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LOW DIMENSIONAL GROTHENDIEK CONSTANTS

by

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Summary. We refine a method of Fournier to obtain improved upper bounds for the two and three dimensional complex Grothendieck constants.

Let $a=(a_{ij})$ be an NxN matrix with complex coefficients. The norm of a in the injective tensor product space $l_1^N \overset{\vee}{\otimes} l_1^N$ is given by

$$\|a\|_{11} = \sup\{|\sum_{i,j=1}^{N} a_{ij} s_i t_j|\},$$

Where the supremum is taken over all choices of complex scalars S_i and t_j $(1 \leq I, j \leq N)$ with absolute value no greater than 1.

A fundamental inequality due to Grothendieck (2, but see 4) asserts that if $x_1,\ldots,x_N$ and $y_1,\ldots,y_N$ are unit vectors in a complex Hilbert space, then

$$|\sum_{i,j=1}^{N} a_{ij}(x_i, y_j)| \leq G \|a\|_{11}$$

WhereG is an absolute constant. (In other words, G is independent of N, a and the vectors, and is chosen to be as small as possible.) The symbol (,) denotes the inner product in the Hilbert space.

Although the exact value of G is unknown, it has been shown that $1.273 = 4/\pi \leq G \leq \exp(1-\gamma) = 1.527$ where γ is Euler's constant (5).

There are many proofs of Grothendieck's inequality. In (1) Fournier gave a proof which relied on an algorithm originally developed by Schur (6). The purpose of this note is to examine Fournier's proof closely and to modify it in order to give new upper bounds for constants related to the Grothendieck constant.

If $x_1,\ldots,x_N$ and $y_1,\ldots,y_N$ are unit vectors in an *r-dimensional* complex Hilbert space, it is clear that there is an absolute constant $G(r) \leq G$ such that

$$|\sum_{i,j=1}^{N} a_{ij}(x_i, y_j)| \leq G(r) \|a\|_{11}.$$

We shall refer to G(r) as the *r-dimensional complex Grothendieck constant.*

Theorem 1. The two dimensional complex Grothendieck constant is bounded above by 4/3.

Theorem 2. The three dimensional complex Grothendieck constant is bounded above by 3/2.

These results should be compared with work of Krivine (3). He investigated the r-dimensional real Grothendieck constants (for which all scalars and Hilbert spaces are taken to be real). Krivine showed that the two dimensional real Grothendieck constant is exactly $\sqrt{2}$ and that the three dimensional real Grothendieck constant is bounded above by approximately 1.517. He also showed how to obtain upper bounds for the r-dimensional real Grothendieck constant when r is greater than 3

It should also be noted that in (7) different methods have been used to improve Theorem 1 by showing that the two dimensional complex Grothendieck constant is bounded above by $3\sqrt{3}/4$.

We now turn to a detailed discussion of Fournier's method. This will yield proofs of Theorems 1 and 2.

First we require some notation. We shall write

$$\omega = (\omega_1,...,\omega_N) = (\exp(i\theta_1),...., (\exp(i\theta_N))$$

for a generic point in the group T^N , the N-fold direct product of the circle group T. The group T^N will be endowed with the usual Haar probability measure $d\omega = d\omega_1...d\omega_N$. We shall write

$$\beta = (\beta_1,...,\beta_N)$$

for a generic point in the dual group Z^N , the N-fold direct product of the integer group Z. We impose a partial order on Z^N by defining

$$\beta \geq 0 = (0,..., 0) \text{ iff } \beta_1 \geq 0,..., \beta_N \geq 0 .$$

For every β in Z^N any integrable function f on T^N has associated with it a Fourier coefficient $\hat{f}(\beta)$.

The space $L^\infty(T^N)$ is the space of all essentially bounded

Haar measurable functions on T^N . For each f in $L^\infty(T^N)$ We define

$$\|f\|_\infty = \operatorname{esssup}[|f(\omega)|: \omega \in T^N] \quad .$$

Finally we shall write H^{N+1} for the (N+1) dimensional complex Hilbert space. The norm of the element $h = (h_0,\dots,h_N)$ is given by

$$\|h\| = (|h_0|^2 + \dots + |h_N|^2)^{\frac{1}{2}} \quad .$$

Strategy. For each N we aim to construct two non-linear maps R^N and S^N from H^{N+1} to $L^\infty(T^N)$

$$R^N : h \to R^N_h \quad \text{and} \quad S^N : k \to S^N_k$$

with the properties that for all h and k in H^{N+1}

$$(1) \qquad (h,k) = \int R^N_h(\omega) \overline{S^N_K(\omega)}\, d\omega$$

$$(2) \qquad \|R^N_h\|_\infty \le C(N)\ \|h\|$$

$$\text{and} \quad (3) \qquad \|S^N_k\|_\infty \le C(N)\ \|k\|$$

where c(N) is a constant independent of h and k .

