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Abstract Let Ω be a simply-connected domain in the complex plane and let π_n denote the n th degree Bieberbach polynomial approximation to the conformal map *f* of Ω onto a disc. In this paper we investigate the asymptotic behaviour (as $n \to \infty$) of the zeros of π_n , π_n' and also of the zeros of certain closely related rational approximants to *f*. Our results show that, in each case, the distribution of the zeros is governed by the location of the singularities of the mapping function *f* in C\ Ω , and we present numerical examples illustrating this.

Keywords : Bieberbach polynomials, Bergman kernel function, conformal mapping, zeros of polynomials.

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1 Introduction

Let Ω be a simply-connected domain of the complex plane C, whose boundary $\partial\Omega$ is a closed Jordan curve, and let $\zeta \in \Omega$. Then, by the Riemann mapping theorem, there exists a unique conformal mapping $w= f_\zeta$ (z) of Ω onto a disc { w: |w| < r_ζ , }, such that

$$f_\zeta(\zeta)=0, \quad f_\zeta'(\zeta)=1.$$

The radius r_ζ of this disc is called the conformal radius of Ω with respect to ζ .

For the inner product

$$(g,h) := \iint_\Omega g(z)\overline{h(z)}\,dm,$$

Where *dm* is the 2-dimensional Lebesgue measure, we consider the Hilbert space

$$L^2(\Omega):=\{g : g \text{ analytic in } \Omega \ , \| g \|^2 = (g , g) < \infty \} .$$

Let *K*(z , ζ) denote the *Bergman kernel function of* Ω which has the reproducing property

$$g(\zeta) = (g, k(\cdot,\zeta)), \quad \forall g \in L^2(\Omega). \tag{1.1}$$

(cf. [1], [3], [4], [8]). Then it is known (cf. [4, p.34]) that $r_\zeta = (\pi K(\zeta,\zeta))^{-1/2}$ and that for z $\in \Omega$

$$f_\zeta'(z)=\frac{K(z,\zeta)}{K(\zeta,\zeta)}, \quad f_\zeta(z)=\frac{1}{K(\zeta,\zeta)} \int_{t=\zeta}^{z} K(t,\zeta)\,dt. \tag{1.2}$$

Next let $Q_n(z) = \gamma_n z^n + \cdots , \gamma_n > 0$, be the sequence of orthonormal polynomials for the inner product $(\cdot,\cdot)$, ie.

$$\iint_\Omega Q_k(z)\overline{Q_1(z)}\,dm = \delta_{k,l}.$$

Since Ω is a Jordan region, it is known (cf. [4, p. 17]) that $\{Q_n\}_0^\infty$ forms a complete orthonormal system for $L^2(\Omega)$ and consequently, from the reproducing property (1.1), that K($\cdot$, ζ) has the $L^2(\Omega)$-convergent Fourier series expansion

$$K(z,\zeta) - \sum_{j=0}^{\infty} \overline{Q_j(\zeta)}\,Q_j(z). \tag{1.3}$$

The *Bieberbach polynomials* π_n for Ω are defined by

$$\pi_n(z) := \frac{1}{K_{n-1}(\zeta,\zeta)} \int_{t=\zeta}^{2} K_{n-1}(t,\zeta)\,dt, \tag{1.4a}$$

Where $K_{n-1}(\cdot,\zeta)$ denotes the partial Fourier sum

$$K_{n-1}(z,\zeta) := \sum_{j=0}^{n-1} \overline{Q_j(\zeta\}}\,Q_j(z). \tag{1.4b}$$

These polynomials satisfy

$$\pi_n(\zeta) = 0, \quad \pi'_n(\zeta) = 1 \quad \| \pi'_n \|^2 = \frac{1}{K_{n-1}(\zeta,\zeta)},$$

and provide approximations to the mapping function f_ζ in the sense that $\pi_n \to f_\zeta$ locally uniformly in Ω (cf. [4, p.34]). The latter is a direct consequence of the fact that convergence in the norm of $L^2(\Omega)$ implies uniform convergence on each compact subset of Ω (cf. [4, p. 26]). More generally, if $\{s_j\}_1^\infty$ is any complete orthonormal system for $L^2(\Omega)$ and

$$\hat{\pi}_n(z) := \frac{1}{\hat{K}_{n-1}(\zeta,\zeta)} \int_{t=\zeta}^{2} \hat{K}_{n-1}(t,\zeta)dt, \tag{1.5a}$$

Where

$$\hat{K}_{n-1}(z,\zeta) = \sum_{j=1}^{n-1} \overline{s_j(\zeta)} s_j(z), \tag{1.5b}$$

then $\hat{\pi}_n \to f_\zeta$ locally uniformly in Ω (cf. [4, p.32]).

In practice the success of the above method for approximating f_ζ depends critically on the choice of the denning orthonormal system. In particular, if the mapping function f_ζ has singularities either on the boundary $\partial\Omega$ or close to $\partial\Omega$ in C \ $\overline{\Omega}$, then it is essential that the orthonormal system contains functions that reflect the corresponding singularities of $K(\cdot,\zeta)$ (cf. [7], [9], [10]). For this reason, the problem of determining the location and nature of the singularities of f_ζ is of considerable practical interest.

The purpose of this paper is to describe the asymptotic behaviour ($as\ n \to \infty$) of the zeros of the Bieberbach polynomials π_n, and also of the zeros of certain rational approximants π_n of the type studied in [7], [9] and [10] (see also [4, p.36]). Our results show that the distributions of these zeros, and also of the zeros of the derivatives π_n' and $\hat{\pi}_n'$, are governed by the location of the singularities of the mapping function f_ξ in C \ Ω. From the more practical point of view, the results of several numerical experiments indicate that the distributions of the zeros of π_n and π_n' can help to determine the approximate location of these singularities.

The paper is organized as follows: Section 2 contains the statements of our main result concerning the zeros of Bieberbach polynomials (Theorem 2.2) and of an intermediate result (Theorem 2.1) which is needed for our proofs. In Section 3 we present three examples illustrating the results stated in Section 2,and make several observations regarding the distributions of the zeros of π_n and π_n' in relation to the singularities of f_ξ . In Section 4 we give the proofs of Theorems 2.1 and 2.2. Finally, in Section 5 we treat the problem of the distribution of zeros of rational approximants $\hat{\pi}_n$ of the type studied in [7], [9], and [10].

Our main results will be given in terms of a *normalized counting measure for zeros,* which is defined as follows: If P is a polynomial of degree n with zeros $z_1, z_2, \ldots z_n$ (some of which may be repeated), then the measure v(P) is defined by

$$\nu(P)(B) := \frac{1}{n} \cdot [\,\#\text{ of zeros of } P \text{ in } B\,], \tag{1.6}$$

for any Borel set $B \subseteq$ C. Thus, v(P) is a probability measure on the Borel subsets of C.

