TR/08/86 June 86

Cylindrical wave diffraction by a rational wedge

by

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Abstract

In this paper new expressions for the field produced by the diffraction of a cylindrical wave source by a wedge, whose angle can be expressed as a rational multiple of π, are given. The solutions are expressed in terms of source terms and real integrals which represent the diffracted field. The general result obtained includes as special cases, Macdonald's solution for diffraction by a half plane, a solution for Carslaw's problem of diffraction by a wedge of open angle $2\pi/3$, and a new representation for the solution of the problem of diffraction by a mixed soft/hard half plane.

1. Introduction

This paper is a sequel to the paper Rawlins (1986), in which the solution to the problem of the diffraction of a plane wave by a rational wedge is given in terms of geometrical acoustic terms, and real integrals representing the diffracted field. Here we shall give an analogous solution to the problem of diffraction of a cylindrical acoustic wave by a wedge whose angle can be expressed as a rational multiple of π.

The exact solution of the problem of diffraction by a soft or hard wedge of any angle, in the two dimensional case of cylindrical acoustic wave incidence, is due to Macdonald (1902). The solution was given in the form of a complex contour integral, which was obtained by summing the Fourier series representation of the Green’s function. For the special case of a wedge which reduces to a half plane, Macdonald showed how the contour integral could be reduced to an elegant form involving real integrals. Though the form of Macdonald's solution is extremely simple the method used in obtaining it required a considerable amount of analysis. The problem of the diffraction of a line source by a half plane had been solved earlier by Carslaw (1899) using a method based on that used by Sommerfeld (1896) in considering diffraction by a plane wave. Carslaw's solution, though equivalent to Macdonald's solution, was of a different form. Sommerfeld's method was heuristic, using the physical method of images in various mathematical Riemann sheets associated with a multivalued function. Although the hybridism of the mathematical and physical concepts was considered abstruse it did produce exact solutions to hitherto insuperable problems in diffraction theory. Carslaw who was an early convert to Sommerfeld's method later gave up using the idea of Riemann surfaces and instead used the more modern approach of using periodic Green's functions. Before giving up the Sommerfeld approach to solve diffraction problems, he espoused the cause of Sommerfeld by writing some fairly long expository papers on his method with applications. In particular he gives in Carslaw (1899), a rather lucid description of Sommerfelds technique by considering the problem of diffraction by a wedge of open angle $2\pi/3$ when the normal method of images breaks down. This particular example is often used elsewhere to describe Sommerfeld's method, see Carslaw (1906), p356, Carslaw and Jaeger (1959) p279. Baker and Copson (1949) p124, however nowhere is the explicit solution given in terms of sources and images and real integrals representing the diffracted field. We shall give such a solution here as a special case of a more general result. Our approach avoids Sommerfeld's use of Riemann surfaces and simply uses the periodic Green's function for an arbitrary angle wedge. We then consider the special case of a wedge whose angle can be expressed as a rational multiple of π. It is then shown, by means of an appropriate integral representation for a Bessel function, that the Green's function for a cylindrical line source can be derived from the plane wave Green's function for a rational wedge. This enables us to obtain a representation for the Green's function for a cylindrical source, in the form of source and image terms and real integrals which are convenient for calculations of the diffracted field. We remark that recently there has been much work done on asymptotics for the wedge, see Deschamps (1985). The results presented here offer a new approach, in that a wedge of any angle can be approximated to any order of accuracy by a rational wedge of angle $p\pi/q$ (p and q integers), and the real integrals obtained in this paper can be asymptotically evaluated without difficulty.

In section 2 we shall give the periodic Green's function for a cylindrical wave source and a wedge of arbitrary angle. The Green's function is in the form of a complex contour integral. Some of the important properties of the Green's function are stated, and appropriate expression, in terms of this Green's function, are given for various diffraction problems. In section 3 we shall consider in detail the special case of evaluating the complex contour integral representation of the Green's function for a wedge whose angle can be expressed as a rational multiple of π. In section 4 we shall give expressions for the Green's function for special cases of wedge angles. Finally in section 5 we shall give solutions to some specific problems in diffraction theory which are special cases of the more general result obtained in section 4. The first problem is the classical problem of diffraction by soft or hard half plane by a cylindrical source, whose solution was given the different forms by Carslaw (1899), and Macdonald (1902), (1915). The second is Carslaw's (1899) didatic problem, used to describe Sommerfeld's technique of diffraction by an open wedge of angle $2\pi/3$, no explicit solution has appeared in the literature for this problem. The last is a new result for the problem of diffraction by a soft/hard plane by a cylindrical source.

In order not to disrupt the flow of the arguments in the main text of the paper, various proofs of results needed have been placed in appendices at the end of the paper. We remark in particular that in appendix A we derive a useful integral representation for the Hankel function $H_\nu^{(2)}(z)$, $|\arg z| < \pi/2$, $\mathrm{Re}\,\nu > -1$. This integral is closely related to a result given by Macdonald (1897), which does not seem to be well known. Macdonalds derivation does not give precise ranges of validity, and Watson's treatise on Bessel functions seems to have overlooked this integral representation.

2. Periodic Green's function for a wedge

The periodic Green's function $G_\alpha(r, \theta, r_0, \theta_0; k)$ for a two dimensional wedge situated in the space $0 < r < \infty$, $2\pi - \alpha \leq \theta \leq 2\pi$, see fig 1, where (r, θ) are cylindrical polar coordinates has been shown by Carslaw (1920) to be given by

$$G\ (r,\theta,r_0,\theta_0;k) = \frac{1}{2\alpha i} \int_c H_0^{(2)}[kR(\zeta)] \frac{\sin\pi\zeta/\alpha}{\cos\pi\zeta/\alpha \ - \ \cos\pi os - \theta_0)/\alpha}\, d\zeta, \qquad (1)$$

where $R(\zeta) = \sqrt{r^2 + r_0^2 - 2rr_0 \cos \zeta o}$ and the square root is defined by $-\pi/2 \leq \arg R(\zeta) \leq \pi/2$. The contour of integration c is such that the starting point is given by $i\infty + c_1$ and the termination point is given by $i\infty + c_2$, where $-\pi < c_1 < 0$, $\pi < c_2 < 2\pi$. The contour of integration c lies below the branch point $\zeta = \alpha = \cosh^{-1}((r^2 + r_0^2)/2rr_0)$, and does not intersect the branch cut : $\mathrm{Re}\,\zeta = 0$ $\alpha < \mathrm{Im}\zeta < \infty$, see fig 2.

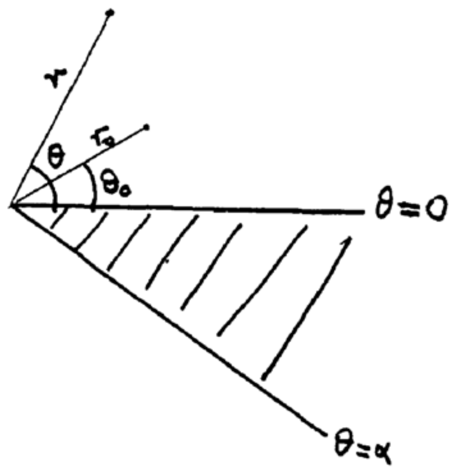

Fig 1

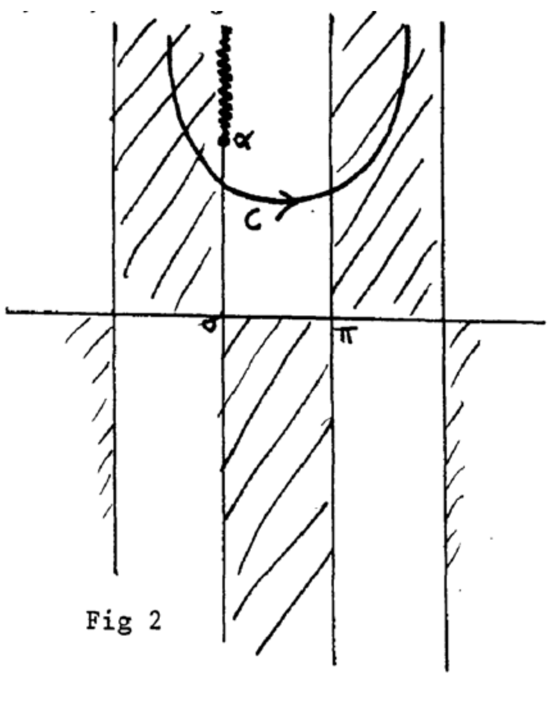

Fig 2

It has been shown by Carslaw that $G_\alpha,(r,\theta,r_0,\theta_0;k)$ has the following properties

(i) $(v^2+k^2)\,G_\alpha = 0$, where $\nabla^2 \equiv \dfrac{\partial^2}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2}{\partial\theta^2}$,

for all points $(r,\theta) \neq (r_0,\theta_0)$,

(ii) $G_\alpha(r,\theta,r_0,\theta_0;k) = G_\alpha(r,\theta+2\alpha, r_0,\theta_0;k)$,

(iii) $G_\alpha(r,\theta,r_0,\theta_0;k)$ is finite and continuous for all $(r,\theta) \neq (r_0,\theta_0)$,

(iv) $G_\alpha(r,\theta,r_0,\theta_0;k) \sim H_0^{(2)}[kR(\theta-\theta_0)]$, as $(r,\theta) \to (r_0,\theta_0)$, (2)

$\sim 0,$ as $r \to \infty$.