Once this is done, the (N+1)-dimensional complex Grothendieck inequality follows immediately. For if we choose an nxn matrix $a = (a_{ij})$ and unit vectors $h_1,\dots,h_N$ and $k_1,\dots,k_N$ in H^{N+1} we have

$$|\sum a_{ij}(h_i,k_j)| = |\sum a_{ij} \int R^N_{h_i}(\omega)\overline{S^N_{k_j}(\omega)}\, d\omega$$

$$\int |\sum a_{ij} R^N_{h_i}(\omega)\overline{S^N_{k_j}(\omega)}|\, d\omega$$

$$\le c(N)^2 \|a\|_{11} \quad .$$

This tells us that $G(N+1) \le c(N)^2$.

A naive algorithm. Consider initially the case N = 1 .A "natural" way to map an element $x = (x_0,x_1)$ of H^2 into $L^\infty(T)$ is

$$x \to x_1 + x_0\,\omega_1 \ .$$

Now consider the case N = 2 . We can start an iterative procedure by mapping $x = (x_0,x_1,x_2)$ in H^3 into $L^\infty(T^2)$ in

in the following way:

$$x \to x_2 + (x_1 + x_0\omega_1)\,\omega_2 \ .$$

For general N we associate with x in H^{N+1} a polynomial P_X^N in $L^\infty(T)^N$. The mapping is

$$x + P_x^N \ ; \quad P_x^N(\omega) = x_N + x_{N-1}\omega_N + \cdots + x_0\omega_N \ldots \omega_1$$

.

If we could find a good in equality of the type

$$\| P_X^N \|_\infty \le c(N) \| x \|$$

then we would simply be able to take $R_h^N = P_h^N$ and $S_k^N = P_k^N$ in the basic strategy. Unfortunately, such inequalities do not give good enough values of c (N) for our purposes.

To get over this difficulty, we shall define

$$R_h^N = P_h^N + U_h^N \quad \text{and} \quad S_k^N = P_k^N + V_k^N$$

where the U_h^N and V_k^N are suitable "tails" chosen to be orthogonal to each other and to the P_h^N and P_k^N. The tails must be very carefully chosen to control the norms of R_h^N and S_k^N .

<u>The Schur algorithm.</u> The Schur algorithm is an inductive process due to Schur (6) whose properties are inherited from those of Blaschke factors. In order to understand what is going on, observe that if we expand the Blaschke factor

$$(a + x_1) / (1 + \overline{x_1 a})$$

where $a = x_0\omega_1 / (1 - |x_1|^2)$ as a formal power series in ω_1, then the first two terms are

$$x_1 + x_0\omega_1 \ .$$

These terms are precisely $P_X^1(\omega)$. It will turn out that the remainder of the power series forms an appropriate tail.

We now describe the Schur algorithm.

Start off with a vector x = b_N in H^{N+1} with components

$$b_{N,0} = x_0, b_{N,1} = x_1, \ldots, b_{N,N} = x_N \ .$$

Form a new vector b_{N-1} in H^N with components

$$b_{N-1,0} = x_0 / (1 - |x_N|^2), \ldots, \ b_{N-1,N-1} = x_{N-1} / (1 - |x_N|^2) \ .$$

Continue inductively. For each $0 \le r \le$ N-1 form a vector b_r

in H^{r-1} with components

$$b_{r,n} = b_{r+1,n} /(1-|b_{r+I,r+1}|^2) \quad (0 \le n \le r).$$

Finally, successively define functions $Q_0, \ldots, Q_N$ on T^N by setting

$$Q_0(\omega) = b_{0,0};$$

$$Q_r(\omega) = (\omega_r Q_{r-1}(\omega) + b_{r,r})/(1+\overline{b_{r,r}}\omega_r Q_{r-1}(\omega)) \quad (1 \le r \le N).$$

Observe that Q_r is a function of $\omega_1, \ldots, \omega_r$ only.

Constraint to be imposed on the algorithm. In order to control the norms of the functions Q_r we shall require

$$|b_{r,r}| < 1$$

for every $0 \le r \le N$. It is at this stage, and at this stage only, that our proof deviates from Fournier's.

Observe that if $\sum_{n=0}^{N} |x_n|^2 \le k$ then provided that $\frac{1}{2} \le k \le 1$

$$\sum_{n=0}^{N-1} |x_n|^2 /(1-|x_N|^2)^2 \le (k-|x_N|^2)/(1-|x_N|^2)^2 \le 1/4(1-k).$$

(This follows from a simple calculus argument.)

