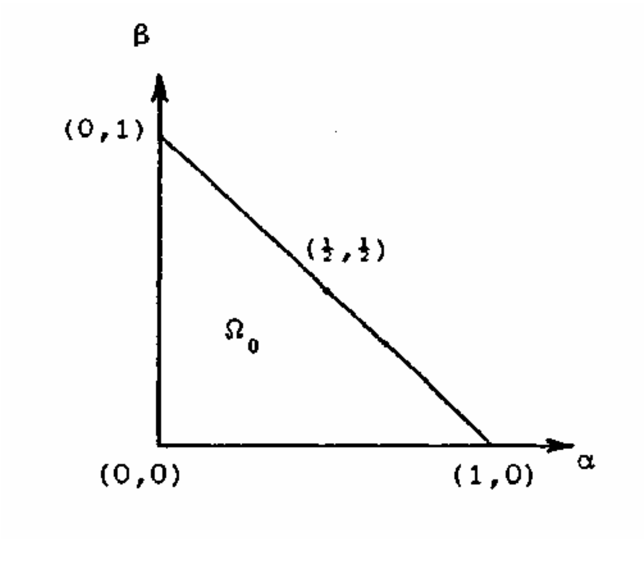

Figure 3.1

is a sufficient condition for a C° limit- In [de Boor '87] a different argument is used to prove convergence for a more general corner cutting process. However, our purpose is to find conditions under which the process (2.1) has a C^1 limit and hence we have found it appropriate to develop a separate C^0 analysis here. The C^1 analysis makes use of the following observation:

Remark 3.3 The parameteric points $\left\{ t_i^k \right\}_{i=0}^{2^k}$ become dense in [a,b].

Proof Since the parameteric points satisfy the corner cutting process, it follows that

$$(3.13) \qquad \begin{cases} \Delta t_{2i}^{k+1} = (1 - \alpha_i^k - \beta_i^k) \Delta t_i^k , \\ \\ \Delta t_{2i+1}^{k+1} = \beta_i^k \Delta t_i^k + \alpha_{i+1}^k \Delta t_i^k , \end{cases}$$

of. (3.7), and that

$$(3.14) \qquad \max_i |\Delta t_i^{k+1}| \le B_k \max_i |\Delta t_i^k$$

of. (3.10). Since (3.12) holds (under the conditions (3.9)) we have $\lim_k \max_i |\Delta t_i^k| = 0$.

4. Cutting corners is C^1

To analyse C^1 convergence, consider the divided difference process defined from (3.7) and (3.13) by

$$(4.1) \qquad \begin{cases} d_{2i}^{k+1} = d_i^k \quad \text{at} \quad t_{2i}^{k+1} , \\ \\ d_{2i+1}^{k+1} = (1 - \theta_i^k) d_i^k + \theta_i^k d_{i+1}^k \quad \text{at} \quad t_{2i+1}^{k+1} , \end{cases}$$

where

$$(4.2) \qquad d_i^k := \Delta f_i^k / \Delta t_i^k ,$$

(4.3) $$\theta_i^k := \alpha_{i+1}^k \Delta t_{i+1}^k / (\beta_i^k \Delta t_i^k + \alpha_{i+1}^k \Delta t_{i+1}^k) \quad .$$

We then have:

Theorem 4.1 If the divided difference scheme converges uniformly to $d \ \varepsilon \ C[a,b]$ (with respect to the parametric points $\left\{ t_i^k \right\}$), then the corner cutting scheme converges uniformly to $f \ \varepsilon \ C^1 \ [a,b]$, where $f' = d$.

Proof Let H_k denote the piecewise cubic Hermite interpolant such that

(4.4) $$H_k(t_i^k) = f_i^k \quad \text{and} \quad H'_k(t_i^k) = d_i^k, \quad i = 0, \ldots, 2^k n.$$

Then for $t \in (t_i^k, t_{i+1}^k)$ with $\theta = (t - t_i^k)/\Delta t_i^k + \theta^2(-2\theta+3) f_{i+1}^k$

(4.5) $$H_k(t) = (1-\theta)^2(2\theta+1) f_i^k + \theta^2(-2\theta+3) f_{i+1}^k$$
$$+ (1-\theta)^2 \theta \Delta t_i^k d_i^k + \theta^2(\theta-1)\Delta t_i^k d_{i+1}^k \quad .$$

and

(4.6) $$H'_k(t) = (-3\theta^2 + 2\theta + 1) d_i^k + (3\theta^2 - 2\theta\theta d_{i+1}^k \quad .$$

