TR/08/90 July 1990

An Alternative Development of Basic Functions

by

D. M. DREW

Department of Mathematics and Statistics
Brunel University
Uxbridge UB8 3PH

An Alternative Development of Basic Functions

The concept of a limit presents considerable problems to many students, yet often the derivative is defined and limits taken with little thought given to the consequences by the instructor, let alone by the student. In this paper we investigate, in a simple problem, the consequence of not proceeding to the limit. Our example indicates how Calculus reduces the range of expressions that occur, yet masks important processes that are of interest and should be returned to later. It reveals a little of the rich area of Mathematics, known as 'basic' functions, a topic which is so frequently dismissed in a few mutterings about taking the limit. The price that has to be paid for taking the limit occurs when reversing the limit process, then difficulties can arise.

'Basic' function arise when solving equations involving q-differences, the q-difference of the function f(x) being defined historically as

$$\Delta f(x) \equiv \frac{f(qx) - f(x)}{(q-1)x}.$$

So for example $\Delta x^n = [n]x^{n-1}$ where the basic number $[n] = \dfrac{q^n - 1}{q - 1}$.

Setting qx = x+h and letting $q \to 1$ $\quad \Delta f(x) \to \dfrac{df}{dx}$.

However, in studying the simple q-difference equation $\Delta y(x) = \dfrac{1}{1+x}$

it became apparent that the results could be obtained more easily if a fundamental change was made. Considerable symmetry and simplicity is achieved if we define

$$\Delta f(x) \equiv \frac{f(Qx) - f\left(\dfrac{x}{Q}\right)}{\left(Q - \dfrac{1}{Q}\right)x}$$

To avoid confusion we will call this the Q-difference, it is closely associated with the q-difference, and it has similer properties, in

particular as $Q \to 1$ $\Delta f(x) \to \frac{df}{dx}$ However, a number of the changes are significant. As before $\Delta x^n - [n]x^{n-1}$ but now the basic number

$$[n] = \frac{Q^n - \frac{1}{Q^n}}{Q - \frac{1}{Q}} = Q^{n-1} + Q^{n-3} + \ldots + \frac{1}{Q^{n-3}} + \frac{1}{Q^{n-1}}.$$

We develop the 'basic' functions that arise in solving the simple Q-difference equation $\Delta y(x) = \frac{1}{1+x}$; in the limit as $Q \to 1$ these functions become $\ell n(1+x)$. We will obtain an explicit expression for the periodic function that relates the two forms of the solution. The resulting relation is both interesting and simple. Moreover the relation could be used to introduce a more advanced topic, elliptic functions.

The solutions of basic equations corresponding to many important differential equations have been investigated. A recent paper by H. Exton [3] examines a basic analogue of Hermite's equation, q-difference equations have attracted the attention of many mathematicians and in particular the Rev. Frank H. Jackson [4]. A useful review paper is that by C. R. Adams [1] which in turn refers to N. E. Norlund's bibliography [5].

The Q-difference Operation.

An interesting extension of many of the fundamental functions that arise in Calculus called 'basic' functions was studied at the turn of the century, notably by Frank H. Jackson. Jackson's papers are listed in his obituary by T. W. Chaundy [2], 'Basic' functions strictly arise when solving equations involving the q-difference operator.

$$\Delta f(x) = \frac{f(qx) - f(x)}{(q-1)x}.$$

However, we propose to extend the term basic function to include the closely related solutions of equations involving our Q-difference operator

$$\Delta f(x)=\frac{f(Qx)-f\left[\frac{x}{Q}\right]}{\left[Q-\frac{1}{Q}\right]x}. \tag{1}$$

On writing $[n] = \frac{Q^n-\frac{1}{Q^n}}{Q-\frac{1}{Q}}$, which equals $Q^{n-1}+Q^{n-3}+-+\frac{1}{Q^{n-3}}+\frac{1}{Q^{n-1}}$

for integer n and tends to n as Q → 1, we readily verify that

$$\Delta x^n=[n]x^{n-1}$$

$$\Delta\left\{\frac{1}{2}\left[Q-\frac{1}{Q}\right]\log Q^X\right\}=\frac{1}{x} \tag{2}$$

$$\Delta\frac{1}{x^n}=-\frac{[n]}{x^{n+1}}$$

Now the differential equation, $\frac{dy}{dx}=\frac{1}{1+x}, y(0)=0$ has solution