The Green's function given above enables one to derive solutions to various diffraction problems in wedge shaped regions. To be specific we shall discuss acoustic waves. The solution U_h or U_s of the problem of a cylindrical wave*

$$U_0 - H_0^{(2)}[kR(0-\theta_0)]\,, \tag{3}$$

diffracted by a rigid wedge ($\partial U_h/\partial\theta = 0$ for $\theta = 0$ and $\theta = \alpha$) or a soft wedge ($U_s = 0$ for $\theta = 0$ and $\theta = \alpha$) is given by

$$U_h = G_\alpha(r,\theta,r_0,\theta_0;k) + G_\alpha(r,\theta,r\ ,-\theta_0;k), \tag{4}$$

or

$$U_s = G_\alpha(r,\theta,r_0,\theta_0;k) - G_\alpha(r,\theta,r_0,-\theta_0;k), \tag{5}$$

respectively.

The solution $U_{s/h}$ of the problem of a cylindrical wave (3) diffracted by a wedge whose face $\theta = 0$ is rigid ($\partial U_{h/s}\,|\,\partial\theta = 0$) and whose face $\theta = \alpha$ is soft ($U_{h/s} = 0$) is given by

$$U_{h/s} = G_{2\alpha}(r,\theta,r_0,\theta_0;k) + G_{2\alpha}(r,-\theta,r_0,\theta_0;k)$$
$$-G_{2\alpha}(r,\theta,r_0,2\alpha-\theta_0;k) - G_{2\alpha}(r,\theta,r_0,-2\alpha+\theta_0;k). \tag{6}$$

(*Footnote: The wave is assumed to have time harmonic variation e^{iwt} , but will not be shown explicitly in the rest of the paper).

3 Line source Green's function for a rational wedge

If the wedge angle a is a rational multiple of π i.e., $\alpha = p\pi/q$ where p and q are integers the line source Green's function (1) becomes

$$G_{\frac{p\pi}{q}}(r,\theta,r_0,\theta_0;k) = \frac{1}{2\pi ip}\int_c H_0^{(2)}[kR(\zeta)]\frac{q\sin(q\zeta/p)}{\cos(\zeta q/p) - \cos((\theta-\theta_0)q/p)}d\zeta\ . \qquad (7)$$

By using the integral representation for the Hankel function, (A.4) of appendix A with $v = 0$, we have

$$H_0^{(2)}[k\,R(\zeta)] = \frac{1}{\pi i}\int_{\infty+ic}^{0} e^{\frac{-i}{2}\left(t+\frac{k^2(r^2+r_0^2)}{t}\right)+\frac{ik^2rr_0}{t}\cos\zeta}\frac{dt}{t} \qquad (8)$$

where c>0 and the contour of integration is as shown in fig 5.

Substituting the representation (8) into the expression (7), and interchanging the order of integration (which is permissible since integrals are uniformly convergent) gives

$$G_{\frac{p\pi}{q}}(r,\theta,r_0\theta_0;k) = \frac{1}{\pi i}\int_{\infty+ic}^{0} e^{\frac{-i}{2}\left(t+\frac{k^2(r^2+r_0^2)}{t}\right)} G_{\frac{p\pi}{q}}\left(r,\theta,\theta_0;\frac{k^2r_0}{t}\right)\frac{dt}{t}, \qquad (9)$$

where

$$G_{\frac{p\pi}{q}}(r,\theta,r_0,\theta_0;k) = \frac{1}{2\pi ip}\int_c e^{ikr\cos\zeta}\frac{q\sin(q\zeta/p)}{\cos(\zeta q/p) - \cos((\theta-\theta_0)q/p)}d\zeta\ , \qquad (10)$$

is the plane wave Green's function for a rational wedge. It has been shown Rawlins (1986) that the integral (10) can be written in the alternative form:

$$G_{\frac{p\pi}{q}}(r,\theta,\theta_0;k) = \sum_{m=0}^{q-1}\sum_{N} H[\pi - |\theta-\theta_0+2\pi mp/q+2\pi pN|]e^{ikr\cos(\theta-\theta_0+2\pi mp/q)}$$

$$+\frac{1}{2p}\sum_{m=0}^{q-1} e^{ikr\cos(\theta-\theta_0+2\pi m p/q)-i\pi/(2p)}\frac{\sin(\theta-\theta_0+2\pi m p/q)\sin(\pi/p)}{\sin((\theta-\theta_0+2\pi m p/q)/p)}$$

$$\int_{\infty}^{kr} e^{-ix\cos(\theta-\theta_0+2\pi m p/q)}\, H^{(2)}_{\frac{1}{p}}(x)\,dx$$

$$+\frac{1}{2ip}\sum_{m=0}^{q-1}\sum_{n=1}^{p-2}\frac{e^{ikr\cos(\theta-\theta_0+2\pi m p/q)}}{\sin((\theta-\theta_0+2\pi m p/q)/p)}\Bigg\{$$

$$+e^{in\pi/(2p)}\sin((n+1)(\theta-\theta_0+2\pi m p/q)/p)\,\sin(n\pi/p)\int_{\infty}^{kr} e^{-ix\cos(\theta-\theta_0+2\pi m p/q)}\,H^{(2)}_{\frac{(p-n)}{p}}(x)dx$$

$$-e^{i(n+1)\pi/(2p)}\sin(n(\theta-\theta_0+2\pi m p/q)/p)\,\sin((n+1)\pi/p)$$

$$\int_{\infty}^{kr} e^{-ix\cos(\theta-\theta_0+2\pi m p/q)}\,H^{(2)}_{\frac{(p-1-n)}{p}}(x)dx\Bigg\}, \qquad (11)$$

where the summation over N is for all integer values of N which can make the argument of the Heaviside step function $\left\{H[x]=\begin{matrix}1 & x>0\\ \frac{1}{2} & x=0\\ 0 & x<0\end{matrix}\right\}$ non negative. Thus

on substituting the expression (11) into (9) and interchanging the order of integrations results in having to evaluate integrals of the form:

$$\frac{1}{\pi i}\int_{\infty+ic}^{0} e^{-\frac{1}{2}\left(t+\frac{k^2(r^2+r_0^2)}{t}\right)} e^{\frac{ik^2 r r_0\cos\psi}{t}}\int_{\infty}^{\frac{k^2 r r_0}{t}} e^{-ix\cos\psi}H^{(2)}_{\nu}(x)dx\,\frac{dt}{t}\ ,$$

which is shown in appendix B to 'be equal to

$$-\frac{2}{\pi}e^{\frac{i\nu\pi}{2}}\int_0^{\infty}\frac{\cosh\,\nu t}{\cosh\,t+\cos\psi}\,H_0^{(2)}\,[kR(\pi-it)]dt\ .$$