2 Statements of Results for Bieberbach Polynomials

Let w = ϕ(z) denote the conformal mapping of $D := \overline{C} \setminus \overline{\Omega}$ onto $\{ w : |w| > 1 \}$, normalized by $\phi(\infty) = \infty$ and $\phi'(\infty) > 0$, and observe that the Green function of D with pole at ∞ is given by $g_D(z,\infty) = \log|\phi(z)|$. Further, for each $\sigma > 1$, let $\Gamma\sigma$ denote genetically the locus

$$\Gamma_\sigma := \{z : |\sigma(z)| = \sigma\} = \{z : g_D(z,\infty) = \log\sigma\}, \tag{2.1}$$

and set $\Gamma_1 := \partial\Omega$. Finally, let Ω_σ denote the collection of points interior to the level curve Γ_σ.

For the compact set $\overline{\Omega_\sigma}, \sigma \geq 1$, there exists a unique probability measure μ_σ supported on Γ_σ that minimizes the energy integral

$$I[\mu] := \iint \log|z-t|^{-1} \, d\mu(z) d\mu(t) \tag{2.2}$$

over all probability measures supported on $\overline{\Omega_\sigma}$ (cf. [5, § 16.4], [13]). The measure μ_σ is called the *equilibrium distribution* for $\overline{\Omega_\sigma}$ and the *logarithmic capacity* of $\overline{\Omega_\sigma}$ is definned by

$$cap(\overline{\Omega}_\sigma) := \exp(-I[\mu_\sigma]). \tag{2.3}$$

In terms of the mapping function Φ we have that

$$\mathrm{cap}(\overline{\Omega_\sigma}) = \sigma / \Phi'(\infty). \tag{2.4}$$

Before giving our main result it is convenient to state

Theorem 2.1 *With the notations and assumptions of Section I,*

$$\limsup_{n\to\infty} |Q_n(\zeta)|^{1/n} = \frac{1}{\rho} (\leq 1), \tag{2.5}$$

where $\rho(\geq 1)$) *is the largest index such that* f_ζ *has an analytic (single-valued) extension throughout* Ω_ρ.

Notice thatif f_ζ has a singularity on $\Gamma_1 = \partial\Omega$, , then $\rho = 1$. Moreover, if $\rho < \infty$, then f_ζ has at least one singularity on Γ_p.

We can now state our main result concerning the zeros of Bieberbach polynomials.

Theorem 2.2 *Suppose that the constant* ρ *of Theorem 2.1 is finite and let* $A \subseteq N$ *be a sequence for which*

$$\lim_{\substack{n\to\infty \\ n\in A}} |Q_n(\zeta)|^{1/n} = \frac{1}{\rho} \cdot \tag{2.6}$$

Then in the weak-star topology of measures, the normalizing counting measures for the zeros of π_n *and* π'_n *satisfy*

$$v(\pi_{n-1}) \xrightarrow{*} \mu_\rho \; and \; v(\pi'_{n-1}) \xrightarrow{*} \mu_\rho \;, \; as \; n \to \infty \;, \; n \in A, \tag{2.7}$$

Where μ_ρ *is the equilibrium distribution for* $\overline{\Omega_{p.}}$

In (2.7), the first convergence means that

$$\lim_{\substack{n\to\infty \\ n\in A}} \int f\, dv(\pi_{n+1}) = \int f\, d\mu_\rho$$

for every function f continuous on C having compact support. From this it follows (cf. [6, pp. 8,9]) that if B is any Borel set, then

$$\mu_\rho\,(\overset{0}{B}) \le \liminf_{n\to\infty} v(\pi_{n+1})(B) \le \limsup_{n\to\infty} v(\pi_{n+1})(B) \le \mu_\rho(\overline{B}),$$

where $\overset{o}{B}$ denotes the interior of B.

Since supp $(\mu_p) = \Gamma_\rho$,, we immediately deduce from (2.7) the following.

Corollary 2.3 *With* p *as in Theorem 2.2, every point on* Γ_ρ *is an accumulation point of the zeros of the Bieberbach polynomials* $\{\pi_n\}_0^\infty$ *and of the derived sequence* $\{\pi'_n\}_1^\infty$ *Consequently,* $\{\pi_n\}\infty_0$ *cannot converge uniformly in any neighbourhood containing a point of* Γ_ρ.

Remark 1 Theorem 2.2 (and Corollary 2.3) does not preclude the possibility that there exist accumulation points of zeros of $\{\pi_n\}$ or $\{\pi'_n\}$ that lie off of the level curve Γ_ρ. However, the number of zeros of $\{\pi_{n+1}\}_{n\in\Lambda}$ and of $\{\pi'_{n+1}\}_{n\in\Lambda}$ that can lie on a given compact set disjoint from Γ_ρ is $o\,(n)$.

Remark 2 For p > 1, Corollary 2.3 also follows from the maximal geometric convergence of the sequences $\{\pi_n\}_0^\infty$ and $\{\pi'_n\}_1^\infty$ and Walsh's extension of the Jentzsch theorem [15]. However, this argument does not apply to the important case p = 1.

3 Examples

3.1 Consider the case where $\Omega = \{ z : |z| < 1 \}$. Then

$$Q_n(z) = \sqrt{\frac{n+1}{\pi}}\, z^n,\ n = 0,1,...,$$

and hence

$$k(z,\zeta) = \frac{1}{\pi}.\sum_{j=0}^{\infty} (j+1)(\bar{\zeta}z)^j = \frac{1}{\pi}.\frac{1}{(1-\bar{\zeta}z)^2} \qquad \zeta, z \in \Omega,$$

$$k_{n-1}(z,\zeta) = \frac{1}{\pi}.\sum_{j=0}^{n-1} (j+1)(\bar{\zeta}z)^j = \frac{1}{\pi}.\frac{n(\bar{\zeta}z)^{n+1} - (n+1)(\bar{\zeta}z)^n + 1}{(1-\bar{\zeta}z)^2}$$

Also,

$$f_\zeta(z) = (1-|\zeta|^2\left(\frac{z-\zeta}{1-\bar{\zeta}z}\right),$$

so that the mapping function f_ζ has a simple pole at the point z = $1/\bar{\zeta}$ but is otherwise analytic in the extended plane. Therefore:

(i) Since Φ (z) = z, the constant p in Theorems 2.1 and 2.2 is

$$p = \frac{1}{|\zeta|}.$$

This is, indeed, the reciprocal of

$$\lim_{n\to\infty} |Q_n(\zeta)|^{1/n} = \lim_{n\to\infty} |\sqrt{\frac{n+1}{\pi}}\zeta^n|^{1/n} = |\zeta|.$$

