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USE OF QUALITY CONTROL METHODS IN MONITORING THE PURCHASING BEHAVIOUR OF CONSUMER PANELS

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CONTENTS

1. Introduction

Consider a consumer panel of n members (which may be individuals, households, etc.). where each member provides information about his weekly purchases of a certain kind of product. We suppose that several brands of the product are available and that we wish to monitor the panel's overall preference for a particular brand B.

We assume that for each member

(i) successive purchase occasions occur as if in an independent Poisson process,

(ii) the brand purchased on a given occasion is chosen as if at random, and independently of previous brand choices, according to certain brand-choice probabilities.

For member i, $i = 1, \ldots, n$, let μ_i denote the mean number of purchase occasions per week, and p_i the probability of choosing brand B on a given purchase occasion. These parameters may be constant or may vary from week to week. It follows that for the panel as a whole

1) successive purchase occasions occur in a Poisson process at a rate of $\mu = \sum_{i=1}^{N} \mu_i$ such occasions per week.

2) the brand purchased on a given occasion is independent of those purchase on previous occasions, and the probability that brand B is purchased is

$$\bar{p} = \sum_{j=1}^{n} \text{pr}\left(\begin{array}{c}\text{individual } i\\ \text{made the purchase}\end{array}\right) \text{pr}\left(\text{B chosen}\,\middle|\,\begin{array}{l}\text{i made}\\ \text{the purchase}\end{array}\right)$$

$$= \sum_{i=1}^{n} \frac{\mu_i}{\mu.} P_i \quad .$$

$\bar{p}$ is thus the expected proportion of a series of purchase occasions for the panel on which brand B is purchased, provided it remains constant over these occasions.

We shall be concerned with detecting changes in the panel's overall preference for B as represented by the expected proportion $\bar{p}$, and shall present an approach for monitoring the value of $\bar{p}$ on a weekly basis using quality control methods.

2. Some Distributional Results

Let X_t = overall number of purchase occasions for the panel in week t,

Z_t = overall number of occasions on which brand B is purchased by the panel in week t.

and $Y_t = \frac{Z_t}{X_t}$ =overall proportion of occasions on which brand B is purchased by the panel in week t ,

where t = 1,2, ...

These random variables are independent between weeks.

The Y's provide estimates of the value of $\bar{p}$ in their respective weeks.

The X's,on the other hand, are ancillary statistics as far as $\bar{p}$ is concerned: they provide information about how accurately $\bar{p}$ can be estimated in their respective weeks but no information about its value.

We therefore consider the distributions of the random variables Z_t and Y_t. conditionally on X_t.

Now, for given $X_t > 0$, Z_t follows the binomial distribution $Bi(X_t, \bar{p})$ and hence

$$E(Y_t \mid X_t) = \bar{p} \text{ , var } (Y_t \mid X_t) = \frac{\bar{P}(1 - \bar{P})}{X_t} .$$

Further, if the realised X_t is sufficiently large, this binomial distribution, and the conditional distribution of Y_t, may be approximated by normal distributions. Hence, for sufficiently large realised X_t, the conditional distribution of Y_t given X_t is approximately

$$N\left(\bar{P}, \frac{\bar{P}(1-\bar{P})}{x_t}\right).$$

Standardising, the conditional distribution of

$$T_t = \frac{Y_1 - \bar{P}}{\sqrt{\frac{\bar{P}(1-\bar{P})}{X_t}}}$$

given X_t is approximation N(0,1) for sufficiently large realised X_t. Further the T's are independent both conditionally on the X's and unconditionally.

Approximate conditions for the validity of the normal approximation to the above binomial distribution are that we require

$$X_t > 5 \text{ and } \frac{1}{\sqrt{X_t}} \left| \left(\frac{\bar{P}}{1-\bar{P}}\right)^{\frac{1}{2}} - \left(\frac{1-\bar{P}}{\bar{P}}\right)^{\frac{1}{2}} \right| < 0.3 \;.$$

For example, if p = 0.02 we require a realised $X_t > 522$, while if p = 0.2 we require a realised $X_t > 25$. The realised values of the X's will be large with high probability if the underlying overall mean weekly purchase rate $\mu = \sum_{i=1}^{n} \mu_j$ of the panel is large in each week, which will be the case if the size n of the panel is sufficiently large.

Note that since the approximate conditional distribution of T_t given X_t does not depend on X_t, it follows that this approximate distribution applies unconditionally also, i.e. for large μ, the unconditional distribution of each T_t is approximately N(0,1), independently for each T_t.

Note also that the above distributional result does not depend on the panel's overall mean weekly purchase rate μ, and is therefore independent of any variation in the value of μ. from week to week such as seasonal variation.