Thus

$$G_{\frac{p\pi}{q}}(r,\theta,r_0,\theta_0;k)=\sum_{m=0}^{q-1}\sum_{N} H[\pi-|\theta-\theta_0+2\pi m p/q+2\pi p N|\,]\,H_0^{(2)}[kR(\theta-\theta_0+2\pi m p/q)]$$

$$-\frac{I}{\pi p}\sum_{m=0}^{q-1}\frac{\sin(\theta-\theta_0+2\pi m p/q)\sin(\pi/p)}{\sin((\theta-\theta_0+2\pi m p/q)/p)}\int_0^{\infty}\frac{\cosh\,(t/p)\,H_0^{(2)}[kR(\pi-it)]}{\cosh t+\cos(\theta-\theta_0+2\pi m p)}\,dt$$

$$-\frac{I}{\pi p}\sum_{m=0}^{q-1}\frac{\sin(\theta-\theta_0+2\pi mp/q)\sin(t/p)}{\sin((\theta-\theta_0+2\pi mp/q)/p)}\int_0^\infty\frac{\cosh(t/p)\,H_0^{(2)}[kR(\pi-it)]}{\cosh t+\cos(\theta-\theta_0+2\pi mp/q)}\,dt$$

$$-\frac{1}{\pi p}\sum_{m=0}^{q-1}\sum_{n=1}^{p-2}\left\{\frac{\sin((n+1)(\theta-\theta_0+2\pi mp/p)/p)\sin(n\pi/p)}{\sin((\theta-\theta_0+2\pi mp/q)/p)}\int_0^\infty\frac{\cosh((p-n)t/p)H_0^{(2)}[kR(\pi-it)]}{\cosh t+\cos(\theta-\theta_0+2\pi mp/q)}\,dt\right.$$

$$\left.-\frac{\sin(n(\theta-\theta_0+2\pi mp/q)/p)\sin(n+1)\pi/p)}{\sin((\theta-\theta_0+2\pi mp/q)/p)}\int_0^\infty\frac{\cosh((p-1-n)t/p)\,H_0^{(2)}[kR(\pi-it)]}{\cosh t+\cos(\theta-\theta_0+2\pi mp/q)}\,dt\right\},\qquad(12)$$

where the summation $\sum_N$ is performed for all values of N which satisfy the inequality $-\pi<\theta-\theta_0+2\pi mp/q+2\pi pN<\pi$.

Thus the solution $U(r,\theta)$ of the problem of diffraction of the cylindrical source $U_0 = H_0^{(2)}[kR(\theta-\theta_0)]$ by a soft or hard wedge of open angle $\alpha=p\pi/q$ is given by

$$U_s(r,\theta) = G_{\frac{p\pi}{q}}(r,\theta,r_0,\theta_0;k) - G_{\frac{p\pi}{q}}(r,\theta,r_0-\theta_0;k),\qquad(13)$$

and

$$U_h(r,\theta) = G_{\frac{p\pi}{q}}(r,\theta,r_0,\theta_0;k) + G_{\frac{p\pi}{q}}(r,\theta,r_0-\theta_0;k),\qquad(14)$$

respectively where $G_{\frac{p\pi}{q}}$ is given by the expression (12). Similarly the solution of the problem of diffraction of the line source $U_0 = H_0^{(2)}[kR(\theta\text{-}\theta_0)]$ by a wedge whose face $\theta = 0$ is soft, and whose other face $\theta = p\pi/q$ is hard is given by

$$\begin{aligned}U_{s/h}(r,\theta) = {}& G_{\frac{2p\pi}{q}}(r,\theta,r_0\theta_0;k) + G_{\frac{2p\pi}{q}}\left(r,\theta,r_0,\frac{2p\pi}{q}-\theta_0;k\right),\\ &-G_{\frac{2p\pi}{q}}\left(r,\theta,r_0,\frac{2p\pi}{q}+\theta_0;k\right) - G_{\frac{2p\pi}{q}}\left(r,\theta,\frac{2p\pi}{q}-\theta_0;k\right),\end{aligned}\qquad(15)$$

where $G_{\frac{2p\pi}{q}}(r,\theta,r_0,\theta_0;k)$ is given by the expression (12) with p replaced by 2p.

An asymptotic expression for $G_{\frac{p\pi}{q}}(r,\theta,r_0,\theta_0;k)$ can be obtained, from the expression (20) of Rawlins (1986), by applying the techniques outlined in the appendix C. Thus for $kr\to\infty$ we have

$$G_{\frac{p\pi}{q}}(r,\theta,r_0,\theta_0;k) = \sum_{m=0}^{q-1}\sum_{N} H[\pi - |\theta-\theta_0 + 2\pi mp/q + 2\pi pN|]H_0^{(2)}[kR(\theta-\theta_0+2\pi mp/q)]$$

$$+\frac{i}{\pi p}\sum_{m=0}^{q-1}\frac{\sin(\theta-\theta_0+2\pi mp/q)\sin(\pi/p)}{\sin(\theta-\theta_0+2\pi mp/q)/p)|\cos((\theta-\theta_0+2\pi mp/q)/2|}\int_{\infty}^{\xi(\theta_0)} e^{-ikR(\theta-\theta_0+2\pi mp/q)\cosh\xi}d\xi$$

$$+\frac{i}{\pi p}\sum_{m=0}^{q-1}\sum_{n=1}^{p-2}\frac{1}{\sin(\theta-\theta_0+2\pi mp/q)/p)|\cos((\theta-\theta_0+2\pi mp/q)/2|}\Bigg\{$$

$$+\sin((n+1)((\theta-\theta_0+2\pi mp/q)/p)\sin(n\pi/p)\int_{\infty}^{\xi(\theta_0)} e^{-ikR(\theta-\theta_0+2\pi mp/q)\cosh\xi}d\xi$$

$$-\sin(n(\theta-\theta_0+2\pi mp/q)/p)\sin((n+1)\pi/p)\int_{\infty}^{\xi(\theta_0)} e^{-ikR(\theta-\theta_0+2\pi mp/q)\cosh\xi}d\xi\Bigg\}$$

$$+0((kR)^{-3/2}), \tag{16}$$

where

$$\xi(\theta_0) = \sinh^{-1}\left\{\frac{2\sqrt{rr_0}\,|\cos(\theta-\theta_0+2\pi mp/q)/2|}{R(\theta-\theta_0+2\pi mp/q)}\right\} \tag{17}$$

The integrals appearing in the above expression (16) can be further expressed in terms of Fresnel integrals, whose properties are well known, for details see Jones, (1986) , p558.

Special cases of wedge angles.

$\boxed{P = 1}$

$$G_{\frac{p\pi}{q}}(r,\theta,r_{0},\theta_{0};k) = \sum_{m=0}^{q-1} \sum_{N} H[\pi-|\theta-\theta_{0}+2\pi mp/q+2\pi pN|]H_{0}^{(2)}[kR(\theta-\theta_{0}+2\pi mp/q)] \quad (18)$$

———— " ————

$\boxed{q = 1}$

$$G_{p\pi}(r,\theta,r_{0},\theta_{0};k) = \sum_{N} H[\pi-|\theta-\theta_{0}+2\pi pN|]\ H_{0}^{(2)}[kR(\theta-\theta_{0})]$$

$$-\frac{1}{\pi p}\ \frac{\sin(\theta-\theta_{0})\sin(\pi/p)}{\sin(\theta-\theta_{0})/p)} \int_{0}^{\infty} \frac{\cosh(t/p)\ H_{0}^{(2)}[kR(\pi-it)]}{\cosh t + \cos(\theta-\theta_{0})}\, dt$$

$$-\frac{1}{\pi p}\sum_{n=1}^{p-2}\left\{\frac{\sin(n+1)(\theta-\theta_{0})/p)\sin(n\pi/p)}{\sin(\theta-\theta_{0})/p)} \int_{0}^{\infty} \frac{\cosh(p-n)(t/p)\ H_{0}^{(2)}[kR(\pi-it)]}{\cosh t + \cos(\theta-\theta_{0})}\, dt\right.$$

$$\left.-\frac{\sin(n(\theta-\theta_{0})/p)\sin((n+1)\pi/p)}{\sin((\theta-\theta_{0})/p)} \int_{.0}^{\infty} \frac{\cosh((p-1-n)t/p\ \ H_{0}^{(2)}[kR(\pi-it)]}{\cosh t + \cos(\theta-\theta_{0})}\, dt\right\}. \quad (19)$$

———— " ————

$\boxed{p = 2}$

$$G_{\frac{2\pi}{q}}(r,\theta,r_{0},\theta_{0};k) = \sum_{m=0}^{q-1}\sum_{N} H[\pi-|\theta-\theta_{0}+4\pi N|]\ H_{0}^{(2)}[kR(\theta-\theta_{0}+4\pi m/q)]$$

$$-\frac{1}{\pi}\sum_{m=0}^{q-1}\cos\ [\frac{(\theta-\theta_{0}+4\pi m/q)}{2}] \int_{0}^{\infty} \frac{\cosh(t/2)\ \ H_{0}^{(2)}\ [kR(\pi-it)]}{\cosh\ t + \cos(\theta-\theta_{0}+4\pi m/p)}\ dt\ . \quad (20)$$

The last expression (20) can be put in an alternative form by using the results of appendix D. Thus

$$G_{\frac{2\pi}{q}}(r,\theta,r_{0},\theta_{0};k) = \sum_{m=0}^{q-1}\sum_{N} H[\pi-|\theta-\theta_{0}+4\pi m/q+4\pi N|]\ H_{0}^{(2)}[kR(\theta-\theta_{0}+4\pi m/q)]$$

$$+\frac{i}{\pi}\sum_{m=0}^{q-1} \mathrm{sgn}[\cos\{(\theta-\theta_{0}+4\pi m/q)/2\}] \int_{\infty}^{\zeta(\theta_{0})} e^{-ikR(\theta-\theta_{0}+4\pi m/q)\cosh\zeta} d\zeta,$$

$$\text{where}\ \xi(\theta_{0}) = \sinh^{-1}\left\{\frac{2\sqrt{rr_{0}}\ |\cos(\theta-\theta_{0}+4\pi m/q)/2}{R(\theta-\theta_{0}+4\pi m/q)}\right\}$$

4. Some Specific problems in diffraction theory

Macdonald's Solution for a half plane.