Also, let d be the divided difference control polygon (piecewise linear interpolant) defined for $t \in [t_i^k, t_{i+1}^k)$ by

(4.7) $$d_k(t) = (1-\theta) d_i^k + \theta d_{i+1}^k \quad ,$$

where $d^k \rightarrow d$ uniformly on C[a,b] by hypothesis. Then subtracting (4.6) from (4.7) leads to

(4.8) $$\| d^k - H'_k \| \leq \frac{3}{4} \max_i | d_{i+1}^k - d_i^k | \quad .$$

Thus

(4.9) $$\lim_k \| d - H'_k \| \leq \lim_k \| d - d^k \| + \lim_k \| d^k - H'_k \| = 0$$

i.e. $H'_k \rightarrow d$ uniformly. (The right hand side of (4.8) converges necessarily to zero if $\lim dk = d \ \varepsilon \ C[a,b]$ and the parametric points become dense in [a,b].) We now show that $\{H_k\}_{k=0}^{\infty}$ converges on $C^1[a,b]$. Assume, without loss of qenerality, that $f_0^0 = f_1^0 = f_2^0 = 0$. Then at the kth step $f_0^k = f_1^k = f_2^k = 0$ and $d_0^k = d_1^k = 0$. (This reflects the "local suppor" nature of the corner cutting process.) Thus, necessarily, $f(a) = d(a) = 0$ and $H_k(a) = 0$.

Define

$$(4.10) \qquad f(t) := \int_a^t d(\bar{t})\,d\bar{t}$$

Then

$$(4.11) \qquad \|f - H_k\| = \max_{a \prec t \leq b} \left| \int_a^t \left\{ d(\bar{t}) - H'_k(\bar{t}) \right\} d\bar{t} \right| \leq (b-a)\,\|d - H'_k\| \ .$$

Hence, H_k converges uniformly to $f \; \varepsilon \; C^1$ [a,b], where $f' = d$. Finally, since

$$(4.12) \qquad \|f - f^k\| \leq \|f - H_k\| + \|H_k - f^k\| \leq \|f - H_k\| + \frac{1}{2} \max_i (\Delta t_i^k)^2 \|H''_k\|$$

(using the Cauchy remainder for linear interpolation), it follows that the control polygon fk of the corner cutting process converges uniformly to $f \in C^1[a,b]$, (where we again note that the parametric points become dense in [a,b], see Remark 3.3).

Theorem 4.1 indicates that, in order to prove C^1 convergence of the corner cutting process, we should find conditions for which the divided difference process (4.1) has a C^0 limit. Now the process (4.1) has the property that the image set

$$(4.13) \qquad I_k = ((t, d^k(t)) \in R^2 \; : \; t \; \varepsilon \; [a,b]\}$$

lies on the initial image set $I_O \; \forall \; k$. It is thus tempting to conclude that $d^k = d^0 \in C[a,b] \; \forall \; k$. However, this is an incorrect argument since $d^k(t)$ has been defined as the piecewise linear interpolant with respect to the partition $t_0^k < t_1^k < \ldots < t_n^k$ of the original corner cutting process. Thus the analysis of C^o convergence of the divided difference process must be constructed with more care and, following the approach of section 3, we have:

Lemma 4.2

(4.14) $$\| d^{k+p} - d^{k} \| \le 2 \max_{i} | \Delta_{di}^{k} | \quad \forall \;\; k,p \ge 0 \; .$$

Theorem 4.3 (C^1 convergence) The divided difference process (4.1) converges uniformly to a C^0 limit (and hence the corner cutting process converges uniformly to a C^1 limit) if

(4.15) $$\bar{\alpha} > 0 \;\; , \;\; \underline{\beta} > 0 \;\; , \;\; 2\bar{\alpha} + \bar{\beta} < 1 \;\; \text{and} \;\; \bar{\alpha} + 2\bar{\beta} < 1 \; .$$

The proof of Lemma 4.2 is identical to that of Lemma 3.1. The proof of Theorem 4.3 requires the following additional lemma:

Lemma 4.4 Let

(4.16) $$r_{i}^{k} \; := \; \Delta t_{i+1}^{k} / \Delta t_{i}^{k}$$

and assume that (4.15) holds. Then there exist r and R such that

(4.17) $$0 < r \le r_{i}^{k} \le R < \infty \qquad \forall \;\; i, k \; .$$

Proof From (3.13) we obtain the following non-linear relations:

