# Recursive Filtering for Stochastic Parameter Systems with Measurement Quantizations and Packet Disorders

Dan Liu, Zidong Wang, Yurong Liu and Fuad E. Alsaadi

*Abstract*—**In this paper, the recursive filtering problem is put forward for stochastic parameter systems subject to quantization effects and packet disorders. Before entering communication networks, measurement outputs are quantized by logarithmic quantizers. The packet disorders result from transmission delays which are provoked by communication constraints and occur randomly in the sensor-to-filter channel. In case of measurement quantizations and packet disorders, the objective of this paper is to devise a novel recursive filter approach that is capable of 1) guaranteeing desired upper bounds on the resultant filtering error covariances; and 2) minimizing such upper bounds by acquiring appropriate filter gains. Furthermore, sufficient conditions are established to ensure the mean-square boundedness of filtering errors by means of stochastic analysis techniques. At last, simulations are given to validate the applicability of our designed approach.**

*Index Terms*—**Recursive filtering, stochastic parameter systems, measurement quantizations, packet disorders.**

## I. Introduction

For decades, the filtering issue has been well recognized as a fundamental yet attractive research topic in various practical realms such as control engineering, target tracking and signal processing. The classical Kalman filtering (KF) algorithm, which was proposed in 1960s, has been deemed to be one of the most powerful tools for addressing state estimation problems for linear systems with exactly known parameters. Unfortunately, in almost all practical applications, the systems are nonlinear by nature and the parameters might be uncertain, and this hinders the broad application of the KF. Therefore, a recurring research interest has been aroused towards the development of alternative estimation schemes (catering for nonlinearities and/or uncertainties) with examples including the extended Kalman filtering [8], [38], $H_\infty$ filtering [40], [42], set-membership filtering [23], [36] and unscented Kalman filtering algorithms [15], [16].

This work was supported by National Natural Science Foundation of China under Grants 61773017, 61873148, 61873230 and 61933007, the Royal Society of the UK, and the Alexander von Humboldt Foundation of Germany.

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In recent years, the stochastic parameter system has shown its wide existence in a variety of application fields such as radar control, mobile robot localization, economic systems and navigation systems [1], [4], [5], [28], [39], and a great many results have been available in the literature. For example, in [12], a nonlinear filtering task has been accomplished in the presence of correlated noises, fading measurements and random parameter matrices. In [33], the distributed $H_\infty$ filtering problem has been considered for stochastic parameter systems with successive packet dropouts.

Within a networked environment, output signals are normally quantized before transmitting through communication networks owing mainly to the limited data length and communication capacity. From a mathematical point of view, quantization can be viewed as a mapping that maps the signals with continuous amplitudes to those with a finite number of discrete amplitudes. Signal quantization would inevitably generate additional errors that are very likely to bring about performance deteriorations, and thus it is of crucial importance to incorporate the side-effect of the quantization procedure in system analysis/synthesis. As such, great research enthusiasm has been consistently generated towards this topic and numerous results have been reported on addressing the quantization problems [11], [21], [22], [24]–[27]. In particular, a popular yet effective way has been proposed in [10] to resolve quantization issues by converting quantization errors into sector-bounded uncertainties, and this approach has since gained a substantial deal of research interest with potential applications in many practical situations [6], [9], [13], [37].

In addition to the signal quantizations, another challenging issue that is unavoidably confronted within the network environment is the packet disorder. The so-called packet disorders indicate that measurements sent earlier (later) could reach their destination later (earlier) due to random transmission delays caused by limited network bandwidth. In other words, the "first sent first arrive" principle might not be followed when packets are going through the sensor-to-filter channel. In the presence of packet disorders, the *latest* arrival packet might not have the *newest* target information, and this gives rise to some new challenges in computation or even seriously worsen the filtering performance.

To date, the packet disorder problem has stirred a rapidly increasing research interest, and some initial yet inspiring results have been published in [14], [19], [20], [35], [41]. For example, the negative effects caused by the phenomenon of packet disorders have been analyzed in [41] for networked control systems and a compensation strategy has been presented to alleviate the negative effects. In [19], a novel mechanism has been put forward for networked filtering problems subject to both packet disorders and transmission delays where the newest signal has been selected as the actual input arriving at the receiver. In addition, for filtering problem of networked systems with measurement quantizations and packet disorders, [2], [3] have presented meaningful research results, in which multipath signal quantizations and data packet dropouts are considered for the first time, respectively.

The occurrence of transmission delays is ubiquitous in the data communication owing primarily to the finite network resource, and much work has been done [7], [17], [18]. According to the way they occur, transmission delays can be categorized as random and deterministic ones. Random transmission delay (RTD), which is the immediate cause for packet disorder, is generally assumed to be regulated by Markov chains of *a priori* known transition probability matrices (TPMs) [19], [35]. Different from such a Markov description, the RTDs considered in our study are characterized by random but bounded variables whose probability distributions are known via statistical tests. This gives rise to extra difficulties in both the mathematical derivation and the filter implementation with regard to the RTD-induced packet disorders. By now, the recursive KF problem with RTD-induced packet disorders has not been comprehensively investigated, let alone the setting where the stochastic parameter systems are involved.

In this paper, we aim at developing a recursive filtering approach for stochastic parameter systems with measurement quantizations and packet disorders. The addressed problem seems to be nontrivial due primarily to the following substantial difficulties: 1) how to characterize the random perturbations of system parameters in the recursive filtering problem; 2) how to describe the actual filter inputs in the presence of measurement quantizations and packet disorders; 3) how to design an appropriate recursive filter that alleviates the effects of packet disorders; and 4) how to ensure the boundedness of the filtering errors. To tackle the above difficulties, the primary contributions we are making are highlighted as follows: 1) a general stochastic parameter model is proposed to tackle the concurrence of packet disorders and measurement quantizations where bounded random variables are introduced to characterize RTDs with their probability distributions known *a priori*; 2) a reasonable mathematical description is presented to account for the measurement outputs that reflect the impacts of the quantization and RTDs; 3) a novel recursive filter, which is capable of reducing the adverse effects of packet disorders, is proposed to ensure certain locally minimized upper bounds on filtering error covariances; and 4) sufficient conditions are established to ensure the mean-square boundedness of filtering errors.

*Notation:* For a matrix $X$, $X^T$, $X^{-1}$ and $\text{tr}\{X\}$ denote, respectively, the transpose, inverse and trace of $X$. $\mathbb{E}\{\cdot\}$ denotes the expectation operator. The notation $\|\cdot\|$ stands for the Euclidean (spectral) norm of real vectors (matrices). $\lfloor r \rfloor$ is the biggest integer no bigger than real number $r$.

## II. Problem Formulation

Consider the following stochastic parameter model:

$$x_{k+1} = A_k x_k + B_k w_k, \quad (1)$$
$$\vec{y}_k = C_k x_k + D_k v_k, \quad (2)$$

where $x_k \in \mathbb{R}^m$ is the system state and $\vec{y}_k \in \mathbb{R}^n$ is the measurement output. $w_k \in \mathbb{R}^w$ and $v_k \in \mathbb{R}^v$ are zero-mean white noises distributed within bounded domains with covariances $Q_k > 0$ and $R_k > 0$, respectively. $A_k \in \mathbb{R}^{m\times m}$ and $B_k \in \mathbb{R}^{m\times w}$ are the random parameter matrices, $C_k \in \mathbb{R}^{n\times m}$ and $D_k \in \mathbb{R}^{n\times v}$ are the deterministic matrices. The initial value $x_0$ and matrices $A_k$ and $B_k$ have the following statistical properties:

$$\mathbb{E}\{x_0\} = \bar{x}_0, \qquad \mathbb{E}\{(x_0-\bar{x}_0)(x_0-\bar{x}_0)^T\} = P_0, \quad (3)$$
$$\mathbb{E}\{A_k\} = \bar{A}_k, \qquad \text{Cov}\{A^k_{ij}, A^k_{st}\} = T_{A^k_{ij}A^k_{st}}, \quad (4)$$
$$\mathbb{E}\{B_k\} = \bar{B}_k, \qquad \text{Cov}\{B^k_{ij}, B^k_{st}\} = T_{B^k_{ij}B^k_{st}}, \quad (5)$$

where $A^k_{ij}$ and $B^k_{ij}$ are the $(i,j)$-th entries of $A_k$ and $B_k$, respectively. $T_{A^k_{ij}A^k_{st}}$ and $T_{B^k_{ij}B^k_{st}}$ are known scalars, and $P_0$ is a known matrix. Denoting $\tilde{A}_k = A_k - \bar{A}_k$ and $\tilde{B}_k = B_k - \bar{B}_k$, we have $\mathbb{E}\{\tilde{A}_k\} = 0$ and $\mathbb{E}\{\tilde{B}_k\} = 0$. Throughout this paper, $x_0, w_k, v_k, A_k$ and $B_k$ are supposed to be mutually independent.