In terms of the Green's function, the solution for the problem of diffraction of a cylindrical wave $U_0(r,\theta) = H_0^{(2)}[kR(\theta-\theta_0)]$ by a soft, or hard half plane is given by

$$\begin{aligned} U_h(r,\theta,r_0,\theta_0,) &= G_{2\pi}(r,\theta,r_0,\theta_0;k) - G_{2\pi}r,\theta,r_0,\theta_0;k) , \\ U_s(r,\theta,r_0,\theta_0,) &= G_{2\pi}(r,\theta,r_0,\theta_0;k) - G_{2\pi}r,\theta,r_0,\theta_0;k) , \end{aligned} \tag{22}$$

respectively.

Putting q - 1 in the expression (21) gives

$$G_{2\pi}(r,\theta,r_0,\theta_0;k) = \sum_N H[\pi-|\theta-\theta_0+4\pi N|]\, H_0^{(2)}[kR(\theta-\theta_0)]$$
$$+\frac{i}{\pi}\mathrm{sgn}[\cos((\theta-\theta_0)/.2)] \int_\infty^{|\xi(\theta_0)|} e^{ikR(\theta-\theta_0)\cosh\xi}\, d\xi, \tag{23}$$

$$\text{where}\quad \xi(\theta_0) = \sinh^{-1}\left\{\frac{2\sqrt{rr_0}\cos(\theta-\theta_0)/2}{R(\theta-\theta_0)}\right\} \tag{24}$$

Now for $0<\theta_0<2\pi$, and $0<\theta<2\pi$, then $|\theta-\theta_0|<2\pi$, so that the argument of the Heaviside step function in (23) can only be positive if N = 0. Hence

$$G_{2\pi}(r,\theta,r_0,\theta_0;k) = H[\pi-|\theta-\theta_0|]H_0^{(2)}[kR(\theta-\theta_0)]$$
$$+\frac{i}{\pi}\,\mathrm{sgn}\left[\cos\frac{(\theta-\theta_0)}{2}\right] \int_\infty^{|\xi(\theta)|} e^{ikR(\theta-\theta_0)\cosh\xi} d\xi ,$$

$$= H\left[\cos\frac{(\theta-\theta_0)}{2}\right] H_0^{(2)}[kR(\theta-\theta_0)] + \frac{i}{\pi}\,\mathrm{sgn}\left[\cos\frac{(\theta-\theta_0)}{2}\right] \int_\infty^{|\xi(\theta)|} e^{ikR(\theta-\theta_0)\cosh\xi} d\xi ,$$

If $\cos((\theta-\theta_0)/2>0$ than

$$G_{2\pi}(r,\theta,r_0,\theta_0;k) = H_0^{(2)}[kR(\theta-\theta_0)] + \frac{i}{\pi}\int_\infty^{\xi(\theta_0)} e^{-ikr(\theta-\theta_0)\cosh\xi} d\xi .$$

Now using the fact that

$$H_0^{(2)}[kR(\theta-\theta_0)] = \frac{i}{\pi}\int_{-\infty}^{\infty} e^{-ikR(\theta-\theta_0)\cosh\xi} d\xi ,$$

we can write

$$G_{2\pi}(r,\theta,r,\theta\ ;k) = \frac{i}{\pi}\left\{\int_{-\infty}^{\infty} + \int_{\infty}^{\xi(\theta_0)}\right\} e^{-ikR(\theta-\theta_0)\cosh\xi} d\xi ,$$

$$= \frac{i}{\pi}\int_{-\infty}^{\xi(\theta_0)} e^{-ikR(\theta-\theta\ \ )\cosh\xi} d\xi \tag{25}$$

If $\cos(\theta-\theta_0)/2) < 0$ then

$$G_{2\pi}(r,\theta,r_0,\theta_0\ ;k) = \frac{i}{\pi}\int_{\infty}^{\xi(\theta_0)} e^{-ikR(\theta-\theta_0)\cosh\xi} d\xi ,$$

$$= \frac{i}{\pi}\int_{-\infty}^{\xi(\theta_0)} e^{-ikR(\theta-\theta\ \ )\cosh\xi} d\xi \tag{26}$$

Hence for any sign of $\cos(\theta-\theta_0)/2$ we have

$$G_{2\pi}(r,\theta,r_0,\theta_0\ ;k) = \frac{i}{\pi}\int_{-\infty}^{\xi(\theta_0)} e^{-ikR(\theta-\theta_0)\cosh\xi} d\xi , \tag{27}$$

The expression for $G_{2\pi}(r,\theta,r_0,-\theta_0{}'k)$ can be found in exactly the same manner for $0 < 0+\theta_0 < 4\pi$ i.e.

$$G_{2\pi}(r,\theta,r_0,-\theta_0;k) = H[\pi[\,|\theta+\theta_0\,|\,]\,H_0^{(2)}[kR(\theta+\theta_0)]$$

$$+ H[\pi-\theta+\theta_0-4\pi|\,]H_0^{(2)}[kR(\theta+\theta_0)] + \frac{i}{\pi}\mathrm{sgn}[\cos((\theta+\theta_0)/2)]\int_{\infty}^{|\xi(-\theta_0)|} e^{-ikR(\theta+\theta_0)\cosh\xi} d\xi \tag{28}$$

where $$\xi(-\theta_0) = \sinh^{-1}\left\{\frac{2\sqrt{rr_0}\cos((\theta+\theta_0)/2}{R(\theta+\theta_0)}\right\} \tag{29}$$

Hence

$$G_{2\pi}(r,\theta,r_0,-\theta_0;k) = H[\cos((\theta+\theta_0)/2)]H_0^{(2)}[kR(\theta+\theta_0)]$$

$$+\frac{i}{\pi}\,\mathrm{sgn}[\cos((\theta+\theta_0)/2)] \int_{\infty}^{\xi(-\theta_0)} e^{-ikR(\theta+\theta_0)\cosh\xi} d\xi$$

$$= \frac{i}{\pi} \int_{-\infty}^{\xi(-\theta_0)} e^{-ikR(\theta+\theta_0)\cosh\xi} d\xi \,. \tag{30}$$

Thus the solution of the problem of diffraction of a cylindrical wave by a hard or soft half plane is given by substituting the expressions (27) and (30) into (22) giving

$$U_h(r,\theta,r_0,\theta_0) = \frac{i}{\pi} \int_{-\infty}^{\xi(\theta_0)} e^{-ikR(\theta-\theta_0\cosh\xi} d\xi + \frac{i}{\pi} \int_{-\infty}^{\xi(\theta_0)} e^{-ikR(\theta-\theta_0\cosh\xi} d\xi \;,$$

$$U_s(r,\theta,r_0,\theta_0;k) = \frac{i}{\pi} \int_{-\infty}^{\xi(\theta_0)} e^{-ikR(\theta-\theta_0)\cosh\xi} d\xi - \frac{i}{\pi} \int_{-\infty}^{\xi(\theta_0)} e^{-ikR(\theta-\theta_0)\cosh\xi} d\xi \;, \tag{31}$$

where $\xi(\pm\theta_0)$ are given by (24) and (29) respectively. This result agrees with that of Macdonald (1915).

Solution of Car slaw's problem for a wedge angle a = 2π/3.