(4.18) $$\begin{cases} r_{2i}^{k+1} = \left[\beta_i^k + \alpha_{i+1}^k r_i^k \right] / \left[1 - \alpha_i^k - \beta_i^k \right] \\ r_{2i}^{k+1} = \left[1 - \alpha_{i+1}^k - \beta_{i+1}^k \right] r_i^k / \left[\beta_i^k + \alpha_{i+1}^k r_i^k \right] \end{cases} .$$

Hence, if $r \le r_i^k \le R \quad \forall \;\; i,k,$ we have

$$\left[\beta_i^k + \alpha_{i+1}^k r \right] / \left[1 - \alpha_i^k - \beta_i^k \right] \le r_{2i}^{k+1} \le \left[\beta_i^k + \alpha_{i+1}^k R \right] \left[1 - \alpha_i^k - \beta_i^k \right] \;\; ,,$$

$$\left[1 - \alpha_{i+1}^k - \beta_{i+1}^k \right] / \left[\beta_i^k / r + \alpha_{i+1}^k \right] \le r_{2i+1}^{k+1} \le \left[1 - \alpha_{i+1}^k - \beta_{i+1}^k \right] / \left[\beta_i^k / R + \alpha_{i+1}^k \right] \; .$$

Thus, we require r and R such that

$$\left[\beta_i^k + \alpha_{i+1}^k R \right] / \left[1 - \alpha_i^k - \beta_i^k \right] \le R \;\; , \quad \left[1 - \alpha_{i+1}^i - \beta_{i+1}^k \right] / \left[\beta_i^k / R + \alpha_{i+1}^k \right] \le R \;\; ,$$

and

$$r \le \left[\beta_i^k + \alpha_{i+1}^k r\right] / \left[1-\alpha_i^k - \beta_i^k\right], \quad r \le \left[1-\alpha_{i+1}^k - \beta_{i+1}^k\right] / \left[\beta_i^k/r + \alpha_{i+1}^k\right].$$

Therefore

$$(4.19) \qquad \max_i \left\{\beta_i^k/\left[1-\alpha_i^k - \beta_i^k - \alpha_{i+1}^k\right], \left[1-\alpha_{i+1}^k \ \beta_{i+1}^k - \beta_i^k\right]/\alpha_{i+1}^k - \alpha_{i+1}^k\right\} \le R,$$

and

$$(4.20) \qquad r \le \max_i \left\{\beta_i^k/\left[1-\alpha_i^k - \beta_i^k - \alpha_{i+1}^k\right], \left[1-\alpha_{i+1}^k - \beta_{i+1}^k - \beta_i^k\right]/\alpha_{i+1}^k\right\}$$

provided

$$(4.21) \qquad 1-\alpha_i^k - \beta_i^k - \alpha_{i+1}^k > 0 \text{ and } 1-\alpha_{i+1}^k - \beta_{i+1}^k - \beta_i^k > 0.$$

Condition (4.17) is then btained under the hypothesis (4.15).

Proof of Theorem 4.3 From (4.1) and (4.3) we obtain

$$(4.22) \qquad \begin{cases} \Delta_{d2i}^{k+1} = \theta_i^k \Delta_{di}^k, \\ \Delta_{d2i+1}^{k+1} = (1-\theta_i^k)\Delta_{di}^k, \end{cases}$$

where $0 < \theta_i^k < 1$. Thus

$$(4.23) \qquad \max_i |\Delta_{di}^{k+1}| \le C_k \max_i |\Delta_{di}^k|,$$

where

$$(4.24) \qquad Cl = \max_i \left\{\theta_i^k, 1 - \theta_i^k\right\}$$

and hence

$$(4.25) \qquad 0 < C_k < 1.$$

Condition (4.25) is not strong enough for our purposes and we wish to show that

$$(4.26) \qquad C_k \le C < 1$$

for some constant C, independent of k. Now, from (4.3),

$$(4.27) \qquad \theta_i^k = \frac{1}{1+1/\hat{r}_i^k} \quad , \quad 1-\theta_i^k = \frac{1}{1+\hat{r}_i^k} \quad ,$$

where

$$(4.28) \qquad \hat{r}_i^k = \alpha_{i+1}^k r_i^k / \beta_i^k$$

Furthermore, by Lemma 4.4 (and the hypothesis (4.15)), there exist $\hat{r}$ and $\hat{R}$ such that

$$(4.29) \qquad 0 < \hat{r} \le \hat{r}_i^k \le \hat{R} < \infty \ .$$

Thus

$$(4.30) \qquad \theta_i^k \le \frac{1}{1+1/R} \quad \text{and} \quad 1-\theta_i^k \le \frac{1}{1+r} \qquad \forall \quad i,k$$

and (4.26) holds. Finally, it now follows from Lemma 4.2 that $\{d^k\}_{k=0}^{\infty}$ defines a Cauchy sequence on C[a,b] and hence has limit d ε C[a,b] say.