*Remark 1:* In a networked environment, system parameters are likely to suffer from random perturbations because of the sudden environment changes and abrupt fluctuations of network loads. The statistical properties of the random parameter perturbations can be determined via statistical experiments, and the underlying system is modeled as the form of (1)-(2) in this paper, where the existence of the stochastic parameters would induce extra difficulties in the subsequent mathematical derivation.

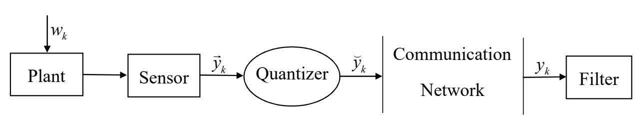


Fig. 1: Diagrammatic sketch of filtering problem with quantization effects.

It can be seen from Fig. 1 that the measurement output $\vec{y}_k$ is quantized before entering the filter through a communication network and the signal after quantization is $\breve{y}_k$. The well-known logarithmic quantizer is used and the mapping of the involved quantization process is

$$q(\vec{y}_k) = \begin{bmatrix} q_1(\vec{y}^{(1)}_k) & q_2(\vec{y}^{(2)}_k) & \cdots & q_n(\vec{y}^{(n)}_k) \end{bmatrix}^T.$$

For each $q_j(\cdot)$, the set of quantization levels is

$$\Upsilon_j = \left\{ \pm\upsilon^{(j)}_l, \upsilon^{(j)}_l = (\rho^{(j)})^l \upsilon^{(j)}_0, l = 0, \pm1, \pm2, \dots \right\} \cup \{0\},$$
$$0 < \rho^{(j)} < 1, \qquad \upsilon^{(j)}_0 > 0,$$

where $\rho^{(j)}$ $(j = 1, 2, \dots, n)$ is the quantization density and $\upsilon^{(j)}_0$ is a scaling constant.

The mapping from the entire segment to different quantization levels is accomplished through

$$q_j(\vec{y}_k^{(j)}) = \begin{cases} \upsilon_l^{(j)}, & \frac{1}{1+\theta_j}\upsilon_l^{(j)} < \vec{y}_k^{(j)} \le \frac{1}{1-\theta_j}\upsilon_l^{(j)}, \\ 0, & \vec{y}_k^{(j)} = 0, \\ -q_j(-\vec{y}_k^{(j)}), & \vec{y}_k^{(j)} < 0, \end{cases}$$

where $\theta_j = (1-\rho^{(j)})/(1+\rho^{(j)})$. Hence,

$$q_j(\vec{y}_k^{(j)}) = (1+\Delta_k^{(j)})\vec{y}_k^{(j)}$$

with $|\Delta_k^{(j)}| \le \theta_j$.

Denoting $\Delta_k \triangleq \text{diag}\{\Delta_k^{(1)}, \Delta_k^{(2)}, \ldots, \Delta_k^{(n)}\}$, the quantized measurement signal is

$$\check{y}_k = (I+\Delta_k)C_k x_k + (I+\Delta_k)D_k v_k. \tag{6}$$

By letting $\Theta \triangleq \text{diag}\{\theta_1, \theta_2, \ldots, \theta_n\}$ and $U_k = \Delta_k \Theta^{-1}$, it is obvious that the unknown real-valued time-varying matrix $U_k$ satisfies $U_k U_k^T = U_k^T U_k \le I$.

It is clear from Fig. 1 that, at time $k$, the measurement before and after transmission through the communication network are $\check{y}_k$ and $y_k$, respectively. Such a phenomenon is due mainly to the occurrence of RTDs in the transmission process. In this paper, the transmission delay is denoted by a random variable $\tau_k$ which satisfies

$$\text{Prob}\{\tau_k = i\} = p_i, \quad i = 0, 1, \cdots, q,$$

where $q$ and $p_i$ are known positive integers satisfying $0 \le p_i \le 1$ and $\sum_{i=0}^{q} p_i = 1$. Hence, the actual measurement arriving at the filter is

$$\begin{cases} y_k = \check{y}_{k-\tau_k}, \\ y_s = \phi_s, \quad s = -q, -q+1, \cdots, 0. \end{cases} \tag{7}$$

*Remark 2:* Generally, data packets are transmitted with time stamps in the sensor-to-filter channel, and this is convenient for the filter to recognize the arrived packets. In practical applications, however, data packets may not have time-stamps when the bandwidth is limited or the network is unacknowledged. For example, the sign of innovations in [30] is transmitted without time stamps. In view of this, the data packets in this paper are assumed to be non-time-stamped during the data transmission. In other words, the filter would have no acknowledgment of the sent time-instant for the arrived packets and, subsequently the delays of packets received by the filter would be unknown. Under this situation, instead of using the time stamps to obtain the delays, we describe the transmission delays as random variables whose probability distributions are known. In addition, we assume that only one packet reaches the filter at each time because of limited network resources.

In this paper, the following filter is put forward:

$$\hat{x}_{k+1|k} = \bar{A}_k \hat{x}_{k|k}, \tag{8}$$

$$\hat{x}_{k+1|k+1} = \hat{x}_{k+1|k} + K_{k+1}(y_{k+1} - C_{k+1-d}\hat{x}_{k+1-d|k-d}), \tag{9}$$

where $\hat{x}_{k|k}$ is the estimate of state $x_k$ with $\hat{x}_{0|0} = \bar{x}_0$, $\hat{x}_{k+1|k}$ is the prediction of $x_k$, $K_{k+1}$ is the filter gain, and the integer $d$ satisfies

$$d = \begin{cases} \lfloor \bar{\tau} \rfloor, & \text{if } \bar{\tau} - \lfloor \bar{\tau} \rfloor < \frac{1}{2}, \\ \lfloor \bar{\tau} \rfloor + 1, & \text{otherwise}, \end{cases} \tag{10}$$

where $\bar{\tau} \triangleq \mathbb{E}\{\tau_{k+1}\} = \sum_{i=0}^{q} i p_i$.

Our purpose in this paper is twofold: 1) design filter (8)-(9) such that certain upper bounds are guaranteed for filtering error covariances, i.e., there exists $\Psi_{k|k} > 0$ satisfying

$$\mathbb{E}\{(x_{k+1} - \hat{x}_{k+1|k+1})(x_{k+1} - \hat{x}_{k+1|k+1})^T\} \le \Psi_{k+1|k+1};$$

and 2) minimize $\Psi_{k|k}$ through appropriately designing $K_k$.

*Remark 3:* Notice that the innovation in the proposed filter structure differs from that in the traditional filter. Specifically speaking, the innovation in (9) depends on the parameter $d$ (rather than the time delay $\tau_k$) because $\tau_k$ is a random variable that should not appear in the designed filter for practical realizability, and thus the expectation value $\bar{\tau}$ is adopted in filter (8)-(9). Obviously, in discrete-time systems, $\bar{\tau}$ might not be an integer while the time subscript must be an integer. Therefore, $\bar{\tau}$ is rounded to an integer $d$ (see (10)) with the help of the floor function. Roughly speaking, the structure of filter (8)-(9) can mitigate the negative effects resulting from packet disorders to some extent, and the robustness of filter (8)-(9) can be improved accordingly.

## III. Main Results

In this section, the recursive filtering problem is considered for stochastic parameter systems with measurement quantizations and packet disorders. Upper bounds are derived for error covariances and then minimized by selecting proper $K_k$. First, we calculate upper bounds $\Psi_{k|k}$. Then, filter gains $K_k$ are obtained by minimizing the acquired $\Psi_{k|k}$. Finally, the boundedness of filtering errors is discussed.