$$y=\ell n(1+x)= x-\frac{x^2}{2}+\frac{x^3}{3}- \quad --- \quad \text{for } |x|<1 \tag{3}$$

$$= \ell nx+\frac{1}{x}-\frac{1}{2x^2}+ \quad --- \quad \text{for } |x|>1.$$

The corresponding Q-difference equation is

$$\Delta y(x)=\frac{1}{1+x}=1-x+x^2-x^3+ \quad --- \quad \text{for } |x|<1 \tag{4.1}$$

$$=\frac{1}{x}-\frac{1}{x^2}+\frac{1}{x^3}-\frac{1}{x^4}+ \quad --- \quad \text{for } |x|>1. \tag{4.2}$$

The solution suggested by the expansion (4.1) is

$$f(x)\equiv x-\frac{x^2}{[2]}+\frac{x^3}{[3]}-\frac{x^4}{[4]}+ \ --- \tag{5}$$

which is convergent for $|x| < \frac{1}{Q}$ when $0 < Q < 1$ or $|x| < Q$ when $Q > 1$.

While the solution suggested by the expansion (4.2) is

$$\frac{1}{2}\left(Q-\frac{1}{Q}\right)\log_Q x+\frac{1}{x}-\frac{1}{[2]x^2}+\frac{1}{[3]x^3}-\frac{1}{[4]x^4}+ \ ---$$

$$=\frac{1}{2}\left\{Q-\frac{1}{Q}\right\}\log_Q x+f\left[\frac{1}{x}\right]$$

which is convergent for $|x| > Q$ when $Q < 1$ or $|x| > \frac{1}{Q}$ when $Q > 1$.

Thus the regions in which these two expansions converge overlap. It is the interrelation between them that we will investigate in this paper.

Solutions of $\Delta y(x) = \frac{1}{1+x}$ can of course differ by more than a constant.

Because $\Delta y(x) = 0$ when

$$y(Qx) = y\left[\frac{x}{Q}\right]$$

Setting $x = Q^{2u}$

$$y\left[Q^{2u+1}\right] = y\left[Q^{2u-1}\right]$$

we see that the solutions may differ by a function periodic in u and with period 1.

Consideration of the value x = 1 leads us to write

$$f(x) = \frac{1}{2}\left[Q - \frac{1}{Q}\right]\log Q^{X} + f\,\frac{1}{x} + D(x,Q). \qquad (7)$$

This function D(x,Q) will turn out to be periodic and we will deduce an interesting consequence of this relation.

Iterative Solutions

The Q-difference equation $\Delta f(x) = \frac{1}{1+x}$, $f(0) = 0$ (8)

$$\text{i.e.}\quad f(Qx) - f\left[\frac{x}{Q}\right] = \left[Q - \frac{1}{Q}\right]\frac{x}{1+x}$$

can be solved iteratively for both $Q<1$ and $Q>1$. We will deal first with the case <u>Q < 1.</u>

$$f(x) = -\left(Q - \frac{1}{Q}\right)\frac{xQ}{1+xQ} + f\left(Q^2x\right)$$

$$= -\left[Q - \frac{1}{Q}\right]\sum_{n=0}^{\infty}\frac{xQ^{2n+1}}{1+xQ^{2n+1}} \qquad \textbf{(9)}$$

which on expansion

$$= x - \frac{x^2}{[2]} + \frac{x^3}{[3]} - \frac{x^4}{[4]} + - - - . \qquad \textbf{(10)}$$

The radius of convergence $R = Q^{-1}$ results from the pole at $X = -\frac{1}{Q}$ in (9).

The relation (7) becomes

$$-\left\{Q - \frac{1}{Q}\right\}\sum_{n=0}^{\infty}\frac{xQ^{2n+1}}{1+xQ^{2n+1}} = \frac{1}{2}\left\{Q - \frac{1}{Q}\right\}\log_Q x - \left\{Q - \frac{1}{Q}\right\}\sum_{n=0}^{\infty}\frac{Q^{2n+1}}{x+Q^{2n+1}} + D(x,Q)$$

Giving

$$\sum_{n=0}^{\infty}\frac{Q^{2n+1}}{x+Q^{2n+1}}-\sum_{n=0}^{\infty}\frac{xQ^{2n+1}}{1+xQ^{2n+1}}=\frac{1}{2}\log_Q x+D_Q(x) \qquad (11)$$

where $D_Q(x)=\dfrac{D(x,Q)}{Q-\dfrac{1}{Q}}$.