(ii) If $\zeta \neq 0$, then according to Theorem 2.2 (which holds with $\Lambda = N$)

$$v(\pi_n)\overset{*}{\rightarrow}\mu_{1/|\zeta|} \quad and \quad v(\pi'_n)\overset{*}{\rightarrow}\mu_{1/|\zeta|},$$

where $\mu_{1/|\zeta|}$ is the equilibrium distribution for the disc $|z| \leq 1/|\zeta|$. That is,

$$d\mu_{1/|\zeta|} = \frac{|\zeta|}{2\pi}ds,$$

where s denotes arc length on the circle $|z| = 1/|\zeta|$. In other words, $d\mu_{1/|\zeta|}$ is the uniform distribution on the circle ζ This limit behaviour can be verified directly from the explicit formulae for $\pi'_n(z) = \{k_{n-1}(z,\zeta)/k_{n-1}(\zeta,\zeta)\}$ *and* $\pi_n(z)$.

3.2 Let Ω be the rectangle Ω = { z =x + *iy* ; \ *x* \ < *2* , \ *y* \ < 1 } and set $\zeta = 0$. Then, the mapping function f_0 is analytic on $\partial\Omega$, but its analytic extension has a simple pole at each of the points

$$z=2(2k+il),\ k\ ,1\ =0,\pm1,\pm2,\dots,\ k+l=\text{odd}.$$

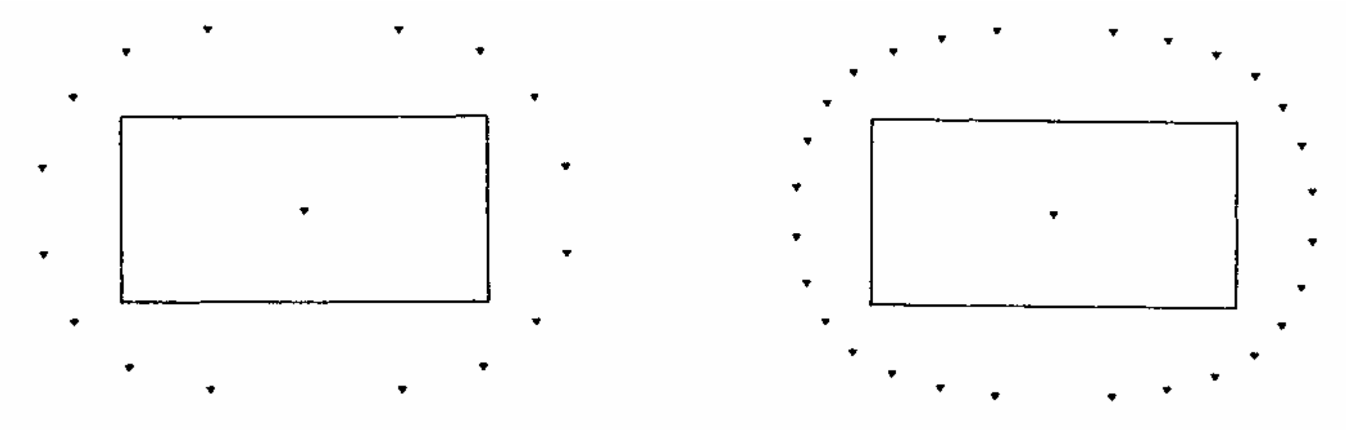
Thus,the singularities of f_0 nearest to $\partial\Omega$ occur at the points z = $\pm 2i$ and, consequently, the value of ρ in Theorems 2.1 and 2.2 is (to 5 significant digits)

$$\rho = \Phi(2i) = 1.4095$$

(cf.[10,p.662]).

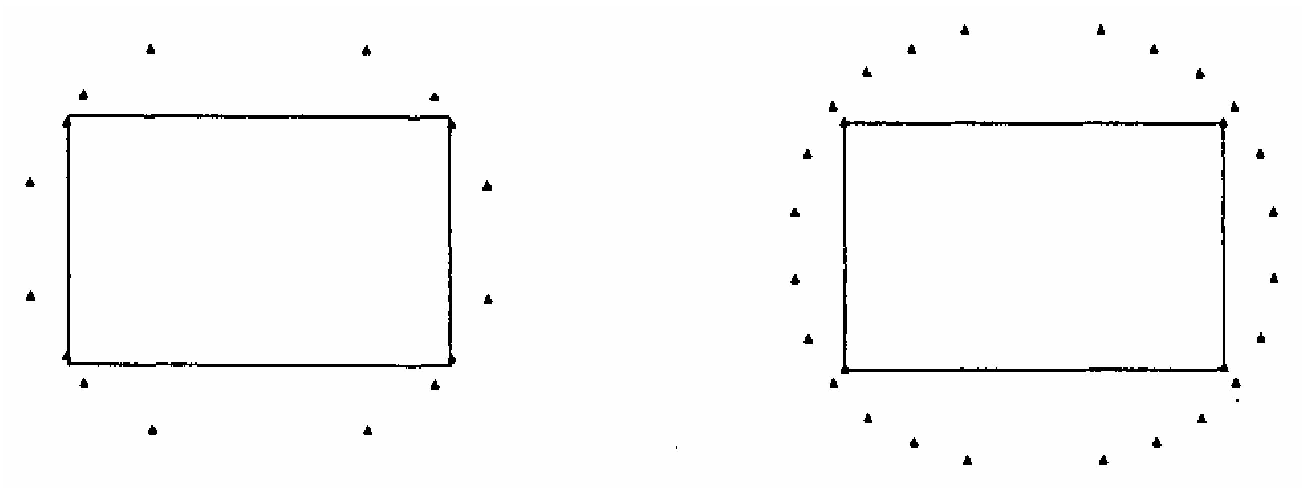
In Figure 1 we have plotted the zeros of the Bieberbach polynomials π_{17} and π_{29} and in Figure 2 those of their derivatives π'_{17} and π'_{29}. These zeros were obtained by using the Fortran conformal mapping package BKMPACK of Warby [16] (for computing the Bieberbach polynomials) and the NAG zero finding subroutine C02AEF.