The solution for the problem of diffraction of a cylindrical wave $U_0(r,\theta,r_0,\theta_0) = H_0^{(2)}[kR(\theta-\theta_0)]$ by a soft or hard wedge of open angle $\alpha = 2\pi/3$ is given by

$$U_s(r,\theta,r_0,\theta_0) = G_{\frac{2\pi}{3}}(r,\theta,r_0,\theta_0;k) - G_{\frac{2\pi}{3}}(r,\theta,r,-\theta_0;k),$$

$$U_h(r,\theta,r_0,\theta_0) \quad G_{\frac{2\pi}{3}}(r,\theta,r_0,\theta_0;k) - G_{\frac{2\pi}{3}}(r,\theta,r,-\theta_0;k), \tag{32}$$

where, from the expression (21) with q = 3,

$$\begin{aligned} G_{\frac{2\pi}{3}}(r,\theta,r_0,\theta_0;k) &= \sum_{N_1} H[\pi-|\theta-\theta_0+4\pi\pi_1|]H_0^{(2)}[kR(\theta-\theta_0)] \\ &+ \sum_{N_2} H[\pi-|\theta-\theta_0+4\pi/3+4\pi N_2|]H_0^{(2)}[kR(\theta-\theta_0+4\pi/3)] \\ &+ \sum_{N_2} H[\pi-|\theta-\theta_0+8\pi/3+4\pi N_3|]H_0^{(2)}[kR(\theta-\theta_0+8\pi/3)] \end{aligned}$$

$$+\frac{i}{\pi}\operatorname{sgn}[\cos((\theta-\theta_0)/2)]\int_\infty^{\sinh^{-1}[2\sqrt{rr_0}\,|\cos((\theta-\theta_0)/2|]} e^{-ikR(\theta-\theta_0)\cosh\xi}\,d\xi$$

$$+\frac{i}{\pi}\operatorname{sgn}[\cos((\theta-\theta_0+4\pi/3)/2)]\int_\infty^{\sinh^{-1}[2\sqrt{rr_0}\,|\cos((\theta-\theta_0+4\pi/3)/2|]} .\, e^{-ikR(\theta-\theta_0+4\pi/3)\cosh\xi}\,d\xi$$

$$+\frac{i}{\pi}\operatorname{sgn}[\cos((\theta-\theta_0+8\pi/3)/2)]\int_\infty^{\sinh^{-1}[2\sqrt{rr_0}\,|\cos((\theta-\theta_0+8\pi/3)/2|]} .\, e^{-ikR(\theta-\theta_0+8\pi/3)\cosh\xi}\,d\xi \tag{33}$$

It is not difficult to show that for $-2\pi/3<\theta-\theta_0<2\pi/3$ then $N_1 = 0$, $N_2 = 0$, $N_3 = -1$. Hence

$$\begin{aligned} G_{\frac{2\pi}{3}}(r,\theta,r_0,\theta_0;k) &= H_0^{(2)}[kR(\theta-\theta_0)] \\ &+ H[\pi-|\theta-\theta_0+4\pi/3|]H_0^{(2)}[kR(\theta-\theta_0+4\pi/3)] \\ &+ H[\pi-|\theta-\theta_0-4\pi/3|]H_0^{(2)}[kR(\theta-\theta_0+4\pi/3)] \end{aligned}$$

$$+\ \frac{i}{\pi}\ \mathrm{sgn}[\cos((\theta-\theta_0)/2)]\ \int_{\infty}^{\sinh^{-1}[2\sqrt{rr_0}\,|\cos((\theta-\theta_0)/2)|]} e^{-ikR(\theta-\theta_0)\cosh\xi} d\xi$$

$$+\ \frac{i}{\pi}\ \mathrm{sgn}[\cos((\theta-\theta_0+4\pi/3)/2)]\ \int_{\infty}^{\sinh^{-1}[2\sqrt{rr_0}\,|\cos((\theta-\theta_0+/4\pi/3)|2|]}\ .$$

$$.\ e^{-ikR(\theta-\theta_0+4\pi\pi/3)\cosh} d\xi$$

$$+\frac{i}{\pi}\,\mathrm{sgn}\ [\cos\ (\ (\theta-\theta_0-4\pi/3)/2)]\ \int_{\infty}^{\sinh^{-1}[2\sqrt{rr_0}\,|\cos\ (\ (\theta-\theta_0\ +4\pi/3\ )/2\ |\ ]}\ .$$

$$.\ e^{-ikR(\theta-\theta_0+8\pi\pi/3)\cosh} d\xi. \qquad (34)$$

We also have from the expression (21),

$$G_{\frac{2\pi}{3}}(r,\theta,r_0,-\theta_0;k)\ =\sum_{N_1}\ H[\pi-|\,\theta+\theta_0+4\pi N_1\,|\,]H_0^{(2)}\ [kR(\theta+\theta_0)\ ]$$

$$+\ \sum_{N_2}\ H[\pi-|\,\theta+\theta_0+4\pi/3+4\pi N_2\,|\,]H_0^{(2)}\ [kR(\theta+\theta_0+4\pi/3)]$$

$$+\ \sum_{N_3}\ H[\pi-|\,\theta+\theta_0+8\pi/3+4\pi N_2\,|\,]H_0^{(2)}\ [kR(\theta+\theta_0+8\pi/3)]$$

$$+\ \frac{i}{\pi}\ \mathrm{sgn}[\cos((\theta+\theta_0)/2)]\ \int_{\infty}^{\sinh^{-1}[2\sqrt{rr_0}\,|\cos((\theta+\theta_0)/2)|]} e^{-ikR(\theta+\theta_0)\cosh\xi} d\xi$$

$$+\ \frac{i}{\pi}\ \mathrm{sgn}[\cos((\theta+\theta_0)+4\pi/3)/2)]\ \int_{\infty}^{\sinh^{-1}[2\sqrt{rr_0}\,|\cos((\theta+\theta_0+/4\pi/3)|2|]}\ .$$

$$.\quad e^{-ikR(\theta+\theta_0+4\pi/3)\cosh\xi} d\xi$$

$$+\frac{i}{\pi}\,\mathrm{sgn}\ [\cos\ (\ (\theta+\theta_0+8\pi/3)/2)]\ \int_{\infty}^{\sinh^{-1}[2\sqrt{rr_0}\,|\cos\ (\ (\theta+\theta_0\ +8\pi/3\ )/2\ |\ ]}\ .$$

$$.\ e^{-ikR(+\theta_0+8\pi\pi/3)\cosh} d\xi. \qquad (35)$$

For the range of values $0<\theta+\theta_0<4\pi/3$ it is not difficult to show that $N_1 = 0$, N_2 takes no values, $N_3 = -1$, so that

$$G_{\frac{2\pi}{3}}(r,\theta,r_0,-\theta_0;k)\ = H[\pi-|\,\theta+\theta_0\,|\,]H_0^{(2)}\ [kR(\theta+\theta_0)\ ]$$

$$+\ H[\pi-|\,\theta+\theta_0-4\pi/3)]H_0^{(2)}\ [kR(\theta+\theta_0-4\pi/3)\ ]$$

$$+\frac{i}{\pi}\,\mathrm{sgn}\ [\cos\ (\ (\theta+\theta_0+)/2)]\ \int_{\infty}^{\sinh^{-1}[2\sqrt{rr_0}|\ \cos\ (\ (\theta+\theta_0\ \ )/2\ |\ ]}$$

$$e^{-ikR(\theta+\theta_0)\cosh\xi} d\xi.$$

$$+\frac{i}{\pi}\,\mathrm{sgn}\,[\cos((\theta+\theta_0+4\pi/3)/2)]\int_{\infty}^{\sinh^{-1}[2\sqrt{rr_0}\,|\cos((\theta+\theta_0+4\pi/3))/2|]} e^{-ikR(+\theta_0+4\pi/3)\cosh\xi}\,d\xi.$$

$$+\frac{i}{\pi}\,\mathrm{sgn}\,[\cos((\theta+\theta_0-4\pi/3)/2)]\int_{\infty}^{\sinh^{-1}[2\sqrt{rr_0}\,|\cos((\theta+\theta_0-4\pi/3))/2|]} .\; e^{-ikR(\theta+\theta_0-4\pi/3)\cosh\xi}\,d\xi\;. \tag{36}$$

Substituting the expressions (34) and (36) into (32) gives the solution to the problem of diffraction of the cylindrical wave $U_{.0}(r,\theta,r_0,\theta_0)$ by a soft or hard wedge of open angle $2\pi/3$.