As an immediate consequence, we have

$$||b_2|| < \sqrt{2}/\sqrt{3} \to ||b_1|| < \sqrt{3}/2 \to ||b_0|| < 1.$$

(This sequence could be traced further back, but there is no point since the results which would be obtained would be weaker than Pisier's (5).) The facts to retain are

(1) $||b_2|| < \sqrt{2}/\sqrt{3} \Rightarrow |b_{r,r}| < 1 \quad (0 \le r \le 2)$

and (2) $||b_1|| < \sqrt{3}/2 \Rightarrow |b_{r,r}| < 1 \quad (0 \le r \le 1)$

Properties of the algorithm.

(1) The condition $|b_{r,r}| < 1$ for every r implies that $\|Q_r\|_\infty \le 1$ for every r.

When $r = 0$ this is obvious. For other values of r this property can be proved by induction. The crucial point is that

because of the condition

$$(z+b_{r,r})/(1+\overline{b_{r,r}z})$$

is an analytic function of z in the region $|z| \leq 1$, and has absolute value 1 on the boundary $|z| = 1$. The induction is now easy if one uses the maximum modulus principle.

(2) For each N, $Q_N = P_X^N + U_X^N$ where $\hat{U}_X^N(\beta) = 0$ unless possibly $\beta \geq 0$ and $\beta \neq (1,\ldots,1)$. (In other words, the Fourier coefficients $\hat{U}_X^N(\beta)$ are zero unless possibly all the β_n's are positive and some of them are greater than or equal to 2 .)

To justify this assertion, one can again work by induction. We aim to show that $Q_r = P_{b_r}^r + U_{b_r}^r$ where $\hat{U}_X^N(\beta) = 0$, unless possibly $\beta \geq 0$ and $\beta \not\leq (1,\ldots,1)$, for each $0 \leq r \leq N$.

If we make the natural interpretation that $P_{b_0}^0(\omega) = b_{0,0}$ then there is nothing to prove when $r = 0$.

Assume that $Q_r = P_{b_r}^r + U_{b_r}^r$ when $0 \leq r \leq s$. Then

$$\begin{aligned}
Q_s(\omega) &= (\omega_s Q_{s-1}(\omega) + b_{s,s})/(1+\overline{b_{s,s}}\,\omega_s Q_{s-1}(\omega)) \\
&= b_{s,s} + \omega_s Q_{s-1}(\omega) - |b_{s,s}|^2 \omega_s Q_{s-1}(\omega) + (\text{terms in } \omega_s^2) \\
&= b_{s,s} + (1-|b_{s,s}|^2)(P_{b_{s-1}}^{s-1}(\omega) + U_{b_{s-1}}^{s-1}(\omega)) \\
&\qquad\qquad + (\text{terms in } \omega_s^2). \\
&= P_{b_s}^s(\omega) + (1-|b_{s,s}|^2)\, U_{b_{s-1}}^{s-1}(\omega) + (\text{terms in } \omega_s^2).
\end{aligned}$$

This is what was required.

<u>Construction of R_h^N</u>. Fix h in H^{N+1}. Choose $c(N) > 2/\sqrt{3}$ if $N = 1$ and $c(N) > \sqrt{3}/\sqrt{2}$ if $N = 2$. Set $x = h/c(N)$.

Now if h has norm 1 then the algorithm is constrained in the required way. Consequently, if we set

$$R_h^N = c(N) Q_N$$

we shall have $\|R_h^N\|_\infty \leq c(N)$.

Construction of S_k^N. Fix a unit vector k in H^{N+1}. The main problem in difining S_k^N is to do so in such a way that its tail is orthogonal to that of R_h^N.

Set $\tilde{k} = (k_N, k_{N-1}, \ldots, k_0)$ and define $\tilde{\omega} = (\overline{\omega_N}, \ldots, \overline{\omega_1})$.

Now choose $c(N)$ as in the construction of R_h^N. Set $x = \tilde{k}/c(N)$. Then the algorithm is constrained in the required way, so if we define

$$S_k^N(\omega) = \omega_1 \ldots \omega_N \, Q_N(\tilde{\omega}) \, c\,(N)$$

we can assert that $\left\| S_k^N \right\|_\infty \leq c(N)$.

Moreover, a moment's reflection shows that S_k^N is the sum of P_k^N and a function whose Fourier coefficients are zero except possibly when $\beta \leq (1,\ldots,1)$ and $\beta \ngeq 0$.

Conclusion. The functions R_h^N, and S_k^N, have all the properties demanded by the strategy, and the Theorems are now proved.

References.

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