Figures 3 and 4 contain, respectively, plots of the images of the zeros of π_{17}, π_{29} (with the exception of z = 0) and of π'_{17}, π'_{29}, under the conformal map ϕ: C \ $\overline{\Omega} \rightarrow \{w : |w| > 1\}$. These images were obtained from an accurate approximation to ϕ, which was sagain computed by using BKMPACK



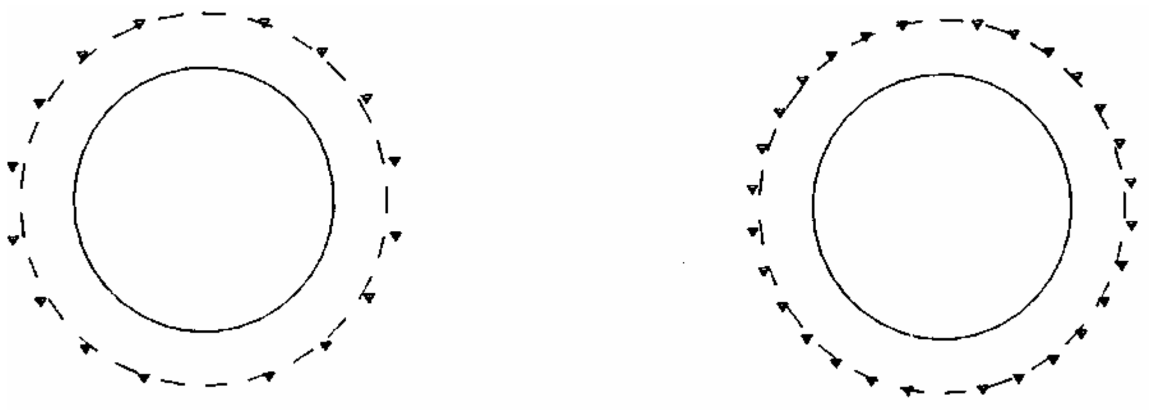
(a) **n =17** (b) **n=29**

Figure 1: Zeros of π_n



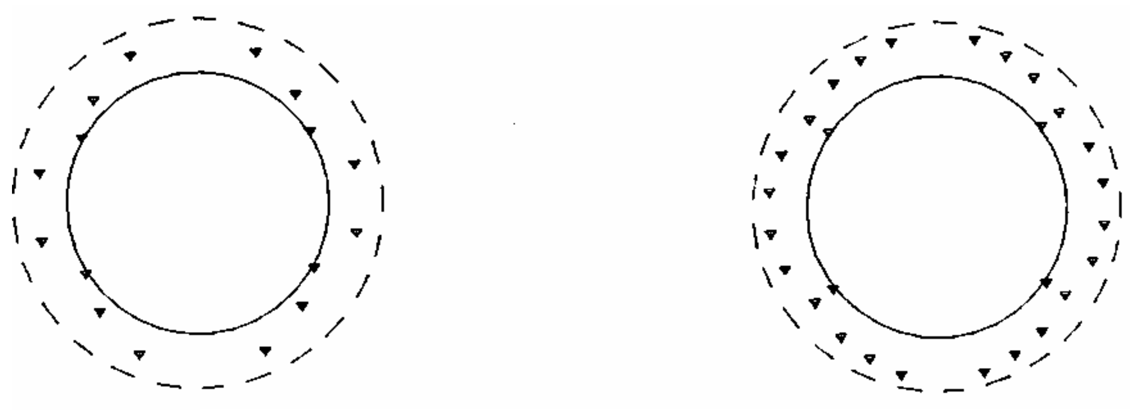
(a) ***n* = 17** **(b) *n* = 29**

Figure 2 : Zeros of π_n'



(a) n=17 (b) n=29

Figure 3 : Images of zeros of π_n



(a) n=17 **(b)** ***n*** **=29**

Figure 4 : Images of zeros of $\pi_n^{'}$

We observe the following in connection with the plots in Figures 1-4 :

(i) The plots in Figures 3 and 4 illustrate the position of the images of the zeros relative to the circles $|w| = 1$ and $|w| = P = 1.4095$, and hence the closeness of the zeros to the level curve Γ_ρ.

(ii) As predicted by Theorem 2.2, the zeros of the Bieberbach polynomials appear to be approaching the level curve Γ_ρ that corresponds to the nearest singularities of f_0. Although the zeros appear to thin out near the two singular points $\pm 2i$, where f_0 becomes unbounded, Theorem 2.2 assures us that they do approach these points as n increases.

(iii) The behaviour of the zeros of the derivatives is similar to that described above, except that now there is always a zero close to each of the four corners of Ω. This reflects the fact that f_0' is zero at each of these points.

3.3 Let Ω be the L-shaped domain illustrated in Figure 5 and take $\zeta = 0$. In this case, the mapping function f_0 has a branch point singularity at the re-entrant corner $z_c = 1$, in the sense that

$$f_0(z) - f_0(z_c) - (z - z_c)^{2/3}, \qquad as\ z \to z_c.$$

In addition, f_0 has simple pole singularities in $C \setminus \overline{\Omega}$, of which the closest to $\partial\Omega$ occur at the points $-1 \pm l$ (cf.[10, p.663]). Since f_0 has a singularity on $\Gamma_1 = \partial\Omega$, it follows that $\rho = 1$ in Theorems 2.1 and 2.2.

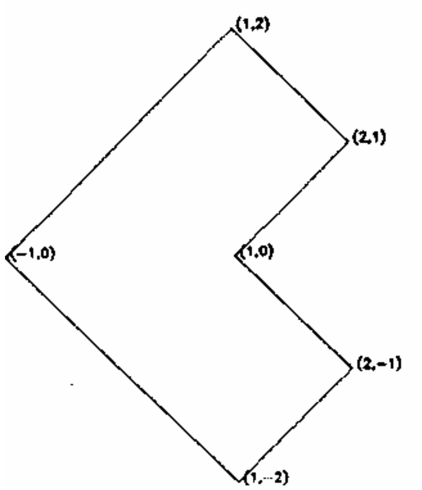
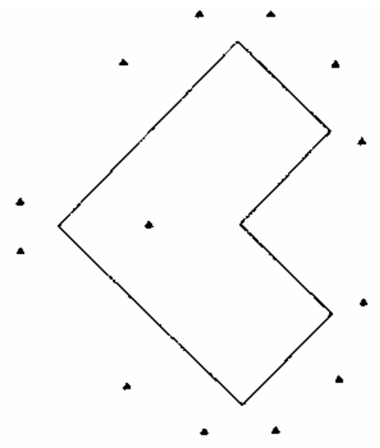
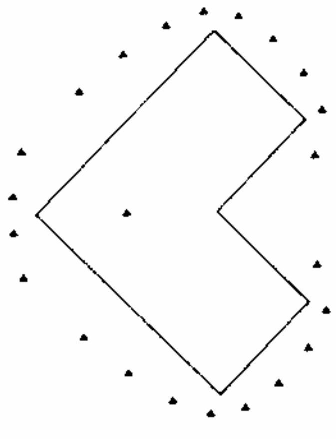


Figure 5 : L-shaped region Ω

In Figure 6 we have plotted the zeros of the Bieberbach polynomials π_{13} and π_{23} and in Figure 7 those of their derivatives π'_{13} and π'_{23}. These zeros were again computed by using the conformal mapping package **BKMPACK** and the **NAG** zero finding routine **C02AEF** .