This function $D_Q(x)$ is remarkably small for Q in the interval $\frac{1}{2}<Q<1$; some computed values of its maximum value for real x are given in the table, these behave as $\left|2\alpha\alpha^{\alpha\pi}\right|$ where $\alpha = \dfrac{\pi}{\ell nQ}$.

Maximum Value of $D_Q(x)$

$u = .75$, $x = Q^{1.5}$

Q^2	$D_Q(x)$
.9	5.15 E-80
.8	2.15 E-37
.7	3.25 E-23
.6	4.06 E-16
.5	7.77 E-12
.4	6.04 E-09
.3	7.91 E-07
.2	3.68 E-05
.1	1.03 E-03
.05	5.77 E-03
.01	3.75 E-02

The error involved in omitting from (11) $D_Q(X)$ is thus quite small unless $Q<.5$.

Interesting approximations to logQx are obtained by taking a finite number of terms from the left hand side of (11).
Further it is readily shown that

$$D_Q\left\{Q^{2(u+1)}\right\}=D_Q\left\{Q^{2u}\right\}$$

So that when treated as a function of u, $D_Q(x)$ is a function periodic in u with period 1. It can be expressed as a Fourier sine series

$$D_Q\left(Q^{2u}\right)=\frac{1}{\ell nQ}\sum_{n=1}^{\infty}(-1)^n\frac{\sin 2n\pi n}{\sinh n\beta}. \qquad (12)$$

where $\beta=\frac{\pi^2}{\ell nQ}$. Noting that β is negative for Q < 1, this sum can be rearranged to give

$$D_Q\left[Q^{2u}\right]=\frac{\pi}{\ell nQ}\sum_{r=0}^{\infty}\frac{\sin 2\pi i}{\cosh(2r+1)\beta+\cos 2\pi o}.$$

$$=\frac{\pi}{\ell nQ}\sum_{r=0}^{\infty}\frac{2e^{(2r+1)\beta}\sin 2\pi u}{1+2e^{(2r+1)\beta}\cos 2\pi u+e^{2(2r+1)\beta}}. \qquad (13)$$

In the case <u>Q > 1</u>, equation (8) can be solved iteratively by writing

$$f(x)=\left(Q-\frac{1}{Q}\right)\frac{\frac{x}{Q}}{1+\frac{x}{Q}}+f\left(\frac{x}{Q^2}\right).$$

$$=\left(Q-\frac{1}{Q}\right)\sum_{n=0}^{\infty}\frac{\frac{X}{Q^{2n+1}}}{1+\frac{X}{Q^{2n+1}}} \qquad (14)$$

which on expansion

$$=x-\frac{x^2}{[2]}+\frac{x^3}{[3]}-\frac{x^4}{[4]}+\;---\;.$$

The relation corresponding to (11) is

$$\sum_{n=0}^{\infty}\frac{x}{Q^{2n+1}+x}-\sum_{n=0}^{\infty}\frac{1}{xQ^{2n+1}+1}=\frac{1}{2}\log_Q x+D_Q(x) \qquad (15)$$

and the effect is simply to replace Q by $\frac{1}{Q}$, since $D_{\frac{1}{Q}}(x) = -D_Q(x)$.

In fact formula (12) for $D_Q[Q^{2u}]$ still holds.

Integral Result

With $x = Q^{2u}$ the relation (11) can clearly be integrated.
Consider the function φ

$$\varphi=\underset{n=0}{\overset{\infty}{\pi}}\left(1+Q^{2n+1}X\right)\left(1+Q^{2n+1}\frac{1}{x}\right)$$

$$=\underset{n=0}{\overset{\infty}{\pi}}\left(1+Q^{2n+1}Q^{2u}\right)\left(1+Q^{2n+1}Q^{-2u}\right) \qquad (16)$$

$$\frac{d}{du}\left[\log_Q \varphi\right] = 2\sum_{n=0}^{\infty} \frac{Q^{2n+1}Q^{2u}}{1+Q^{2n+1}Q^{2u}} - 2\sum_{n=0}^{\infty}\frac{Q^{2n+1}Q^{-2u}}{1+Q^{2n+1}Q^{2u}}$$