*Lemma 1:* [34] Let the scalar $\eta > 0$, symmetric matrix $M > 0$, matrices $A, B, C$ and $D$ be given such that $CC^T \le I$ and $\eta^{-1}I - DMD^T > 0$ hold. Then, we have

$$\begin{aligned} &(A+BCD)M(A+BCD)^T \\ &\qquad \le A(M^{-1} - \eta D^T D)^{-1}A^T + \eta^{-1}BB^T. \end{aligned}$$

*Lemma 2:* The state variance $X_{k+1} \triangleq \mathbb{E}\{x_{k+1}x_{k+1}^T\}$ can be recursively calculated as

$$\begin{aligned} &X_{k+1} \\ &\quad = \bar{A}_k X_k \bar{A}_k^T + \Xi_{\tilde{A}_k\tilde{A}_k}(X_k) + \bar{B}_k Q_k \bar{B}_k^T + \Xi_{\tilde{B}_k\tilde{B}_k}(Q_k), \end{aligned}$$

where

$$\begin{aligned} \left(\Xi_{\tilde{A}_k\tilde{A}_k}(X_k)\right)_{st} &\triangleq \mathbb{E}\{\tilde{A}_k X_k \tilde{A}_k^T\}_{st} \\ &= \sum_{j=1}^{m}\sum_{i=1}^{m} T_{A_{tj}^k A_{si}^k} X_{ij}^k, \ (s,t = 1,2,\ldots,m), \\ \left(\Xi_{\tilde{B}_k\tilde{B}_k}(Q_k)\right)_{lh} &\triangleq \mathbb{E}\{\tilde{B}_k Q_k \tilde{B}_k^T\}_{lh} \\ &= \sum_{j=1}^{w}\sum_{i=1}^{w} T_{B_{hj}^k B_{li}^k} Q_{ij}^k, \ (l,h = 1,2,\ldots,m). \end{aligned}$$

Moreover, the initial value is $X_0 = \bar{x}_0\bar{x}_0^T + P_0$.

*Proof:* Rewrite (1) as

$$x_{k+1} = (\bar{A}_k + \tilde{A}_k)x_k + (\bar{B}_k + \tilde{B}_k)w_k.$$

In light of $\mathbb{E}\{\tilde{A}_k\} = 0$ and $\mathbb{E}\{\tilde{B}_k\} = 0$, one has

$$\begin{aligned} X_{k+1} &= \bar{A}_k\mathbb{E}\{x_kx_k^T\}\bar{A}_k^T + \bar{A}_k\mathbb{E}\{x_kx_k^T\tilde{A}_k^T\} \\ &\quad + \mathbb{E}\{\tilde{A}_kx_kx_k^T\}\bar{A}_k^T + \mathbb{E}\{\tilde{A}_kx_kx_k^T\tilde{A}_k^T\} \\ &\quad + \bar{B}_k\mathbb{E}\{w_kw_k^T\}\bar{B}_k^T + \bar{B}_k\mathbb{E}\{w_kw_k^T\tilde{B}_k^T\} \\ &\quad + \mathbb{E}\{\tilde{B}_kw_kw_k^T\}\bar{B}_k^T + \mathbb{E}\{\tilde{B}_kw_kw_k^T\tilde{B}_k^T\} \\ &= \bar{A}_kX_k\bar{A}_k^T + \bar{B}_kQ_k\bar{B}_k^T + \mathbb{E}\{\tilde{A}_kx_kx_k^T\tilde{A}_k^T\} \\ &\quad + \mathbb{E}\{\tilde{B}_kw_kw_k^T\tilde{B}_k^T\}. \end{aligned}$$

According to the property of conditional expectation and (4)-(5), we obtain

$$\begin{aligned} \mathbb{E}\{\tilde{A}_kx_kx_k^T\tilde{A}_k^T\} &= \mathbb{E}\left\{\mathbb{E}\{\tilde{A}_kx_kx_k^T\tilde{A}_k^T|\tilde{A}_k\}\right\} \\ &= \mathbb{E}\left\{\tilde{A}_k\mathbb{E}\{x_kx_k^T\}\tilde{A}_k^T\right\} \\ &= \mathbb{E}\{\tilde{A}_kX_k\tilde{A}_k^T\} \\ &\triangleq \Xi_{\tilde{A}_k\tilde{A}_k}(X_k) \end{aligned}$$

and

$$\begin{aligned} \mathbb{E}\{\tilde{B}_kw_kw_k^T\tilde{B}_k^T\} &= \mathbb{E}\left\{\mathbb{E}\{\tilde{B}_kw_kw_k^T\tilde{B}_k^T|\tilde{B}_k\}\right\} \\ &= \mathbb{E}\left\{\tilde{B}_k\mathbb{E}\{w_kw_k^T\}\tilde{B}_k^T\right\} \\ &= \mathbb{E}\{\tilde{B}_kQ_k\tilde{B}_k^T\} \\ &\triangleq \Xi_{\tilde{B}_k\tilde{B}_k}(Q_k). \end{aligned}$$

The proof is now complete. ■

Denoting

$$e_{k+1|k} \triangleq x_{k+1} - \hat{x}_{k+1|k}$$

and subtracting (8) from (1), one has

$$e_{k+1|k} = \bar{A}_ke_{k|k} + \tilde{A}_kx_k + (\bar{B}_k + \tilde{B}_k)w_k. \tag{11}$$

Similarly, defining

$$e_{k+1|k+1} \triangleq x_{k+1} - \hat{x}_{k+1|k+1}$$

and subtracting (9) from (1), one has

$$e_{k+1|k+1} = e_{k+1|k} - K_{k+1}(y_{k+1} - C_{k+1-d}\hat{x}_{k+1-d|k-d}). \tag{12}$$

Adding

$$K_{k+1}C_{k+1}e_{k+1|k} - K_{k+1}C_{k+1}e_{k+1|k} = 0$$

to both sides of (12) yields

$$\begin{aligned} e_{k+1|k+1} &= (I - K_{k+1}C_{k+1})e_{k+1|k} + K_{k+1}C_{k+1}e_{k+1|k} \\ &\quad - K_{k+1}(I + \Delta_{k+1-\tau_{k+1}})C_{k+1-\tau_{k+1}}x_{k+1-\tau_{k+1}} \\ &\quad - K_{k+1}(I + \Delta_{k+1-\tau_{k+1}})D_{k+1-\tau_{k+1}}v_{k+1-\tau_{k+1}} \\ &\quad + K_{k+1}C_{k+1-d}\hat{x}_{k+1-d|k-d}. \end{aligned} \tag{13}$$

Denote

$$P_{k+1|k} \triangleq \mathbb{E}\{e_{k+1|k}e_{k+1|k}^T\},$$

$$P_{k|k} \triangleq \mathbb{E}\{e_{k|k}e_{k|k}^T\}.$$

*Lemma 3:* The one-step prediction error covariance $P_{k+1|k}$ obeys the following recursion:

$$\begin{aligned} &P_{k+1|k} \\ &\quad = \bar{A}_kP_{k|k}\bar{A}_k^T + \Xi_{\tilde{A}_k\tilde{A}_k}(X_k) + \bar{B}_kQ_k\bar{B}_k^T + \Xi_{\tilde{B}_k\tilde{B}_k}(Q_k). \end{aligned}$$