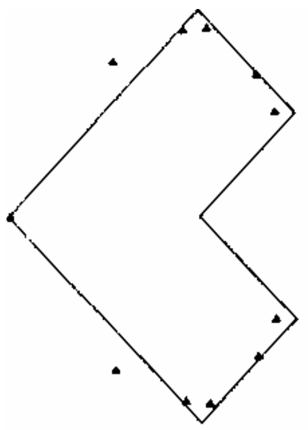


(a) $n = 13$

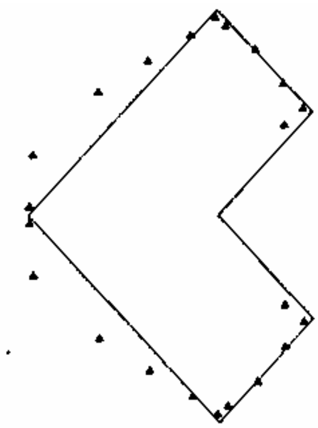


(b) $n = 23$

Figure 6 : Zeros of π_n



(a) $n = 13$



(b) $n = 23$

Figure 7: Zeros of $\pi_n^{'}$

As predicted by Theorem 2.2, the zeros of both the Bieberbach polynomials and their deriveatives appear to be approaching the boundary $\partial\Omega$. In both cases, the zeros appear to thin out near the re-entrant corner $z_c = 1$, where f_0' becomes unbounded. They do, however, approach z_c as n increases. A similar, but less pronounced, thinning out occurs near the parts of $\partial\Omega$ which are close to the two points $-1 \pm i$, where f_0 becomes unbounded. Finally, in the case of the derivatives, there are always zeros close to each of the right-angled corners, reflecting the fact that f_0' is zero there.

4 Proofs of Theorems 2.1 and 2.2

To establish Theorems 2.1 and 2.2 we shall make use of several lemmas. The first is due to S.N. Bernstein and J.L. Walsh.

Lemma 4.1 ([14, p.77]) *Let $E \subset$ C be a compact set whose complement $U := \overline{C} \setminus E$ is connected and regular with respect to the Dirichlet problem. Let $g_U(z, \infty)$ denote the Green function for U with pole at ∞. P_n is a polynomial of degree at most n and*

$$\| P_n \|_{L\infty(E)} := \max_{Z \in E} | P_n(z) | \leq M,$$

then

$$|P_n(z)| \leq M \exp\{n\, g_U(z,\infty)\}, \quad z \in U. \tag{4.1}$$

Lemma 4.2 ([14, p.28]) *If G is a bounded simply-connected domain and $R > 1$ is given, then there exists a closed Jordan region $E \subset G$ such that the closed region $\overline{G}$ lies interior to the level curve*

$$l_R := \{ z : g_{\overline{C}\setminus E}(z,\infty) = \log R \}. \tag{4.2}$$

Combining the above lemmas we shall establish

Lemma 4.3 *The orthonormal polynomials Q_n of Section 1 satisfy*

$$\lim_{n\to\infty} \| Q_n \|^{1/n}_{L_\infty(\overline{\Omega})} = 1. \tag{4.3}$$

Proof The argument is similar to that in [14, p.96]. Let $R > 1$ be given. Then, by Lemma 4.2, there exists a closed Jordan region $E \subset \Omega$ such that $\overline{\Omega}$ lies interior to the level curve l_R of (4.2). Let $r := \text{dist}(E, \partial\Omega)$. From the well-known estimate

$$| g(z) |^2 \leq \frac{1}{\pi r^2} \iint_\Omega | g |^2 \, dm, \quad z \in E, \tag{4.4}$$

which holds for every $g \in L^2(\Omega)$ (cf. [4, p.4]), we get

$$\| Q_n \|^2_{L\_(E)} \le \frac{1}{\pi r^2} \iint_{\Omega} |Q_n|^2 \, dm = \frac{1}{\pi r^2},$$

and so, by Lemma 4.1,

$$\| Q_n \|_{L_\infty(l_R)} \le \frac{R^n}{r\sqrt{\pi}}. \tag{4.5}$$

Since $\overline{\Omega}$ lies interior to l_R, we have by the maximum principle that

$$\| Q_n \|_{L_{-\infty}(\overline{\Omega})} \le \| Q_n \|_{L_\infty(l_R)} .$$

Thus, from (4.5),

$$\limsup_{n\to\infty} \| Q_n \|^{1/n}_{L_\infty(\overline{\Omega})} \le R.$$

But as R is arbitrary, letting $R \downarrow 1$ yields

$$\limsup_{n\to\infty} \| Q_n \|^{1/n}_{L_\infty(\overline{\Omega})} \le 1.$$

Moreover, the inequality

$$\liminf_{n\to\infty} \| Q_n \|^{1/n}_{L_\infty(\overline{\Omega})} \ge 1$$

is an easy consequence of the fact that the Q_n 's are orthonormal. Thus, (4.3) holds. □

In the terminology of [11, §3], Lemma 4.3 shows that the measure dm on $\overline{\Omega}$ is *completely regular*. Consequently (cf. [11, Proposition 3.2]), the leading coefficients γ_n of the polynomials Q_n satisfy

$$\lim_{n\to\infty} \gamma_n^{1/n} = \frac{1}{cap(\overline{\Omega})}. \tag{4.6}$$

We remark that in the special case when Ω is bounded by an analytic Jordan curve, then estimates finer than that in (4.6) can be obtained for the γ_n's (cf. [4, p.12]).