Diffraction by a hard/soft half plane

In terms of the Green's function the solution for the problem of the diffraction of the line source $U_0(r,\theta,r_0,\theta_0) = H_0^{(2)}[kR(\theta-\theta_0)]$ by a hard/soft half plane is given by

$$U_{h/s}(r,\theta,r_0,\theta_0) = G_{4\pi}(r,\theta,r_0,\theta_0;k) + G_{4\pi}(r,\theta,r_0,-\theta_0;k)$$
$$- G_{4\pi}(r,\theta,r_0,4\pi-\theta_0;k) - G_{4\pi}(r,\theta,r_0,-4\pi+\theta_0;k)\,. \tag{37}$$

By putting p = 4 in the expression (19) we obtain

$$G_{4\pi}(r,\theta,r_0,4\pi-\theta_0;k) = \sum_N H[\pi-|\theta-\theta_0+8\pi N|\,]H_0^{(2)}\,[kR(\theta-\theta_0)]$$
$$-\frac{1}{4\pi\sqrt{2}}\cdot\frac{\sin(\theta-\theta_0)}{\sin((\theta-\theta_0)/4)}\int_0^\infty \frac{\cosh(t/4)H_0^{(2)}[kR(\pi-it)]}{\cosh t+\cos(\theta-\theta_0)}\,dt$$
$$-\frac{1}{4\pi}\left\{\frac{\sin((\theta-\theta_0)/2)}{\sqrt{2}\sin((\theta-\theta_0)/4)}\int_0^\infty \frac{\cosh(3t/4)H_0^{(2)}[kR(\pi-it)]}{\cosh t+\cos(\theta-\theta_0)}\,dt\right.$$
$$\left.-\int_0^\infty \frac{\cosh(t/2)H_0^{(2)}[kR(\pi-it)]}{\cosh t+\cos(\theta-\theta_0)}\,dt\right\}$$
$$-\frac{1}{4\pi}\left\{\frac{\sin(3(\theta-\theta_0)/4)}{\sin((\theta-\theta_0)/4)}\int_0^\infty \frac{\cosh(t/2)H_0^{(2)}[kR(\pi-it)]}{\cosh t+\cos(\theta-\theta_0)}\,dt\right.$$
$$\left.-\frac{\sin((\theta-\theta_0)/2)}{\sqrt{2}\sin((\theta-\theta_0)/4)}\int_0^\infty \frac{\cosh(t/4)H_0^{(2)}[kR(\pi-it)]}{\cosh t+\cos(\theta-\theta_0)}\,dt\right\}. \tag{38}$$

For $-2\pi<\theta-\theta_0<2\pi$ the only value of N which satisfies $-\pi<\theta-\theta_0+8\pi N<\pi$ is N = 0. Hence

$$G_{4\pi}(r,\theta,r_0,\theta_0;k) = H[\pi-|\theta-\theta_0|\,]H_0^{(2)}\,[kR(\theta-\theta_0)]$$

$$+\frac{1}{4\pi}\left(1-\frac{\sin(3(\theta-\theta_0)/4)}{\sin((\theta-\theta_0)/4)}\right)\int_0^\infty \frac{\cosh(t/2)H_0^{(2)}[kR(\pi-it)]}{\cosh t+\cos(\theta-\theta_0)}\,dt$$

$$-\frac{\sqrt{2}}{4\pi}\cos((\theta-\theta_0)/4)\int_0^\infty \frac{\cosh(3t/4)H_0^{(2)}[kR(\pi-it)]}{\cosh t+\cos(\theta-\theta_0)}\,dt$$

$$+\frac{\sqrt{2}}{4\pi}\cos((\theta-\theta_0)/4)\,(1-2\cos((\theta-\theta_0)/2))\int_0^\infty \frac{\cosh(t/4)H_0^{(2)}[kR(\pi-it)]}{\cosh t+\cos(\theta-\theta_0)}\,dt\,, \qquad (39)$$

$$= H[\pi-|\theta-\theta_0 N|\,]H_0^{(2)}\,[kR(\theta-\theta_0)]$$

$$-\frac{1}{2\pi}\cos((\theta-\theta_0)/2)\int_0^\infty \frac{\cosh(t/4)H_0^{(2)}[kR(\pi-it)]}{\cosh t+\cos(\theta-\theta_0)}\,dt$$

$$-\frac{\sqrt{2}}{4\pi}\cos((\theta-\theta_0)/4)\int_0^\infty \frac{\cosh(3t/4)H_0^{(2)}[kR(\pi-it)]}{\cosh t+\cos(\theta-\theta_0)}\,dt$$

$$+\frac{\sqrt{2}}{4\pi}\cos((\theta-\theta_0)/4)\,(1-2\cos((\theta-\theta_0)/2))\int_0^\infty \frac{\cosh(t/4)H_0^{(2)}[kR(\pi-it)]}{\cosh t+\cos(\theta-\theta_0)}\,dt\,. \quad (40)$$

In a similar manner it is not difficult to show that

$$G_{4\pi}(r,\theta,r_0,-\theta_0;k) = H[\pi-|\theta+\theta_0|\,]H_0^{(2)}\,[kR(\theta+\theta_0)]$$

$$-\frac{1}{2\pi}\cos((\theta+\theta_0)/2)\int_0^\infty \frac{\cosh(t/2)H_0^{(2)}[kR(\pi-it)]}{\cosh t+\cos(\theta+\theta_0)}\,dt$$

$$-\frac{\sqrt{2}}{4\pi}\cos((\theta+\theta_0)/4)\int_0^\infty \frac{\cosh(3t/4)H_0^{(2)}[kR(\pi-it)]}{\cosh t+\cos(\theta+\theta_0)}\,dt$$

$$+\frac{\sqrt{2}}{4\pi}\cos((\theta+\theta_0)/4)\,(1-2\cos((\theta+\theta_0)/2))\int_0^\infty \frac{\cosh(t/4)H_0^{(2)}[kR(\pi-it)]}{\cosh t+\cos(\theta+\theta_0)}\,dt\,. \quad (41)$$

$$G_{4\pi}(r,\theta,4\pi-\theta_0;k) = H[\pi-|\theta+\theta_0-4\pi|\,]H_0^{(2)}\,[kR(\theta+\theta_0)]$$

$$-\frac{1}{2\pi}\cos((\theta+\theta_0)/2)\int_0^\infty \frac{\cosh(t/2)H_0^{(2)}[kR(\pi-it)]}{\cosh t+\cos(\theta+\theta_0)}\,dt$$

$$+\frac{\sqrt{2}}{4\pi}\cos((\theta+\theta_0)/4)\int_0^\infty \frac{\cosh(3t/4)H_0^{(2)}[kR(\pi-it)]}{\cosh t+\cos(\theta+\theta_0)}\,dt$$

$$-\frac{\sqrt{2}}{4\pi}\cos((\theta+\theta_0)/4)\,(1-2\cos((\theta+\theta_0)/2))\int_0^\infty \frac{\cosh(t/4)H_0^{(2)}[kR(\pi-it)]}{\cosh t+\cos(\theta+\theta_0)}\,dt\,. \tag{42}$$

$$G_{4\pi}(r,\theta,r_0,-4\pi+\theta_0;k) = -\frac{1}{2\pi}\cos((\theta-\theta_0)/2)\int_0^\infty \frac{\cosh(t/2)H_0^{(2)}[kR(\pi-it)]}{\cosh t+\cos(\theta-\theta_0)}\,dt$$

$$+\frac{\sqrt{2}}{4\pi}\cos((\theta-\theta_0)/4)\int_0^\infty \frac{\cosh(3t/4)H_0^{(2)}[kR(\pi-it)]}{\cosh t+\cos(\theta-\theta_0)}\,dt$$

$$-\frac{\sqrt{2}}{4\pi}\cos((\theta-\theta_0)/4)\,(1-2\cos((\theta-\theta_0)/2))\int_0^\infty \frac{\cosh(t/4)H_0^{(2)}[kR(\pi-it)]}{\cosh t+\cos(\theta-\theta_0)}\,dt\,. \tag{43}$$

By substituting the expressions (40) to (43) into (37) gives the solution for diffraction by a hard/soft half plane as :

$$U_{h/s}(r,\theta,r_0,\theta_0) = H[\pi-|\theta-\theta_0|\,]H_0^{(2)}\,[kR(\theta-\theta_0)^{\cdot}] + H[\pi-|\theta+\theta_0|\,]H_0^{(2)}\,[kR(\theta+\theta_0)]$$

$$- H[\pi-|\theta+\theta_0-4\pi|\,]H_0^{(2)}\,[kR(\theta+\theta_0)]$$

$$-\frac{1}{\sqrt{2\pi}}\cos((\theta-\theta_0)/4)\int_0^\infty \frac{\cosh(3t/4)H_0^{(2)}[kR(\pi-it)]}{\cosh t+\cos(\theta-\theta_0)}\,dt$$

$$-\frac{1}{\sqrt{2\pi}}\cos((\theta+\theta_0)/4)\int_0^\infty \frac{\cosh(3t/4)H_0^{(2)}[kR(\pi-it)]}{\cosh t+\cos(\theta+\theta_0)}\,dt$$

$$+\frac{1}{\sqrt{2\pi}}\cos((\theta-\theta_0)/4)\,(1-2\cos((\theta-\theta_0)/2))\int_0^\infty \frac{\cosh(t/4)H_0^{(2)}[kR(\pi-it)]}{\cosh t+\cos(\theta-\theta_0)}\,dt$$

$$+\frac{1}{\sqrt{2\pi}}\cos((\theta+\theta_0)/4)\,(1-2\cos((\theta+\theta_0)/2))\int_0^\infty \frac{\cosh(t/4)H_0^{(2)}[kR(\pi-it)]}{\cosh t+\cos(\theta+\theta_0)}\,dt\,. \tag{44}$$

Appendix A.