Conditions (4.15) for C^1 convergence require $(\underline{\alpha},\underline{\beta})$ and $(\bar{\alpha},\bar{\beta})$ to lie strictly within the region Ω_1 depicted in Figure 4.1 (cf. Figure 3.1).

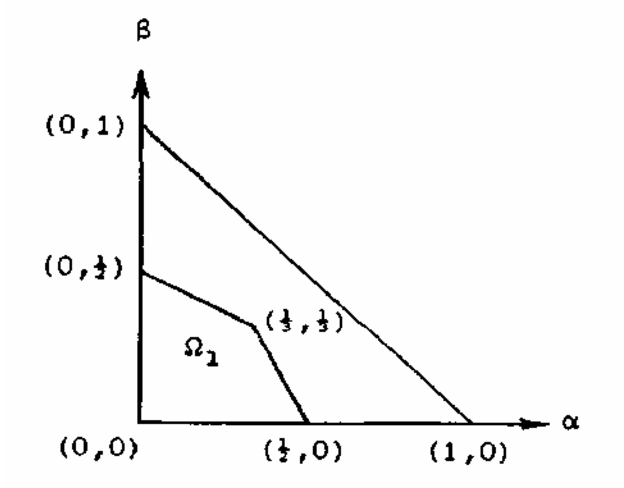

Figure 4.1

In particular, $(0,0) < (\underline{\alpha},\underline{\beta}) < (\alpha,\beta) < (\frac{1}{3},\frac{1}{3})$ is a sufficient condition for a C^1 limit (i.e. corner cutting of proportions strictly less than one third ensures a C^1 limit). If $(\bar{\alpha},\bar{\beta})$ lies strictly outside the region Ω_1 , then convergence to a C^1 limit is no longer guaranteed. For example, with

$\alpha_i^k = \alpha$ and $\beta_i^k = \beta \quad \forall \quad i, k,$ it can be shown that (see (4.22))

$$\Delta_{d0}^{k+1} = \theta_0^k \Delta_{d0}^k = \theta_0^k \theta_0^{k-1} \cdots \theta_0^0 \Delta_{d0}^0$$

will not converge to zero if $2\alpha + ß > 1$. This violates a necessary C^1 convergence condition. Similarly, by symmetry, $\alpha + 2ß > 1$ is not allowable.

We have shown that the corner cutting process has a C^1 limit, under the conditions (4.15), with respect to a parameterization which is itself defined by the corner cutting method. We conclude by showing that this parameterization is regular in the case R^N, $N > 1$.

Theorem 4.5 (Regular parameterization) In the case of corner cutting in R^N, $N > 1$, the C^1 limit curve f of Theorem 4.3 is regular, i.e. $f'(t) = d(t) \neq 0 \;\forall\; t \quad [a,b]$, except for the singular cases, where, for some i,

(i) $f_i^0 = f_{i+1}^0$,

or

(ii) $f_{i+1}^0 = (1-\theta) \dot{f}_i^0 + \theta \dot{f}_{i-1}^0$ for some $\theta > 0$.

Proof Let

(4.31) $$J_k = \left\{ d^k(t) \in R^N : t \in [a,b] \right\}$$

be the image set of $d^k(t)$. Then, following an earlier argument, we have $J_k \subset J_0 \;\; \forall\; k$, for a process of the form (4.1). Thus $d(t) = 0$ for some $t \in [a,b]$ implies that $d^0(t) = O$ for some $t \in [a,b]$ and this can only occur if (i) $d_i^0 = 0$ for some i or (ii) for $d_i^0 = -\theta d_{i-1}^0$ some i and $\theta > 0$.

As a final comment, it should be noted that, because of the local nature of the corner cutting process (2.1), the convergence arguments presented here also apply to the case of closed polygons in R^N, $N > 1$. This case can be treated as periodic date, where $f_n^0 = f_{n+1}^0 = f_1^0$ and in the process $\alpha_{2^k n}^k = \alpha_0^k$, $\beta_{2^k n}^k = \beta_0^k$.

References

de Boor, C. (1987), Cutting corners always works. Computer Aided Geometric Design 4, 125-131.

Dyn, N., Gregory, J.A. and Levin, D. (1988), Analysis of uniform binary subdivision schemes for curve design, preprint.

Micchelli, C.A. and Prautzsch, H. (1987), Uniform refinement of curves, preprint.