We can now give the :

Proof of Theorem 2.1 Again the proof is essentially the same as that given by Walsh [14, p.130]. From (1.2) and (1.3), we have

$$K(\zeta,\zeta) f_\zeta'(z) - \sum_{n=0}^{\infty} \overline{Q_n(\zeta)} Q_n(z). \tag{4.7}$$

ie. the constants $\overline{Q_n(\zeta)}$ are the Fourier coefficients of the function $K(\zeta,\zeta) f_\zeta$ '(z):

$$\overline{Q_n(\zeta)} = K(\zeta,\zeta) \iint_\Omega f_\zeta' \overline{Q_n} \, dm. \tag{4.8}$$

Now suppose that p>1 so that f_ζ (and hence f_ζ') is analytic on $\Omega_\rho \supset \overline{\Omega},$ and let p_n denote the polynomials of respective degrees at most n of best uniform approximation to f_ζ' on $\overline{\Omega}$,. From a result of Walsh [14, p.90], we have

$$\limsup_{n\to\infty} \| f_\zeta' - p_{n-1} \|^{1/n}_{L_\infty(\overline{\Omega})} \le \frac{1}{\rho}. \tag{4.9}$$

Furthermore, by the orthogonality property of the Q_n's, equation (4.8) can be written as

$$\overline{Q_n(\zeta)} = K(\zeta,\zeta)\int\int_{\Omega}(f_{\zeta}^{'} - p_{n-1})\overline{Q_n}\,dm \tag{4.10}$$

Thus, from (4.10) and the Cauchy-Schwarz inequality, we get

$$\limsup |Q_n(\zeta)|^{1/n} \leq \frac{1}{\rho}. \tag{4.11}$$

Note that in the case when f_{ζ} is not analytic on $\overline{\Omega}$, that is $\rho = 1$, inequality (4.11) remains valid.

Next, we suppose that

$$\limsup_{n\to\infty} |Q_n(\zeta)|^{1/n} = \frac{1}{\sigma} \leq \frac{1}{\rho}. \tag{4.12}$$

and show that this leads to a contradiction. Indeed, from (4.3) and Lemma 4.1, we have for $\sigma > \tau > \rho$

$$\limsup_{n\to\infty} \|Q_n\|_{L_\infty(\overline{\Omega_\tau})}^{1/n} \leq \tau,$$

and so, by (4.12),

$$\limsup_{n\to\infty} \|Q_n(\zeta)Q_n(\cdot)\|_{L_\infty(\Omega_\tau)}^{1/n} \leq \frac{\tau}{\sigma} < 1.$$

Thus, the series in (4.7) converges uniformly on Ω_τ to an analytic extension of $K(\zeta,\zeta)f_{\zeta}^{'}$ But this contradicts the definition of ρ as being the largest index for which f_{ζ} (and, equivalently,) is analytic throughout Ω_ρ □

In the proof of Theorem 2.2 we shall make use of the following result due to Blatt, Saff and Simkani, which generalizes an earlier theorem of G. Szegö [12].

Lemma 4.4 ([2]) *Let S be a compact set with positive capacity and suppose that the monic polynomials $P_n(z) = z^n + \ldots$, which are given for a subsequence of indices n, say $n \in \wedge \subseteq N$, satisfy*

(a) $\limsup_{n\to\infty} \|P_n\|_{L_\infty(S)}^{1/n} \leq cap(S), n \in \wedge,$

and

(b) $\lim_{n\to\infty} \nu(P_n)(A) = 0,\ n \in \wedge,$ *for all closed sets A contained in the (2-dimensional) interior of the polynomial convex hull of the set S.*

Then, in the weak-star sense,

$$\nu(P_n) \xrightarrow{*} \mu_s,\ as\ n \to \infty,\ n \in \wedge,$$

where μ_s is the equilibrium distribution for S.

By the *polynomial convex hull* of *S* we mean the complement of the unbounded component of $\overline{C} \setminus S$.

Proof of Theorem 2.2 With the subsequence $\wedge$ as in (2.6) and with the assumption that $\rho < \infty$, we shall first establish that

$$v(\pi'_{n+1}) \overset{*}{\rightarrow} \mu_\rho, \ as\, n \to \infty, \ n \in \wedge \tag{4.13}$$

Since

$$\pi'_{n+1}(z) = \frac{1}{K_n(\zeta,\zeta)} \sum_{j=0}^{n} \overline{Q_j(\zeta)} Q_j(z) \tag{4.14}$$

we see that the leading coefficient of π'_{n+1} is given by

$$\lambda_n := \frac{\overline{Q_n(\zeta)\gamma_n}}{K_n(\zeta,\zeta)}, \tag{4.15}$$

that is, $\pi'_{n+1} = \lambda_n z^n + \cdots$. Notice that since $\rho < \infty$, (2.6) gives $\lambda_n \neq 0$ for n $n \in \wedge$ sufficiently large. Setting $P_n := \pi'_{n+1} / \lambda_n, n \in \wedge$, we shall show that the hypotheses (a) and (b) of Lemma 4.4 are valid with $S = \overline{\Omega_\rho}$.

Since $K_n(\zeta,\zeta) \to K(\zeta,\zeta) > 0$ as $n \to \infty$, we deduce from (4.15), (2.6) and (4.6) that

$$\lim_{\substack{n\to\infty \\ n\in\wedge}} |\lambda_n|^{\frac{1}{n}} = \frac{1}{\rho \operatorname{cap}(\overline{\Omega})} = \frac{1}{\operatorname{cap}(\overline{\Omega_p})}. \tag{4.16}$$

Furthermore, it can be seen from the proof of Theorem 2.1 that π'_{n+1} converges locally uniformly in Ω_p to f'_ζ and that

$$\limsup_{n\to\infty} \|\pi'_{n+1}\|^{1/n}_{L_\infty(\overline{\Omega_\rho})} \leq 1. \tag{4.17}$$

Thus, from (4.16) and (4.17) we have

$$\limsup_{\substack{n\to\infty \\ n\in\wedge}} \|P_n\|^{1/n}_{L_\infty(\overline{\Omega_\rho})} \leq \operatorname{cap}(\overline{\Omega_\rho}), \tag{4.18}$$

which establishes property (a) of Lemma 4.4. Also, since f'_ζ can have at most a finite number of zeros in any compact subset $A \subset \Omega_\rho$, property (b) of Lemma 4.4 follows from the theorem of Hurwitz. Hence

$$v(P_n) = v(\pi'_{n+1}) \overset{*}{\rightarrow} \mu_\rho, \ as\, n \to \infty, \ n \in \wedge.$$

Finally, we note that the leading coefficient of π_{n+1} differs from that of π_{n+1} only by a factor $1/(n+1)$ which does not affect the n th root estimates needed for applying Lemma 4.4. Thus, by the same reasoning as above, we get that

$$v(\pi_{n+1}) \overset{*}{\rightarrow} \mu_\rho, \ as\ n \to \infty, \ n \in \wedge. \ \square$$

5 Rational Approximants

Suppose that the mapping function f_ζ *is* analytic on $\overline{\Omega}$ (so that the constant ρ of Theorem 2.1 is greater than one), and assume that f_ζ is analytic on Γ_ρ except for simple poles at the ***l*** points $\alpha_j \in \Gamma_\rho$, $j = 1, 2, ..., l$. Let $\hat{\rho}(>\rho)$ denote the largest index such that

$$\hat{f}_\zeta(z) := f_\zeta(z)\prod_{j=1}^{l}(z-\alpha_j) \tag{5.1}$$

has an analytic (single-valued) extension throughout $\Omega_{\hat{\rho}}$. Then, as discussed in [10], improved rates of convergence can be obtained in the Bergman kernel method when the defining orthonormal system is constructed by orthonormalizing the set consisting of the monomials 1, z, z^2,... and the ***l*** rational functions

$$\eta_j(z) := \frac{1}{(z-\alpha_j)^2},\ j = 1,2,...,l, \tag{5.2}$$

that reflect the singularities of $K(\cdot,\zeta)$ at the points α_j, ***j***= 1,2,..., ***l*** . Our goal in this section is to study the zeros of the resulting approximations to f_ζ.