*Proof:* Substituting (11) into $P_{k+1|k}$ yields

$$\begin{aligned} P_{k+1|k} &= \mathbb{E}\big\{[\bar{A}_ke_{k|k} + \tilde{A}_kx_k + (\bar{B}_k + \tilde{B}_k)w_k] \\ &\qquad \times [\bar{A}_ke_{k|k} + \tilde{A}_kx_k + (\bar{B}_k + \tilde{B}_k)w_k]^T\big\} \\ &= \bar{A}_kP_{k|k}\bar{A}_k^T + \Xi_{\tilde{A}_k\tilde{A}_k}(X_k) + \bar{B}_kQ_k\bar{B}_k^T \\ &\quad + \Xi_{\tilde{B}_k\tilde{B}_k}(Q_k), \end{aligned}$$

and the proof is thus complete. ■

*Lemma 4:* The filtering error covariance $P_{k+1|k+1}$ is given as follows:

$$\begin{aligned} P_{k+1|k+1} &= (I - K_{k+1}C_{k+1})P_{k+1|k}(I - K_{k+1}C_{k+1})^T \\ &\quad + K_{k+1}C_{k+1}P_{k+1|k}C_{k+1}^TK_{k+1}^T + \mathcal{A}_{k+1} \\ &\quad + \mathcal{B}_{k+1} + \mathcal{C}_{k+1} + \mathcal{D}_{1,k+1} + \mathcal{D}_{1,k+1}^T \\ &\quad - \mathcal{D}_{2,k+1} - \mathcal{D}_{2,k+1}^T - \mathcal{D}_{3,k+1} - \mathcal{D}_{3,k+1}^T \\ &\quad + \mathcal{D}_{4,k+1} + \mathcal{D}_{4,k+1}^T - \mathcal{D}_{5,k+1} - \mathcal{D}_{5,k+1}^T \\ &\quad - \mathcal{D}_{6,k+1} - \mathcal{D}_{6,k+1}^T + \mathcal{D}_{7,k+1} + \mathcal{D}_{7,k+1}^T \\ &\quad - \mathcal{D}_{8,k+1} - \mathcal{D}_{8,k+1}^T - \mathcal{D}_{9,k+1} - \mathcal{D}_{9,k+1}^T, \end{aligned}$$

where

$$\begin{aligned} \mathcal{A}_{k+1} &\triangleq K_{k+1}\mathbb{E}\big\{(I + \Delta_{k+1-\tau_{k+1}})C_{k+1-\tau_{k+1}}x_{k+1-\tau_{k+1}} \\ &\qquad \times x_{k+1-\tau_{k+1}}^TC_{k+1-\tau_{k+1}}^T(I + \Delta_{k+1-\tau_{k+1}})^T\big\}K_{k+1}^T, \\ \mathcal{B}_{k+1} &\triangleq K_{k+1}\mathbb{E}\big\{(I + \Delta_{k+1-\tau_{k+1}})D_{k+1-\tau_{k+1}}v_{k+1-\tau_{k+1}} \\ &\qquad \times v_{k+1-\tau_{k+1}}^TD_{k+1-\tau_{k+1}}^T(I + \Delta_{k+1-\tau_{k+1}})^T\big\}K_{k+1}^T, \\ \mathcal{C}_{k+1} &\triangleq K_{k+1}C_{k+1-d}\hat{x}_{k+1-d|k-d}\hat{x}_{k+1-d|k-d}^TC_{k+1-d}^TK_{k+1}^T, \\ \mathcal{D}_{1,k+1} &\triangleq (I - K_{k+1}C_{k+1})\mathbb{E}\{e_{k+1|k}e_{k+1|k}^T\}C_{k+1}^TK_{k+1}^T, \\ \mathcal{D}_{2,k+1} &\triangleq (I - K_{k+1}C_{k+1})\mathbb{E}\big\{e_{k+1|k}x_{k+1-\tau_{k+1}}^T \\ &\qquad \times C_{k+1-\tau_{k+1}}^T(I + \Delta_{k+1-\tau_{k+1}})^T\big\}K_{k+1}^T, \\ \mathcal{D}_{3,k+1} &\triangleq (I - K_{k+1}C_{k+1})\mathbb{E}\big\{e_{k+1|k}v_{k+1-\tau_{k+1}}^T \\ &\qquad \times D_{k+1-\tau_{k+1}}^T(I + \Delta_{k+1-\tau_{k+1}})^T\big\}K_{k+1}^T, \\ \mathcal{D}_{4,k+1} &\triangleq (I - K_{k+1}C_{k+1})\mathbb{E}\{e_{k+1|k}\hat{x}_{k+1-d|k-d}^T\} \\ &\qquad \times C_{k+1-d}^TK_{k+1}^T, \\ \mathcal{D}_{5,k+1} &\triangleq K_{k+1}C_{k+1}\mathbb{E}\big\{e_{k+1|k}x_{k+1-\tau_{k+1}}^TC_{k+1-\tau_{k+1}}^T \\ &\qquad \times (I + \Delta_{k+1-\tau_{k+1}})^T\big\}K_{k+1}^T, \\ \mathcal{D}_{6,k+1} &\triangleq K_{k+1}C_{k+1}\mathbb{E}\big\{e_{k+1|k}v_{k+1-\tau_{k+1}}^TD_{k+1-\tau_{k+1}}^T \\ &\qquad \times (I + \Delta_{k+1-\tau_{k+1}})^T\big\}K_{k+1}^T, \\ \mathcal{D}_{7,k+1} &\triangleq K_{k+1}C_{k+1}\mathbb{E}\{e_{k+1|k}\hat{x}_{k+1-d|k-d}^T\}C_{k+1-d}^TK_{k+1}^T, \\ \mathcal{D}_{8,k+1} &\triangleq K_{k+1}\mathbb{E}\big\{(I + \Delta_{k+1-\tau_{k+1}})C_{k+1-\tau_{k+1}}x_{k+1-\tau_{k+1}} \\ &\qquad \times \hat{x}_{k+1-d|k-d}^T\big\}C_{k+1-d}^TK_{k+1}^T, \\ \mathcal{D}_{9,k+1} &\triangleq K_{k+1}\mathbb{E}\big\{(I + \Delta_{k+1-\tau_{k+1}})D_{k+1-\tau_{k+1}}v_{k+1-\tau_{k+1}} \\ &\qquad \times \hat{x}_{k+1-d|k-d}^T\big\}C_{k+1-d}^TK_{k+1}^T. \end{aligned}$$

*Proof:* According to (12), the expression of filtering error covariance $P_{k+1|k+1}$ can be obtained. Hence, the proof can be derived readily. ■

*Theorem 1:* Let positive scalars $\gamma_{i,k+1}$ $(i = 0, 1, \ldots, q)$ and $\varepsilon_r$ $(r = 1, 2, \ldots, 9)$ be given. With initial condition $\Psi_{0|0} = P_{0|0} > 0$, if the following equalities

$$\begin{aligned}\Psi_{k+1|k} &= \bar{A}_k\Psi_{k|k}\bar{A}_k^T + \Xi_{\tilde{A}_k\tilde{A}_k}(X_k) + \bar{B}_kQ_k\bar{B}_k^T + \Xi_{\tilde{B}_k\tilde{B}_k}(Q_k),\end{aligned}\tag{14}$$

$$\begin{aligned}\Psi_{k+1|k+1} &= \epsilon_1(I - K_{k+1}C_{k+1})\Psi_{k+1|k}(I - K_{k+1}C_{k+1})^T \\ &\quad + K_{k+1}\Big[\epsilon_2C_{k+1}\Psi_{k+1|k}C_{k+1}^T + \sum_{i=0}^{q}p_i\big(\epsilon_3\mathrm{tr}\{C_{k+1-i} \\ &\quad \times X_{k+1-i}C_{k+1-i}^T\} + \epsilon_4\mathrm{tr}\{D_{k+1-i}R_{k+1-i}D_{k+1-i}^T\}\big) \\ &\quad \times \big((I - \gamma_{i,k+1}\Theta\Theta)^{-1} + \gamma_{i,k+1}^{-1}I\big) + \epsilon_5C_{k+1-d} \\ &\quad \times \hat{x}_{k+1-d|k-d}\hat{x}_{k+1-d|k-d}^TC_{k+1-d}^T\Big]K_{k+1}^T,\end{aligned}\tag{15}$$

where

$$\begin{aligned}\epsilon_1 &\triangleq 1 + \varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4, \\ \epsilon_2 &\triangleq 1 + \varepsilon_1^{-1} + \varepsilon_5 + \varepsilon_6 + \varepsilon_7, \\ \epsilon_3 &\triangleq 1 + \varepsilon_2^{-1} + \varepsilon_5^{-1} + \varepsilon_8, \\ \epsilon_4 &\triangleq 1 + \varepsilon_3^{-1} + \varepsilon_6^{-1} + \varepsilon_9, \\ \epsilon_5 &\triangleq 1 + \varepsilon_4^{-1} + \varepsilon_7^{-1} + \varepsilon_8^{-1} + \varepsilon_9^{-1},\end{aligned}$$

have solutions $\Psi_{k+1|k} > 0$ and $\Psi_{k+1|k+1} > 0$ such that constraints

$$\gamma_{i,k+1}^{-1}I - \Theta\Theta > 0 \quad (i = 0, 1, \ldots, q)$$

are satisfied, then $\Psi_{k+1|k+1}$ is an upper bound of $P_{k+1|k+1}$, i.e., $P_{k+1|k+1} \le \Psi_{k+1|k+1}$.