Hence, with $x = Q^{2u}$, the relation (11) becomes

$$-\frac{d}{du}\left[\log_Q \varphi\right] = \log_Q Q^{2u} + 2D_Q\left(Q^{2u}\right)$$

$$= 2u + 2D_Q(Q^{2u})$$

and on integrating using the form (13) for $D_Q\,(Q^{2u})$

$$-\log_Q \varphi = u^2 - \log_Q \theta + \text{consant} \qquad (17)$$

where the function $\theta = \pi_{r=0}^{\infty} \left[1+2e^{(2r+1)\beta}\cos 2\pi o + e^{2(2r+1)\beta}\right]$

and $\beta = \dfrac{\pi^2}{\ell nQ}$.

The significance of this result is clarified if we define $q = e^{\beta}$,

$$\theta = \pi_{r=0}^{\infty} \left[1+2q^{2r+1}\cos 2\pi u + q^{4r+2}\right] \qquad (18)$$

a Jacobi theta function . Then from (17), with $\ell nq.\ell nQ = \pi^2$

$$\theta = AQ^{U^2}\varphi \cdot \qquad (19)$$

An examination of the zeros of θ and of φ will disclose the nature of the relation between the two sides of this equation.

On putting u = 0, the constant A is given by

$$A = \frac{\pi_{r=0}^{\infty}\left[1+q^{2r+1}\right]^2}{\pi_{n=0}^{\infty}\left[1+Q^{2n+1}\right]^2} \qquad (20)$$

when $q = Q = e^{-\pi}$, A = 1.

Infinite Product form of Q^{u^2} .

When $Q < 1$ the infinite products in (16) for φ are absolutely convergent and because $\ell nq.\ell nQ = \pi^2$ it follows $q < 1$.

$$\theta = \pi_{r=0}^{\infty}\left(1+q^{2r+1}e^{i2\pi u}\right)\left(1+q^{2r+1}e^{-i2\pi u}\right) \quad ,$$

and again both infinite products in θ are absolutely convergent.

Q^{u^2} can thus be expressed as the ratio of these infinite products for θ and φ. By suitably changing the variable u, formula (19) provides a useful method of calculating all four of Jacobi's theta functions especially for q near 1.

For Q > 1 we simply replace Q by $\frac{1}{Q}$ and q by $\frac{1}{q}$ and (19) becomes

$$\theta = AQ^{-u^2}\varphi . \tag{21}$$

In particular when Q = e and $q = e^{\pi^2}$

$$e^{u^2} = A\frac{\pi_{n=0}^{\infty}\left(1+e^{-(2n+1)}e^{-2u}\right)\left(1+e^{-(2n+1)}e^{2u}\right)}{\pi_{r=0}^{\infty}\left(1+e^{-(2r+1)\pi^2}e^{i2\pi 2}\right)\left(1+e^{-(2r+1)\pi^2}e^{-i2\pi 2}\right)} \tag{22}$$

with

$$A = \frac{\pi_{r=0}^{\infty}\left(1+e^{-(2r+1)\pi^2}\right)^2}{\pi_{n=0}^{\infty}\left(1+e^{-(2r+1)}\right)^2} .$$

In this expression for e^{u^2} there appear to be finite poles and zeros, However, they all cancel and there are in fact none.

The solution of q- difference equations or Q- difference equations such as (8) seem to be used surprisingly little in Mathematics. [Although elliptic functions and partition function theory certainly are used]. The reason could be due to a lack of awareness of ‘basic’ functions by mathematicians, especially Numerical Analysts and Engineers; it could be the development of the theory was stunted by the lack of computing facilities. Certainly there are useful basic functions that tend to each of the elementary functions of Calculus.

References

[1] Adams C.R. Linear q-difference equations
Bull. Am. Maths. Soc. 1931 37, 361-400

[2] Chaundy T.W. Frank Hilton Jackson
J. London Math. Soc. 1962 37, 126-8

[3] Exton H. A basic analogue of Hermite's equation
J. Inst. Maths. Applics. 1980 26, 315-320

[4] Jackson F.H. q-difference equation Amer. J. of
Maths. 1910 32, 305-314

[5] Nörlund N.E. Vorlesungen über differenzenrechnung, J. Springer, Berlin
1924, 9 + 551 pp.