Consider the nested spaces

$$S_k := \text{span}\{\eta_1,\dots,\eta_k\},\ 1 \le k \le l,$$

$$S_{l+j} := \text{span}\{\eta_1,...,\eta_l,1,z,...,z^{j-1}\},\ j = 1,\ 2,\dots .$$

Let $\{s_1, s_2,... ,s_n\}$ denote an orthonormal basis for S_n, n = 1, 2,... and, for each n, write

$$s_n(z) = \frac{P_n(z)}{q(z)^2}, \tag{5.3}$$

where

$$q(z) := \prod_{j=1}^{l}(z-\alpha_j) \tag{5.4}$$

and P_n is a polynomial. We note the following regarding the polynomials p_n:

(i) For n > l, P_n has the form

$$P_n(z) = \hat{\gamma}_n\, q(z)^2\, z^{n-l-1} + \cdots = \hat{\gamma}_n\, z^{n+l-1} + \cdots, \tag{5.5}$$

where we can assume $\hat{\gamma}_n > 0$.

(ii) For $n > l$, P_n satisfies the l linear homogeneous constraints

$$\text{Res}\ (\frac{P_n}{q_2}, \alpha_k) = \frac{d}{d_z}\{\frac{P_n(z)}{\prod_{\substack{j=1\\ j\neq k}}^{l}(z-\alpha_j)^2}\}\Big|_{z=\alpha_k} = 0,\ k = 1,...,l, \tag{5.6a}$$

which, for brevity, we denote by

$$L\ (P_n) = 0,\quad n > l. \tag{5.6b}$$

(For $l = 1$ the empty product in (5.6a) is taken to equal 1 .)

(iii) Additional constraints hold for $n = 1,\dots,l$, and P_n has degree at most $2l - 2$ for such n.

(iv) Since

$$(s_n, s_k) = \iint_\Omega \frac{P_n}{q^2}\left(\overline{\frac{P_k}{q^2}}\right)dm = \iint_\Omega P_n \overline{P_k}\frac{dm}{|q|^4}, \tag{5.7}$$

the polynomials P_n are orthonormal with respect to the weighted measure $dm/|q|^4$ on Ω. Hence, for $n > l$,

$$\hat{P}_n(z) := \frac{P_n(z)}{\hat{\gamma}_n} = z^{n+1-1} + \cdots \tag{5.8}$$

solves the extremal problem

$$\min_p \iint_\Omega |p|^2 \frac{dm}{|q|^4}, \quad p(z) = z^{n+1-1} + \cdots, \quad L(p) = 0. \tag{5.9}$$

We next observe that the rational functions $s_n = P_n / q^2$, $n = 1,2,\dots$, form a complete orthonormal system for $L^2(\Omega)$. Therefore, the function $K(\zeta,\zeta) f_\zeta' = K(\cdot,\zeta)$ has the Fourier series expansion

$$K(\zeta,\zeta) f_\zeta'(z) \sim \sum_{j=1}^{\infty} \left(\overline{\frac{p_j(\zeta)}{q(\zeta)^2}}\right) \frac{p_j(z)}{q(z)^2} \tag{5.10}$$

and, as in [10], this leads us to consider rational function approximations to f_ζ given by

$$\hat{\pi}_n(z) := \frac{1}{\hat{K}_{n-1}(\zeta,\zeta)} \int_{t=\zeta}^{z} \hat{K}_{n-1}(t,\zeta)\,dt , \tag{5.11}$$

where

$$\hat{K}_{n-1}(z,\zeta) := \sum_{j=1}^{n-1} \left(\overline{\frac{P_j(\zeta)}{q(\zeta)^2}}\right) \frac{P_j(z)}{q(z)^2}. \tag{5.12}$$

We observe that $\hat{\pi}_n$ is rational, since

$$\hat{\pi}_n'(z) = \frac{\hat{K}_{n-1}(z,\zeta)}{\hat{K}_{n-1}(\zeta,\zeta)} \tag{5.13}$$

is clearly rational and has zero residue at each of its poles. Indeed, $\hat{\pi}_n$ 'and $\hat{\pi}_n$ have, respectively, the forms

$$\hat{\pi}_n'(z) = \frac{g_n(z)}{q(z)^2}, \text{ and } \quad \hat{\pi}_n(z) = \frac{h_n(z)}{q(z)}, \tag{5.14}$$

where g_n and h_n are polynomials. Thus, for the study of the zeros of $\hat{\pi}_n$ and $\hat{\pi}_n'$ we define the normalized counting measures $\nu(\hat{\pi}_n)$ and $\nu(\hat{\pi}_n')$ by

$$\nu(\hat{\pi}_n') := \nu(g_n), \quad \nu(\hat{\pi}_n') := \nu(h_n), \tag{5.15}$$

where g_n and h_n are the polynomials in (5.14).

We can now establish the following analogues of Theorems 2.1 and 2.2.

Theorem 5.1 *With the assumptions of this section,*

$$\limsup_{n\to\infty}\left|\frac{P_n(\zeta)}{q(\zeta)^2}\right|^{1/n}=\frac{1}{\hat{\rho}}<\frac{1}{\rho}. \tag{5.16}$$

Theorem 5.2 *Suppose that the constant* $\hat{\rho}$ *is finite and let* $\wedge\subseteq \mathrm{N}$ *be a sequence for which*

$$\lim_{\substack{n\to\infty\\ n\in\wedge}}\left|\frac{P_n(\zeta)}{q(\zeta)^2}\right|^{1/n}=\frac{1}{\hat{\rho}} \tag{5.17}$$

Then

$$\mathrm{v}(\mu_{n+1})\xrightarrow{*}\mu_{\hat{\rho}} \quad \text{and}\, \mathrm{v}(\mu_{n+1})\xrightarrow{*}\mu_{\hat{\rho}}\ \ , as\, n\to\infty,\ \ n\in\wedge, \tag{5.18}$$

where μ_ρ . Is the equilibrium distribution for $\overline{\Omega_{\rho.}}$

Since the proofs of the above results are similar to those of Theorems 2.1 and 2.2, we shall provide only a sketch of the details.