*Proof:* The initial condition implies $P_{0|0} \le \Psi_{0|0}$. Assuming $P_{k|k} \le \Psi_{k|k}$, then $P_{k+1|k+1} \le \Psi_{k+1|k+1}$ needs to be proved.

It follows from Lemma 3 and $P_{k|k} \le \Psi_{k|k}$ that

$$\begin{aligned}P_{k+1|k} &\le \bar{A}_k\Psi_{k|k}\bar{A}_k^T + \Xi_{\tilde{A}_k\tilde{A}_k}(X_k) + \bar{B}_kQ_k\bar{B}_k^T + \Xi_{\tilde{B}_k\tilde{B}_k}(Q_k) \\ &\triangleq \Psi_{k+1|k}.\end{aligned}\tag{16}$$

Next, we are set to show that $P_{k+1|k+1} \le \Psi_{k+1|k+1}$. By utilizing the elementary inequality $ab^T + ba^T \le \varepsilon aa^T + \varepsilon^{-1}bb^T$ (where $\varepsilon$ is an arbitrary positive scalar, and $a$ and $b$ are arbitrary vectors), one has

$$\begin{aligned}&\mathcal{D}_{1,k+1} + \mathcal{D}_{1,k+1}^T \\ &\quad \le \varepsilon_1(I - K_{k+1}C_{k+1})P_{k+1|k}(I - K_{k+1}C_{k+1})^T \\ &\qquad + \varepsilon_1^{-1}K_{k+1}C_{k+1}P_{k+1|k}C_{k+1}^TK_{k+1}^T, \\ &-\mathcal{D}_{2,k+1} - \mathcal{D}_{2,k+1}^T \\ &\quad \le \varepsilon_2(I - K_{k+1}C_{k+1})P_{k+1|k}(I - K_{k+1}C_{k+1})^T \\ &\qquad + \varepsilon_2^{-1}\mathcal{A}_{k+1}, \\ &-\mathcal{D}_{3,k+1} - \mathcal{D}_{3,k+1}^T \\ &\quad \le \varepsilon_3(I - K_{k+1}C_{k+1})P_{k+1|k}(I - K_{k+1}C_{k+1})^T \\ &\qquad + \varepsilon_3^{-1}\mathcal{B}_{k+1}, \\ &\mathcal{D}_{4,k+1} + \mathcal{D}_{4,k+1}^T \\ &\quad \le \varepsilon_4(I - K_{k+1}C_{k+1})P_{k+1|k}(I - K_{k+1}C_{k+1})^T \\ &\qquad + \varepsilon_4^{-1}\mathcal{C}_{k+1}, \\ &-\mathcal{D}_{5,k+1} - \mathcal{D}_{5,k+1}^T \\ &\quad \le \varepsilon_5K_{k+1}C_{k+1}P_{k+1|k}C_{k+1}^TK_{k+1}^T + \varepsilon_5^{-1}\mathcal{A}_{k+1}, \\ &-\mathcal{D}_{6,k+1} - \mathcal{D}_{6,k+1}^T \\ &\quad \le \varepsilon_6K_{k+1}C_{k+1}P_{k+1|k}C_{k+1}^TK_{k+1}^T + \varepsilon_6^{-1}\mathcal{B}_{k+1}, \\ &\mathcal{D}_{7,k+1} + \mathcal{D}_{7,k+1}^T \\ &\quad \le \varepsilon_7K_{k+1}C_{k+1}P_{k+1|k}C_{k+1}^TK_{k+1}^T + \varepsilon_7^{-1}\mathcal{C}_{k+1}, \\ &-\mathcal{D}_{8,k+1} - \mathcal{D}_{8,k+1}^T \le \varepsilon_8\mathcal{A}_{k+1} + \varepsilon_8^{-1}\mathcal{C}_{k+1}, \\ &-\mathcal{D}_{9,k+1} - \mathcal{D}_{9,k+1}^T \le \varepsilon_9\mathcal{B}_{k+1} + \varepsilon_9^{-1}\mathcal{C}_{k+1}.\end{aligned}$$

Combining Lemma 1 and $\Delta_{k+1} = U_{k+1}\Theta$, we obtain that

$$\begin{aligned}\mathcal{A}_{k+1} &= K_{k+1}\sum_{i=0}^{q}p_i(I + \Delta_{k+1-i})C_{k+1-i}X_{k+1-i} \\ &\quad \times C_{k+1-i}^T(I + \Delta_{k+1-i})^TK_{k+1}^T \\ &\le K_{k+1}\sum_{i=0}^{q}p_i\mathrm{tr}\{C_{k+1-i}X_{k+1-i}C_{k+1-i}^T\} \\ &\quad \times (I + \Delta_{k+1-i})(I + \Delta_{k+1-i})^TK_{k+1}^T \\ &= K_{k+1}\sum_{i=0}^{q}p_i\mathrm{tr}\{C_{k+1-i}X_{k+1-i}C_{k+1-i}^T\} \\ &\quad \times (I + U_{k+1-i}\Theta)(I + U_{k+1-i}\Theta)^TK_{k+1}^T \\ &\le K_{k+1}\sum_{i=0}^{q}p_i\mathrm{tr}\{C_{k+1-i}X_{k+1-i}C_{k+1-i}^T\} \\ &\quad \times \Big((I - \gamma_{i,k+1}\Theta\Theta)^{-1} + \gamma_{i,k+1}^{-1}I\Big)K_{k+1}^T\end{aligned}$$

and

$$\begin{aligned}\mathcal{B}_{k+1} &= K_{k+1}\sum_{i=0}^{q}p_i(I + \Delta_{k+1-i})D_{k+1-i}R_{k+1-i} \\ &\quad \times D_{k+1-i}^T(I + \Delta_{k+1-i})^TK_{k+1}^T \\ &\le K_{k+1}\sum_{i=0}^{q}p_i\mathrm{tr}\{D_{k+1-i}R_{k+1-i}D_{k+1-i}^T\} \\ &\quad \times (I + \Delta_{k+1-i})(I + \Delta_{k+1-i})^TK_{k+1}^T \\ &\le K_{k+1}\sum_{i=0}^{q}p_i\mathrm{tr}\{D_{k+1-i}R_{k+1-i}D_{k+1-i}^T\} \\ &\quad \times \Big((I - \gamma_{i,k+1}\Theta\Theta)^{-1} + \gamma_{i,k+1}^{-1}I\Big)K_{k+1}^T.\end{aligned}$$

Summarizing the above discussion, it is deduced that

$$\begin{aligned}P_{k+1|k+1} &\le \epsilon_1(I - K_{k+1}C_{k+1})P_{k+1|k}(I - K_{k+1}C_{k+1})^T \\ &\quad + K_{k+1}\Big[\epsilon_2C_{k+1}P_{k+1|k}C_{k+1}^T \\ &\quad + \sum_{i=0}^{q}p_i\big(\epsilon_3\mathrm{tr}\{C_{k+1-i}X_{k+1-i}C_{k+1-i}^T\} \\ &\quad + \epsilon_4\mathrm{tr}\{D_{k+1-i}R_{k+1-i}D_{k+1-i}^T\}\big) \\ &\quad \times \big((I - \gamma_{i,k+1}\Theta\Theta)^{-1} + \gamma_{i,k+1}^{-1}I\big) \\ &\quad + \epsilon_5C_{k+1-d}\hat{x}_{k+1-d|k-d}\hat{x}_{k+1-d|k-d}^T \\ &\quad \times C_{k+1-d}^T\Big]K_{k+1}^T.\end{aligned}\tag{17}$$

Inequality (16) combined with (17) implies that $P_{k+1|k+1} \leq \Psi_{k+1|k+1}$. ■