Here we derive a contour integral representation for $H_\nu^{(2)}(z)$ for $\nu > -1,\ -\pi/2 \le \arg z \le \pi/2$, namely

$$H_\nu^{(2)}(z) = z^\nu e^{\frac{i\nu\pi}{2}} \frac{1}{\pi i} \int_{\infty+ic}^{oe^{i(\arg z - \pi/2)}} e^{-i(t+z^2/4t)} \frac{dt}{t^{\nu+1}} \tag{A.1}$$

From Watson (1944) p 179 we have the integral representation

$$H_\nu^{(2)}(z) = -\frac{1}{\pi i} \int_{0e^{i0}}^{\infty e^{-i\pi}} e^{\frac{z}{2}(u-u^{-1})} \frac{du}{u^{\nu+1}}\ , \quad -\pi/2 \le \arg z \le \pi/2\ , \tag{A.2}$$

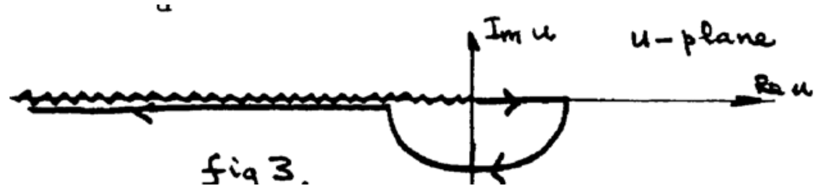

fig 3.

where the contour of integration is shown in fig 3. Let $zu = 2te^{-i\pi/2}$ then

$$H_\nu^{(2)}(z) = -\left(\frac{z}{2}\right)^\nu e^{\frac{i\nu\pi}{2}} \frac{1}{\pi i} \int_{0e^{i(\arg z + \pi/2)}}^{\infty e^{(\arg z - \pi/2)}} e^{-i(t+z^2/4t)} \frac{dt}{t^{\nu+1}} \tag{A.3}$$

Since $-\pi/2 < \arg z < \pi/2$, then $-\pi < \arg z < -\pi/2 < 0$ and $0 < \arg z + \pi/2 < \pi$, which means that the upper limit of integration lies in the lower half t - plane, and the lower limit of integration lies in the upper half t-plane.

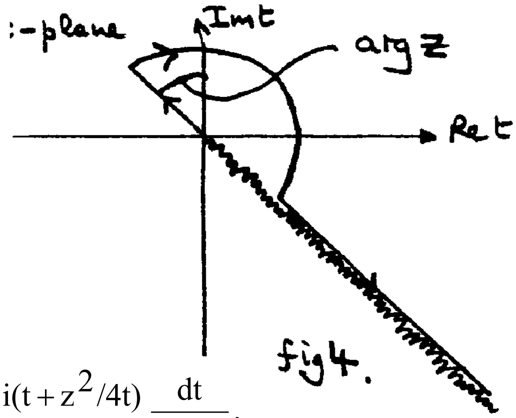

fig4.

Provided Re $\nu > -1$ we can apply Jordan's lemma to distort the path of integration to run along a path parallel to the real axis at a distance $c > 0$ as $t \to \infty$, see Jeffreys and Jeffreys (1956) p 392. Thus

$$H_\nu^{(2)}(z) = -\left(\frac{z}{2}\right)^\nu e^{\frac{i\nu\pi}{2}} \frac{1}{\pi i} \int_{\infty+ic}^{0e^{i(\arg z - \pi/2)}} e^{-i(t+z^2/4t)} \frac{dt}{t^{\nu+1}},$$

$$= z^{\nu} e^{\frac{i\nu\pi}{2}} \frac{1}{\pi i} \int_{\infty + ic}^{0e^{(\arg z \, + \, \pi/2)}} e^{-\frac{i}{2}(t + z^2/t)} \frac{dt}{t^{\nu+1}}, \qquad \text{(A.4)}$$

$-\pi/2 < \arg z < \pi/2,\ c > 0,\ \mathrm{Re}\,\nu > -1\,.$

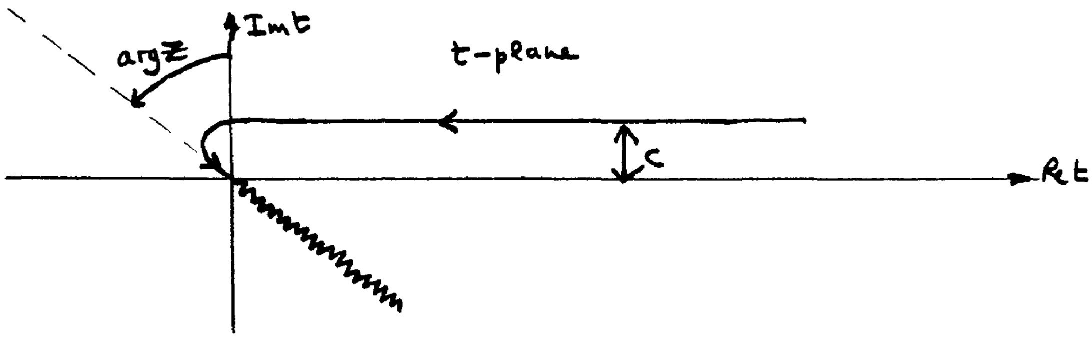

fig 5.

Appendix B

Here we derive an alternative representation for the double integral

$$I = \frac{1}{\pi i} \int_{\infty + ic}^{0} e^{-\frac{i}{2}(t + k^2 (r^2 + r_0^{\,2})/t)} \left\{ e^{\frac{i k^2 r r_0 \cos\psi}{t}} \int_{\infty}^{\frac{k^2 r r_0}{t}} e^{-ix\cos\psi} H_\nu^{(2)}(x) dx \right\} \frac{dt}{t}\,, \; 0 < \nu < 1\,. \qquad \text{(B.1)}$$

Let $I_1 = e^{\frac{k^2 r r_0 \cos\psi}{t}} \int_{\infty}^{\frac{k^2 r r_0}{t}} e^{-ix\cos\psi} H_\nu^{(2)}(x) dx\,;$ (B.2)

then by using the integral representation, Lebedev (1965) p117-118

$$H_\nu^{(2)}(z) = -2 \frac{e^{\frac{i\nu\pi}{2}}}{\pi i} \int_0^{\infty} e^{-ixcoshu} \cosh \nu u\, du,\ \mathrm{Im}\, z \le 0\,,\ |\mathrm{Re}\,\nu| < 1\,, \qquad \text{(B.3)}$$

the expression (B.2) can be written (since $\mathrm{Im}\left(\frac{k^2 rr_0}{t}\right) \le 0$)

as

$$I_1 = -\frac{2e^{\frac{i\nu\pi}{2}}}{\pi i} \int_0^{\infty} \left\{ e^{\frac{ik^2 rr_0}{t}} e^{-ix(\cos\psi + \cosh u)} dx \right\} \cosh \nu u du\,,$$

$$= -\frac{2}{\pi i} e^{\frac{\nu\pi i}{2}} \int_0^\infty e^{-\frac{\frac{ik^2 rr_0}{t} \cosh u}{(\cosh u + \cos\psi o}} \cosh \nu u \quad du \ . \qquad \text{(B.4)}$$