We first observe from (5.7) and (5.10) that the constants $\overline{p_n(\zeta)}\ /\overline{q(\zeta)^2},$ which are the Fourier coefficients of $K\ (\zeta,\zeta)f_\zeta'$ n the basis $\{s_n\}_1^\infty$, are the same as the Fourier coefficients in the expansion of the function

$$F(z):=\ K(\zeta,\zeta)q(z)f_\zeta{}'(z) \tag{5.19}$$

in terms of the orthonormal basis of polynomials $\{P_n\}_1^\infty$ (with respect to the measure $dm/\ q\ |^4$ on $\overline{\Omega}$). The latter expansion can easily be shown to converge maximally to F on $\overline{\Omega}$. in the sense of Walsh [14, p. 79]. (We note that the weight $1\ /|\ q\ |^4$ does not present any difficulties, since it is bounded from above and below by positive constants on Ω.) Thus,

$$\limsup_{n\to\infty}\|F-g_n\|_{L_\infty(\Omega)}^{1/n}=\frac{1}{\hat{\rho}}, \tag{5.20}$$

from which it follows (by the same reasoning as that used in the proof of Theorem 2.1) that (5.16) holds.

Next, to prove Theorem 5.2, we can imitate the proof of Theorem 2.2 provided that we first establish the analogue of (4.6), ie.

$$\lim_{n\to\infty}\hat{\gamma}_n{}^{1/n}=\frac{1}{cap(\overline{\Omega})}. \tag{5.21}$$

This can be seen as follows. Let T_n (z) = z^n + • • • be the monic polynomial of degree n that satisfies

$$\|T_n\|_{L_\infty(\overline{\Omega})}=\min\|P\|_{L_\infty(\overline{\Omega})},$$

where the minimum is taken over all monic polynomials $p=z^n$ + • • • of degree n . As is well-known (cf. [13, § III.5])

$$\lim_{n\to\infty}\|T_n\|_{L_\infty(\bar{\Omega})}^{1/n} = cap(\bar{\Omega}). \qquad (5.22)$$

since $L(q^2 T_{n-l-1}) = 0$ for $n > l$, we see from the extremal property (5.8)-(5.9) of $\hat{P}_n$ that

$$\frac{1}{\hat{\gamma}_n^{\,2}} = \iint_\Omega |\hat{P}_n|^2 \frac{dm}{|q|^4} \le \iint_\Omega |q^2 T_{n-l-1}|^2 \frac{dm}{|q|^4},$$

and so from (5.22) we get that

$$\liminf_{n\to\infty} \hat{\gamma}_n^{\,1/n} \ge \frac{1}{cap(\bar{\Omega})}. \qquad (5.23)$$

On the other hand, it is known (cf, [13, § III.5]) that for any monic polynomial $p\ (z) = z^n + \ldots$of degree n ,

$$[cap(\bar{\Omega})]^n \le \|P\|_{L_\infty(\bar{\Omega})}.$$

Thus,

$$[cap\ (\bar{\Omega})]^{n+l-1} \le \|\hat{P}_n\|_{L_\infty(\bar{\Omega})} = \frac{1}{\hat{\gamma}_n}\|P_n\|_{L_\infty(\bar{\Omega})},$$

and so

$$\limsup_{n\to\infty} \hat{\gamma}_n^{\,1/n} \le \frac{1}{cap(\bar{\Omega})} \cdot \limsup_{n\to\infty} \|P_n\|_{L_\infty(\bar{\Omega})}^{1/n}. \qquad (5.24)$$

Finally, it is easily verified that the orthonormal polynomials P_n satisfy

$$\lim_{n\to\infty}\|P_n\|_{L_\infty(\bar{\Omega})}^{1/n} = 1.$$

Hence, (5.24) yields

$$\limsup_{n\to\infty} \hat{\gamma}_n^{\,1/n} \le \frac{1}{cap(\bar{\Omega})},$$

and this, together with (5.23), establishes (5.21).

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References

[1] S. Bergman , *The Kernel Function and Conformal Mapping* , Math. Surveys, No. 5 (Amer. Math. Soc., Providence, R.I. 1970).

[2] H.-P. Blatt, E.B. Saff and M. Simkani , *Jentzsch*-Szegö *type theorems for the zeros of best approximants* , J. London Math. Soc. 38 (1988) 307 - 316 .

[3] D. Gaier, *Konstruktive Methoden der konformen Abbildung* (Springer, Berlin 1964).

[4] D. Gaier, *Lectures on Complex Approximation (* Birkhäuser, Boston 1987).

[5] E. Hille , *Analytic Function Theory,* vol. II (Ginn and Co., Boston 1962).

[6] N.S. Landkof, *Foundations of Modern Potential Theory (*Springer-Verlag, Berlin 1972).

[7] D. Levin, N. Papamichael and A. Sideridis , *The Bergman kernel method for the numerical conformal mapping of simply-connected domains,* J. Inst. Maths Applis **22** (1978) 171 -187 .

[8] Z. Nehari, *Conformal Mapping (* McGraw-Hill, New York 1952).

[9] N. Papamichael and C.A. Kokkinos, *Two numerical methods for the conformal mapping of simply-connected domains* , Comput. Meths Appl. Mech. Engrg **28** (1981) 285 - 307.

[10] N. Papamichael and M.K. Warby , *Stability and convergence properties of Bergman kernel methods in numerical conformal mapping* , Numer. Math. **48** (1986) 639 - 669 .

[11] E.B. Saff, *Orthogonal polynomials from a complex perspective* , pp. 363 - 393 of : P. Nevai (ed.), *Orthogonal Polynomials : Theory and Practice* (Kluwer Academic Publ., Dordrecht 1990).

[12] G. Szegö , *Über die Nullstellen von Polynomen, die in einem Kreis gleichmässig konvergieren* , Sitzungsber. Ber. Math. Ges. **21** (1922) 59 - 64 .

[13] M. Tsuji, *Potential Theory in Modern Function Theory (*Dover, New York 1959).

[14] J.L. Walsh , *Interpolation and Approximation by Rational Functions in the Complex Domain* , 5th ed., Amer. Math. Soc. Colloq. Publ., Vol. **20** (Amer. Math. Soc., Providence, **R.I.** 1969).

[15] J.L. Walsh , *The analogue for maximally convergent polynomials of Jentzsch's theorem* , Duke Math.J. **26**(1959)605-616.

[16] M.K. Warby , *BKMPACK* , *User's Guide* , Tecnical Report , Dept of Maths and Stats , Brunel University, 1990.