*Theorem 2:* The filter gain is

$$K_{k+1} = \epsilon_1 \Psi_{k+1|k} C_{k+1}^T \Pi_{k+1}^{-1}, \tag{18}$$

where

$$\begin{aligned}\Pi_{k+1} \triangleq & (\epsilon_1 + \epsilon_2) C_{k+1} \Psi_{k+1|k} C_{k+1}^T \\ & + \sum_{i=0}^{q} p_i \big( \epsilon_3 \mathrm{tr}\{C_{k+1-i} X_{k+1-i} C_{k+1-i}^T\} \\ & + \epsilon_4 \mathrm{tr}\{D_{k+1-i} R_{k+1-i} D_{k+1-i}^T\} \big) \\ & \times \big( (I - \gamma_{i,k+1} \Theta\Theta)^{-1} + \gamma_{i,k+1}^{-1} I \big) \\ & + \epsilon_5 C_{k+1-d} \hat{x}_{k+1-d|k-d} \hat{x}_{k+1-d|k-d}^T C_{k+1-d}^T.\end{aligned}$$

Furthermore, the minimal upper bound matrix $\Psi_{k+1|k+1}$ is

$$\Psi_{k+1|k+1} = -\epsilon_1^2 \Psi_{k+1|k} C_{k+1}^T \Pi_{k+1}^{-1} C_{k+1} \Psi_{k+1|k} + \epsilon_1 \Psi_{k+1|k}. \tag{19}$$

*Proof:* The upper bound matrix shown in (15) is rewritten as

$$\begin{aligned}\Psi_{k+1|k+1} = & K_{k+1} \Pi_{k+1} K_{k+1}^T - \epsilon_1 \Psi_{k+1|k} C_{k+1}^T K_{k+1}^T \\ & - \epsilon_1 K_{k+1} C_{k+1} \Psi_{k+1|k} + \epsilon_1 \Psi_{k+1|k} \\ = & (K_{k+1} - \epsilon_1 \Psi_{k+1|k} C_{k+1}^T \Pi_{k+1}^{-1}) \Pi_{k+1} \\ & \times (K_{k+1} - \epsilon_1 \Psi_{k+1|k} C_{k+1}^T \Pi_{k+1}^{-1})^T \\ & - \epsilon_1^2 \Psi_{k+1|k} C_{k+1}^T \Pi_{k+1}^{-1} C_{k+1} \Psi_{k+1|k} \\ & + \epsilon_1 \Psi_{k+1|k}.\end{aligned}$$

It is easily known that $\Psi_{k+1|k+1}$ are minimized through setting $K_{k+1} = \epsilon_1 \Psi_{k+1|k} C_{k+1}^T \Pi_{k+1}^{-1}$. ■

*Definition 1:* [29] The stochastic process $\xi_k$ is exponentially mean-square bounded if there exist real numbers $\varrho > 0$, $\iota > 0$ and $0 < \sigma < 1$ such that

$$\mathbb{E}\{\|\xi_k\|\} \leq \varrho \|\xi_0\|^2 \sigma^k + \iota$$

holds for all $k \geq 0$.

*Theorem 3:* If there exist positive real numbers $\overline{a} < 1, \underline{b}, \overline{b}, \underline{c}, \overline{c}, \overline{d}, \underline{q}, \overline{q}, \overline{r}, \underline{\varphi}, \overline{\varphi}, \overline{l}_1, \overline{l}_2, \overline{\theta}, \overline{\chi}_1, \overline{\chi}_2$ such that

$$\begin{aligned}& \|\bar{A}_k\| \leq \overline{a}, \quad \underline{b} \leq \|\bar{B}_k\| \leq \overline{b}, \quad \underline{c}^2 I \leq C_k C_k^T \leq \overline{c}^2 I, \\ & \|\bar{D}_k\| \leq \overline{d}, \quad \underline{q} I \leq Q_k \leq \overline{q} I, \quad R_k \leq \overline{r} I, \\ & \underline{\varphi} I \leq \Psi_{k+1|k} \leq \overline{\varphi} I, \quad X_k \leq \overline{\chi}_1 I, \\ & \mathrm{tr}\big\{\Xi_{\tilde{A}_k \tilde{A}_k}(X_k)\big\} \leq \overline{l}_1, \quad \mathrm{tr}\big\{\Xi_{\tilde{B}_k \tilde{B}_k}(Q_k)\big\} \leq \overline{l}_2, \\ & \|\Theta\| \leq \overline{\theta}, \quad \|\hat{x}_{k+1|k}\| \leq \overline{\chi}_2\end{aligned}$$

hold, then filtering errors in (13) are exponentially mean-square bounded.

*Proof:* (13) is rewritten as

$$e_{k+1|k+1} = \bar{A}_k e_{k|k} + f_{k+1} + h_{k+1}, \tag{20}$$

where

$$\begin{aligned}f_{k+1} \triangleq & \tilde{A}_k x_k + (\bar{B}_k + \tilde{B}_k) w_k, \\ h_{k+1} \triangleq & -K_{k+1}(I + \Delta_{k+1-\tau_{k+1}}) C_{k+1-\tau_{k+1}} x_{k+1-\tau_{k+1}} \\ & - K_{k+1}(I + \Delta_{k+1-\tau_{k+1}}) D_{k+1-\tau_{k+1}} v_{k+1-\tau_{k+1}} \\ & + K_{k+1} C_{k+1-d} \hat{x}_{k+1-d|k-d}.\end{aligned}$$

One derives from (18) that

$$\begin{aligned}\|K_{k+1}\| = & \|\epsilon_1 \Psi_{k+1|k} C_{k+1}^T \Pi_{k+1}^{-1}\| \\ \leq & \|\epsilon_1 \Psi_{k+1|k} C_{k+1}^T [\epsilon_1 C_{k+1} \Psi_{k+1|k} C_{k+1}^T]^{-1}\| \\ \leq & \frac{\overline{\varphi}\, \overline{c}}{\underline{\varphi}\, \underline{c}^2} \triangleq \overline{k}.\end{aligned}$$

By utilizing properties of the matrix trace, we obtain

$$\begin{aligned}\mathbb{E}\{f_{k+1}^T f_{k+1}\} = & \mathbb{E}\{x_k^T \tilde{A}_k^T \tilde{A}_k x_k\} + \mathbb{E}\{w_k^T \tilde{B}_k^T \tilde{B}_k w_k\} \\ & + \mathbb{E}\{w_k^T \bar{B}_k^T \bar{B}_k w_k\} \\ = & \mathrm{tr}\big\{\Xi_{\tilde{A}_k \tilde{A}_k}(X_k)\big\} + \mathrm{tr}\big\{\Xi_{\tilde{B}_k \tilde{B}_k}(Q_k)\big\} \\ & + \mathrm{tr}\big\{\mathbb{E}\{w_k w_k^T\} \bar{B}_k^T \bar{B}_k\big\} \\ \leq & \overline{l}_1 + \overline{l}_2 + w\overline{q}\overline{b}^2 \triangleq \overline{f}\end{aligned}$$

and

$$\begin{aligned}& \mathbb{E}\{h_{k+1}^T h_{k+1}\} \\ \leq & (1 + \varsigma_1) \mathbb{E}\big\{x_{k+1-\tau_{k+1}}^T C_{k+1-\tau_{k+1}}^T (I + \Delta_{k+1-\tau_{k+1}})^T \\ & \times K_{k+1}^T K_{k+1} (I + \Delta_{k+1-\tau_{k+1}}) C_{k+1-\tau_{k+1}} x_{k+1-\tau_{k+1}}\big\} \\ & + (1 + \varsigma_2) \mathbb{E}\big\{v_{k+1-\tau_{k+1}}^T D_{k+1-\tau_{k+1}}^T (I + \Delta_{k+1-\tau_{k+1}})^T \\ & \times K_{k+1}^T K_{k+1} (I + \Delta_{k+1-\tau_{k+1}}) D_{k+1-\tau_{k+1}} v_{k+1-\tau_{k+1}}\big\} \\ & + (1 + \varsigma_1^{-1} + \varsigma_2^{-1}) \mathbb{E}\big\{\hat{x}_{k+1-d|k-d}^T C_{k+1-d}^T K_{k+1}^T \\ & \times K_{k+1} C_{k+1-d} \hat{x}_{k+1-d|k-d}\big\} \\ = & (1 + \varsigma_1) \sum_{i=0}^{q} p_i \mathrm{tr}\big\{X_{k+1-i} C_{k+1-i}^T (I + \Delta_{k+1-i})^T K_{k+1}^T \\ & \times K_{k+1} (I + \Delta_{k+1-i}) C_{k+1-i}\big\} \\ & + (1 + \varsigma_2) \sum_{i=0}^{q} p_i \mathrm{tr}\big\{R_{k+1-i} D_{k+1-i}^T (I + \Delta_{k+1-i})^T K_{k+1}^T \\ & \times K_{k+1} (I + \Delta_{k+1-i}) D_{k+1-i}\big\} \\ & + (1 + \varsigma_1^{-1} + \varsigma_2^{-1}) \mathrm{tr}\big\{\hat{x}_{k+1-d|k-d}^T C_{k+1-d}^T K_{k+1}^T \\ & \times K_{k+1} C_{k+1-d} \hat{x}_{k+1-d|k-d}\big\} \\ \leq & (1 + \varsigma_1) m \overline{\chi}_1 \overline{c}^2 (1 + \overline{\theta})^2 \overline{k}^2 + (1 + \varsigma_2) v \overline{r} \overline{d}^2 \\ & \times (1 + \overline{\theta})^2 \overline{k}^2 + (1 + \varsigma_1^{-1} + \varsigma_2^{-1}) \overline{\chi}_2^2 \overline{c}^2 \overline{k}^2 \\ \triangleq & \overline{h},\end{aligned}$$

where $\varsigma_1$ and $\varsigma_2$ are positive scalars.