Substituting the last expression (B.4) into (B.1) and interchanging the order of integration gives

$$I = \frac{-2e^{\frac{\nu\pi i}{2}}}{\pi} \int_0^\infty \frac{\cosh \nu u}{\cosh u + \cos\psi} \left\{ \frac{I}{\pi i} \int_{\infty+ic}^0 e^{\frac{i}{2}\left(\frac{t+k^2(r^2+r_0^2+2rr_0\cosh u)}{t}\right)} \cdot \frac{dt}{t} \right\} du,$$

$$= \frac{-2e^{\frac{\nu\pi i}{2}}}{\pi} \int_0^\infty \frac{\cosh \nu u}{\cosh u + \cos\psi} H_0^{(2)} \, [k\sqrt{r^2 + r_0^2 + 2rr_0\cosh u}] \ du \ ,$$

$$= -\frac{2e^{\frac{\nu\pi i}{2}}}{\pi} \int_0^\infty \frac{\cosh \nu u}{\cosh u + \cos\psi} H_0^{(2)} \, [kR(\pi - iu)] \ du. \qquad \text{(B.5)}$$

Appendix C

Here we evaluate the integral

$$I = \frac{1}{\pi} \int_{\infty+ic}^0 e^{-\frac{i}{2}(t + k^2(r^2 + r_0^2)/t)} e^{\frac{ik^2 rr_0 \cos\psi}{t}} \int_\infty^{k\sqrt{2rr_0/t}|\text{coz}\psi/2|} e^{-iv^2} dv \ \frac{dt}{t} \qquad \text{(1.C)}$$

We can rewrite this as ($v = k\sqrt{2rr_0/t}\,|\cos\psi 2|\, u$)

$$I = \frac{1}{\pi i} \int_{\infty+ic}^0 e^{-\frac{1}{2}\ (t + k^2(r^2 + r_0^2 - 2rr_0\cos\psi o/\ t)} |\cos\frac{\psi}{2}| k\sqrt{2rr_0} \int_\infty^1 e^{\frac{-i2k^2 rr_0 \text{cod}^2}{t}(\psi/2)u^2} \cdot du \frac{dt}{t^{3/2}} \ ,$$

$$= k\sqrt{2rr_0}\ |\cos\frac{\psi}{2}|\ \int_\infty^1 \frac{1}{\pi i} \int_{\infty+ic}^0 e^{-\frac{i}{2}\ (t + k^2(R^2(\psi) + 4rr_0\cos^2(\psi/2)u^2)/t)} \frac{dt}{t^{3/2}} \quad du \ ,$$

$$= k\sqrt{2rr_0}\ |\cos\frac{\psi}{2}|\ \int_\infty^1 e^{-i\pi/4} \frac{H_{\frac{1}{2}}^{(2)}[k(R^2(\psi) \ + \ 4rr_0\cos^2(\psi/2)u^2)^{\frac{1}{2}}]}{k^{\frac{1}{2}}(R^2(\psi) \ + \ 4rr_0\cos^2(\psi/2)u^2)^{\frac{1}{4}}]} du \ .$$

But since $H^{(2)}_{\frac{1}{2}}(z) = i\left(\frac{2}{\pi z}\right)^{\frac{1}{2}} e^{-iz}$ then we get

$$I = e^{i\pi/4} 2\sqrt{\frac{rr_0}{\pi}} \,|\cos\frac{\psi}{2}|\, \int_{\infty}^{1} \frac{e^{-ik(r^2(\psi)+4rr_0\cos^2(\psi/2)u^2)^{\frac{1}{2}}}}{(r^2(\psi)+4rr_0\cos^2(\psi/2)u^2)^{\frac{1}{2}}}$$

Now let $2\sqrt{rr_0}\,|\cos\psi/2|\,u = R(\psi)\sinh\xi$, then

$$I = \frac{e^{i\pi/4}}{\sqrt{\pi}} \int_{\infty}^{\xi_0} e^{-ikR(\psi)\cosh\xi} d\xi \quad , \tag{2.C}$$

$$\text{where } \xi_0 = \sinh^{-1}\left\{\frac{2\sqrt{rr_0}\,|\cos\psi/2|}{R(\psi)}\right\} . \tag{3.C}$$

<u>Appendix D.</u>

We shall here give an alternative representation for the integral

$$I = \int_0^\infty \frac{\cosh t/2}{\cosh t + \cos\psi}\, H_0^{(2)}[k(r^2 + r_0^2 + 2rr_0\cosh t)]\, dt \ ,$$

$$= 2\int_0^\infty \frac{H_0^{(2)}[k(r^2 + r_0^2 + 2rr_0\cosh t)]}{\cosh t + \cos\psi} \frac{d}{dt}(\sinh t/2)\, dt$$

Let $v = 2\sqrt{rr_0}\sinh(t/2)$ then since $\cosh t = 1 + 2\sinh^2 t/2$ we get

$$I = 2\sqrt{rr_0} \int_0^\infty \frac{H_0^{(2)}[k\sqrt{(r+r_0)^2 + v^2}]}{v^2 + 4rr_0\cos^2(\psi/2)}\, dv \tag{D.1}$$

We now use the representation, see appendix A,

$$H_0^{(2)}(x) = \frac{1}{\pi i}\int_{c-i\infty}^{0} e^{\frac{1}{2}(t - x^2/t)} \frac{dt}{t} \ ,$$

in the expression (D.l) giving

$$I = \frac{1}{\pi i} \int_{c-i\infty}^{0} e^{\frac{1}{2}t - k^2(r+r_0)^2/2t} \, 2\sqrt{rr_0} \int_0^\infty \frac{e^{-k^2v^2/2t}dt}{v^2 + 4rr_0\cos^2(\psi/2)} \frac{dt}{t} \, .$$

Now $\int_0^\infty \frac{e^{-\alpha u^2}du}{u^2+A^2} = \int_0^\infty e^{-\alpha u^2}du \int_0^\infty e^{-(u^2+A^2)t}dt = \int_0^\infty e^{-A^2t}dt \int_0^\infty e^{-(t+\alpha)u^2}du$

$$= \frac{\sqrt{\pi}}{2} \int_0^\infty \frac{e^{-A^2t}}{\sqrt{t+\alpha}} dt - \sqrt{\frac{\pi}{2}} \, e^{A^2\alpha} \int_\alpha^\infty e^{-A^2X} \frac{dx}{\sqrt{x}} = \frac{\sqrt{\pi\alpha}}{|A|} e^{A^2\alpha} \int_{|A|}^\infty e^{-\alpha w^2} dw \, .$$

Hence

$$2\sqrt{rr_0}\int_0^\infty \frac{e^{-k^2v^2/2t}dv}{v^2+4rr_0\cos^2(\psi/2)} = \frac{\sqrt{\pi}}{2} \frac{ke^{4k^2rr_0\cos^2(\psi/2)/2}}{t^{\frac{1}{2}}|\cos\psi/2|} \int_{2\sqrt{rr_0}|\cos\psi/2|}^\infty e^{-k^2w^2/2t} dw \, ,$$

so that

$$I\sqrt{\frac{\pi}{2}} \frac{k}{|\cos\psi/2|} \int_{2\sqrt{rr_0}|\cos\psi/2|}^\infty \frac{1}{\pi i} \int_{c-i\infty}^0 e^{\frac{1}{2}(t-k^2\{r^2+r_0^2-2rr_0\cos\psi+w^2\}/t)} \frac{dt}{t^{3/2}} \, ,$$

$$I\sqrt{\frac{\pi}{2}} \frac{k}{|\cos\psi/2|} \int_{2\sqrt{r_0}|\cos\psi/2|}^\infty \frac{H_{\frac{1}{2}}^{(2)}\{k(r^2+r_0{}^2-2rr_0\cos\psi+w^2)\}}{k^{\frac{1}{2}}(r^2+r_0^2-2rr_0\cos\psi+w^2)^{\frac{1}{4}}} dw,$$

(See appendix A). Thus

$$I = \frac{i}{|\cos\psi/2} \int_{2\sqrt{rr_0}|\cos\psi/2}^\infty \frac{e^{-ik(r^2+r_0^2-2rr_0\cos\psi+w^2)}}{(r^2+r_0^2-2rr_0\cos\psi+w^2)} dw$$

In the last integral we make the change of variable $w = R(\psi) \sinh\xi$ so that

$$I = \frac{i}{|\cos\psi/2|} \int_{\xi_0}^\infty e^{-ikR(\psi)\cosh\xi}d\xi \, ,$$

Where $\xi_0 = \sinh^{-1}\left[\frac{2\sqrt{rr_0}\,|\cos\psi/2|}{R(\psi)}\right]$

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