Subsequently, consider the following iterative matrix equation in regard to $\Phi_k$:

$$\Phi_{k+1} = \bar{A}_k \Phi_k \bar{A}_k^T + \bar{B}_k Q_k \bar{B}_k^T + \kappa I, \tag{21}$$

where the scalar $\kappa > 0$ and

$$\Phi_0 = \bar{B}_0 Q_0 \bar{B}_0^T + \kappa I.$$

On the one hand, it can be inferred from (21) that

$$\begin{aligned}\|\Phi_{k+1}\| \leq & \|\bar{A}_k\|^2 \|\Phi_k\| + \|\bar{B}_k Q_k \bar{B}_k^T\| + \|\kappa I\| \\ \leq & \overline{a}^2 \|\Phi_k\| + \overline{b}^2 \overline{q} + \kappa.\end{aligned}$$

By further iteration, one has

$$\|\Phi_k\| \leq \overline{a}^{2k} \|\Phi_0\| + (\overline{b}^2 \overline{q} + \kappa) \sum_{i=0}^{k-1} \overline{a}^{2i}.$$

On the basis of $0 < \overline{a} < 1$, we have

$$\begin{aligned}\|\Phi_k\| &\leq \|\Phi_0\| + (\overline{b}^2\overline{q} + \kappa)\sum_{i=0}^{\infty}\overline{a}^{2i}\\ &= \|\Phi_0\| + \frac{\overline{b}^2\overline{q} + \kappa}{1-\overline{a}^2}. \end{aligned} \tag{22}$$

On the other hand, it follows from (21) that

$$\Phi_k \geq \kappa I. \tag{23}$$

Resorting to (22)-(23), there exists a scalar $\overline{\phi} > 0$ such that $\kappa I \leq \Phi_k \leq \overline{\phi} I$ for all $k \geq 0$.

Let $V_k(e_{k|k}) \triangleq e_{k|k}^T \Phi_k^{-1} e_{k|k}$. For a positive scalar $\delta$, we know from (20) that

$$\begin{aligned}&\mathbb{E}\{V_{k+1}(e_{k+1|k+1})|e_{k|k}\} - (1+\delta)V_k(e_{k|k})\\ &= \mathbb{E}\big\{(\bar{A}_k e_{k|k} + f_{k+1} + h_{k+1})^T \Phi_{k+1}^{-1}(\bar{A}_k e_{k|k}\\ &\quad + f_{k+1} + h_{k+1})\big\} - (1+\delta)e_{k|k}^T\Phi_k^{-1}e_{k|k}\\ &\leq (1+\delta)\mathbb{E}\big\{e_{k|k}^T(\bar{A}_k^T\Phi_{k+1}^{-1}\bar{A}_k - \Phi_k^{-1})e_{k|k}\big\}\\ &\quad + (1+\delta^{-1})\mathbb{E}\{h_{k+1}^T\Phi_{k+1}^{-1}h_{k+1}\}\\ &\quad + \mathbb{E}\{f_{k+1}^T\Phi_{k+1}^{-1}f_{k+1}\}. \end{aligned} \tag{24}$$

By employing the matrix inversion lemma, one has

$$\begin{aligned}&\bar{A}_k^T\Phi_{k+1}^{-1}\bar{A}_k - \Phi_k^{-1}\\ &= \bar{A}_k^T(\bar{A}_k\Phi_k\bar{A}_k^T + \bar{B}_kQ_k\bar{B}_k^T + \kappa I)^{-1}\bar{A}_k - \Phi_k^{-1}\\ &= -\big[\Phi_k + \Phi_k\bar{A}_k^T(\bar{B}_kQ_k\bar{B}_k^T + \kappa I)^{-1}\bar{A}_k\Phi_k\big]^{-1}\\ &= -\big[I + \bar{A}_k^T(\bar{B}_kQ_k\bar{B}_k^T + \kappa I)^{-1}\bar{A}_k\Phi_k\big]^{-1}\Phi_k^{-1}\\ &\leq -\left(1 + \frac{\overline{a}^2\overline{\phi}}{\underline{b}^2\underline{q}}\right)^{-1}\Phi_k^{-1}. \end{aligned} \tag{25}$$

Substituting (25) into (24) yields

$$\begin{aligned}&\mathbb{E}\{V_{k+1}(e_{k+1|k+1})|e_{k|k}\} - (1+\delta)V_k(e_{k|k})\\ &\leq -(1+\delta)\left(1 + \frac{\overline{a}^2\overline{\phi}}{\underline{b}^2\underline{q}}\right)^{-1}V_k(e_{k|k}) + \frac{\overline{f}}{\kappa} + (1+\delta^{-1})\frac{\overline{h}}{\kappa}. \end{aligned} \tag{26}$$

Inequality (26) implies

$$\mathbb{E}\{V_{k+1}(e_{k+1|k+1})|e_{k|k}\} \leq \alpha V_k(e_{k|k}) + \mu, \tag{27}$$

where

$$\begin{aligned}\alpha &\triangleq (1+\delta)\left[1 - \left(1 + \frac{\overline{a}^2\overline{\phi}}{\underline{b}^2\underline{q}}\right)^{-1}\right],\\ \mu &\triangleq \frac{\overline{f}}{\kappa} + (1+\delta^{-1})\frac{\overline{h}}{\kappa}. \end{aligned}$$

For some $\delta > 0$, we know $\alpha \in (0,1)$. Furthermore, we derive that

$$\begin{aligned}\mathbb{E}\{\|e_{k+1|k+1}\|^2\} &\leq \frac{\overline{\phi}}{\kappa}\mathbb{E}\{\|e_{0|0}\|^2\}\alpha^{k+1} + \mu\overline{\phi}\sum_{i=0}^{k}\alpha^i\\ &\leq \frac{\overline{\phi}}{\kappa}\mathbb{E}\{\|e_{0|0}\|^2\}\alpha^{k+1} + \mu\overline{\phi}\sum_{i=0}^{\infty}\alpha^i\\ &= \frac{\overline{\phi}}{\kappa}\mathbb{E}\{\|e_{0|0}\|^2\}\alpha^{k+1} + \frac{\mu\overline{\phi}}{1-\alpha}. \end{aligned}$$

Noting Definition 1, our proof is now complete. ■

*Remark 4:* A recursive filtering problem is proposed in this paper for stochastic parameter systems where a new observation model is put forward to model output signals suffering from quantization effects and RTDs. Based on an integer-valued function, we develop a novel filter so as to offset the side-effects of packet disorders. In Theorem 1, upper bounds are derived for filtering error covariances by virtue of (14)–(15) which are later minimized in Theorem 2 by properly selecting filter gains. Moreover, the mean-square boundedness of the resultant filtering errors is analyzed in Theorem 3.

*Remark 5:* The Kalman filter (and its variants) for linear/nonlinear systems has attracted an ever-increasing research attention due mainly to the demand for system monitoring and the prevalence of system/measurement noises, and a number of excellent filtering algorithms have been reported in the literature, see e.g. [8], [15], [16], [38]. Compared to existing literature, the main results developed in this paper exhibit the following distinctive features: 1) the underlying stochastic parameter model is quite general to take into simultaneous account the packet disorders and the measurement quantizations; 2) random but bounded variables are introduced to characterize the RTDs with their probability distributions known *a priori*; 3) the devised recursive filter is capable of resisting the adverse effects from packet disorders and also locally minimized certain upper bounds on filtering error covariances; and 4) the mean-square boundedness of filtering errors is established by virtual of the matrix analysis techniques.

## IV. Simulation Example

Consider system (1)-(2) with parameters:

$$\begin{aligned}&\bar{A}_k = \begin{bmatrix}-0.6 & 0.39\\ 0.48 & 0.53\end{bmatrix}, && \tilde{A}_k = \beta_{1,k}\begin{bmatrix}0.5 & 0\\ 0 & 0.2\end{bmatrix},\\ &\bar{B}_k = \begin{bmatrix}0.05\\ 0.05\end{bmatrix}, && \tilde{B}_k = \beta_{2,k}\begin{bmatrix}0.1\\ 0\end{bmatrix},\\ &C_k = \begin{bmatrix}0.01 & 1.03\end{bmatrix}, && D_k = 0.2,\end{aligned}$$

where $\beta_{1,k} \in \mathbb{R}$ and $\beta_{2,k} \in \mathbb{R}$ are Gaussian white sequences with zero means and unity variances.

Set $\hat{x}_{0|0} = \bar{x}_0 = \begin{bmatrix}0.5 & 0.4\end{bmatrix}^T$, $P_0 = I_2$, $q = 3$, $X_{-2} = X_{-1} = X_0 = \bar{x}_0\bar{x}_0^T + P_0$, $\hat{x}_{-2|-3} = \hat{x}_{-1|-2} = \hat{x}_{0|-1} = \bar{x}_0$ and $Q_k = R_k = 0.5$. In addition, let $\upsilon_0^{(1)} = 2$, $\rho^{(1)} = 0.2$, $\gamma_{0,k+1} = \gamma_{1,k+1} = \gamma_{2,k+1} = \gamma_{3,k+1} = 0.5, \varepsilon_1 = 0.6, \varepsilon_2 = 0.48, \varepsilon_3 = 0.18, \varepsilon_4 = 0.05, \varepsilon_5 = 0.9, \varepsilon_6 = 0.1, \varepsilon_7 = 0.15, \varepsilon_8 = 0.1$ and $\varepsilon_9 = 0.5$.

Denote the mean square error (MSE) of the $i$-th state estimate as

$$\mathrm{MSE}_i = \frac{1}{M}\sum_{j=1}^{M}(x_{i,k}^{(j)} - \hat{x}_{i,k|k}^{(j)})^2$$

where $M = 100$ is the number of independent simulation tests.

The detailed simulation results are given in Figs. 2–5. Figs. 2–3 depict trajectories of states $x_{i,k}$ and their estimates $\hat{x}_{i,k}$ $(i = 1,2)$, which illustrate that the developed filter can estimate the actual state well. To quantify the accuracy of estimation, Figs. 4–5 plot the upper bounds of $\Psi_{i,k|k}$ and $\mathrm{MSE}_i$ $(i = 1, 2)$ of state estimation, where $\mathrm{MSE}_i^0$ and upper $\mathrm{bound}_i^0$ represent the MSE and upper bound with respect to the $i$-th $(i = 1, 2)$ state when $d = 0$, respectively. Similarly, the $\mathrm{MSE}_i^3$ and upper $\mathrm{bound}_i^3$ represent the MSE and upper bound with respect to the $i$-th $(i = 1, 2)$ state when $d = 3$, respectively. It is easily seen that the MSE curves always stay below that of the upper bound. In addition, these upper bounds and MSEs become larger with the increase of $d$, which shows the influences on the filter performance brought by the RTD-induced packet disorders. The above simulation results confirm that the filtering performance of our proposed scheme is acceptable for the addressed system subject to random parameter matrices, measurement quantizations and packet disorders.

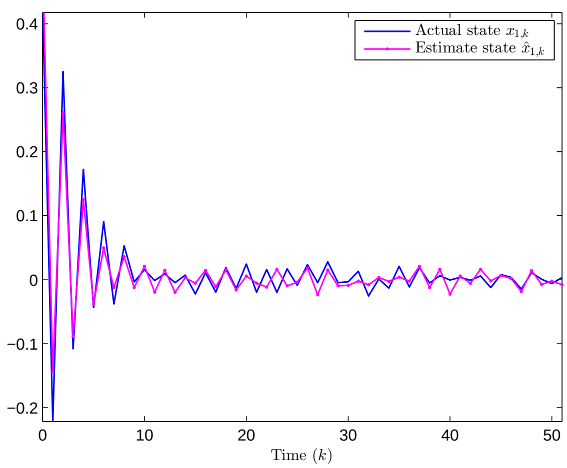


Fig. 2: $x_{1,k}$ and $\hat{x}_{1,k}$.

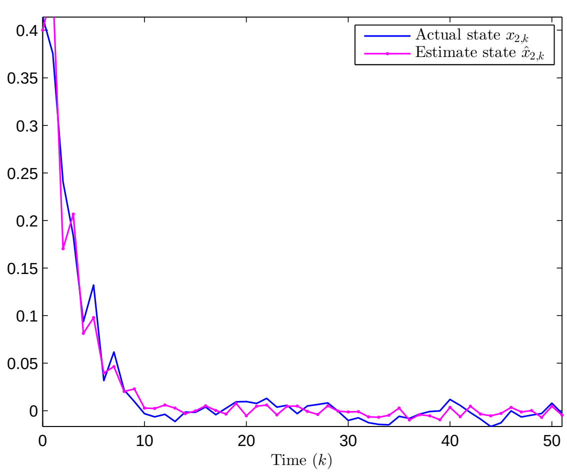


Fig. 3: $x_{2,k}$ and $\hat{x}_{2,k}$.

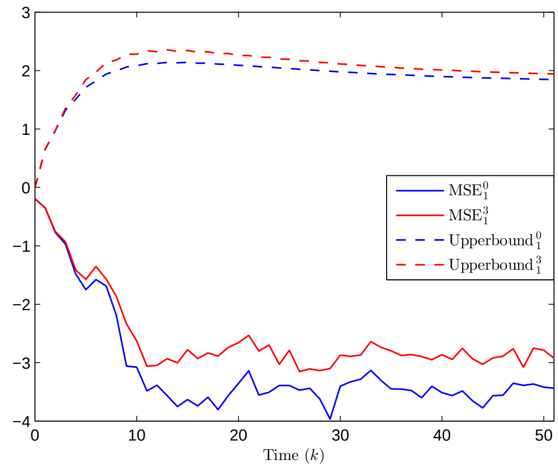


Fig. 4: $\mathrm{MSE}_1$ and its bounds when $d = 0$ and $d = 3$.

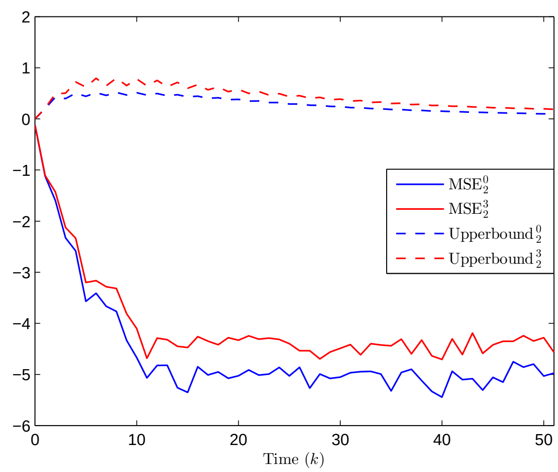


Fig. 5: $\mathrm{MSE}_2$ and its bounds when $d = 0$ and $d = 3$.

## V. Conclusion

This paper has coped with the recursive filtering problem for stochastic parameter systems with quantization effects and packet disorders. A logarithmic quantizer has been employed to cope with the measurement outputs before the transmission. Transmission delays occurred within the sensor-to-filter channel have been modeled by random variables of known distributions. Upper bounds on filtering errors have been acquired and filter gains have been determined through minimizing these bounds. Subsequently, the boundedness of filtering errors has also been analyzed. Future research topics would include the extension of the main results to some practical applications such as networked control systems [31] and [32], where the quantized control problem has been investigated.

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