3. Detecting a change in $\bar{p}$ from an Established Value

Suppose in a given week t = 0 it is established from past data that $\bar{p}$ has the value $\bar{p}_0$. Then to detect a change from this established value we may monitor the panel's weekly purchasing behaviour using Shewhart and cumulative sum (cusum) charts of the statistics

$$T_t = \frac{Y_t - \bar{P}_o}{\sqrt{\frac{\bar{P}_o(1-\bar{P}_o)}{X_t}}}, \; t = ,2,\dots \;.$$

Assuming that the value of p does not change from week to week then, conditionally on the X's or unconditionally, these statistics are independent and each is distributed approximately N(0,1). We shall assume that the overall weekly purchase rates are sufficiently large for the normal approximations to be valid.

3.1 The Shewhart Chart

For this chart T_t is plotted against t. While $\bar{p}$ has the value $\bar{p}_0$ the T's have zero expectations, but if $\bar{p}$ increases or decreases then they have positive or negative expectations, respectively. Large positive or negative values the T's are therefore evidence of a change in $\bar{p}$. To decide whether or not a value of T_t is significantly large in absolute value, the chart is provided with decision limits, which may be chosen to give an acceptable sensitivity.

We suggest that decision limits are placed at ± 2.58; for unchanged $\bar{p}$, each T_t has a probability of only 0.01 of falling outside these limits (a probability of 0.005 of falling beyond each limit). If, then, a value of T_t exceeds 2.58 or is less than - 2.58, this is taken as evidence of an increase or decrease, respectively, in the value of $\bar{p}$.

The sensitivity of the Shewhart chart in detecting changes in $\bar{p}$ may be analysed as follows.

Suppose at the start of a given week the value of $\bar{p}$ changes from $\bar{p}_0$ to $\bar{p}_1$ and remains at the new value thereafter. We shall only consider changes $\Delta\bar{p}_0 = \bar{p}_1 - \bar{p}_0$ which are small compared with $\bar{p}_0$, but which are comparable in

magnitude with

$$\sigma_0 = \sqrt{\frac{\bar{p}_0(1-\bar{p}_0)}{\mu.}},$$

which is the approximate unconditional standard deviation of Y when $\bar{p} = \bar{p}_0$. We shall also assume that u remains constant in the weeks following the change in $\bar{p}$.

For such a change in $\bar{p}$, and for large μ, the distribution of T_t is approximately $N(\frac{\Delta \bar{p}_0}{\sigma_0}, 1)$ independently for each T_t, both unconditionally and conditionally on the realised X_t (see Appendix 1).

Suppose $\bar{p}$ increases and let

$$\theta = \text{pr}\left(T_t > 2.58 \middle| \bar{p}_1 \right) = 1 - \phi(2.58 - \frac{\Delta \bar{p}_0}{\sigma_0}).$$

Let R denote the number of weeks that elapse following the increase before an increase is detected, i.e. before the upper decision limit is reached; R has the geometric distribution

$$\Pr(R = r) = (1 - \theta)^{r-1}\theta \ , r = 1,2,\ldots \ .$$

Hence the probability that the increase is detected by the rth week is

$$\Pr(R \le r) = 1 - (1-\theta)^r \ , r = 1,2,\ldots,$$

and the expected number of weeks that elapse before the increase is detected is

$$E(R) = \frac{1}{\theta},$$

which is called the average run length (ARL) in quality control.

Corresponding results apply for the detection of a decrease in $\bar{p}$.

The sensitivity of the chart in detecting a change in $\bar{p}$ depends on the value of $|\Delta \bar{p}_0|/\sigma_0$, i.e. on the change in $\bar{p}$ expressed as a multiple of σ_0 . Table 1 gives approximate values for the probability of the detection of an increase or a decrease, as appropriate, by the rth week, and of the ARL, for $|\Delta \bar{p}_0|/\sigma_0 = 0, 0.5, 1, 2, 3$. Table 2 gives the values of $|\Delta \bar{p}_0|$ corresponding to these multiples of σ_0 for two pairs of values of p_0 and μ typical of those found in practice.

In the case of no change in the value of $\bar{p}$ ($\Delta \bar{p}_0/\sigma_0 = 0$), the starting point for calculating the detection probabilities is arbitrary. In such a case, of course, the detection of an increase or a decrease constitutes an error. The ARL both between consecutive detections of an increase and consecutive detections of a decrease is 200 weeks; hence if $\bar{p}$ remains unchanged the ARL between false alarms is 100 weeks. (These ARL's do not depend on the assumption of constant u between weeks.) The choice of decision limits was in fact based on this choice of the ARL between false alarms.

TABLE 1

Detection Probabilities and ARL's for the Shewhart Chart

$\frac{\lvert\Delta\bar{P}_0\rvert}{\sigma_0}$	r = 2	4	6	8	10	15	20	ARL
	P(R<r)							
0	0.010	0.020	0.030	0.039	0.049	0.072	0.095	200
0.5	0.037	0.073	0.11	0.14	0.17	0.25	0.32	53
1	0.11	0.21	0.30	0.38	0.44	0.59	0.69	18
2	0.48	0.73	0.86	0.93	0.96	0.99	1.0	3.6
3	0.89	0.99	1.0	1.0	1.0	1.0	1.0	1.5

TABLE 2

Values of $\lvert\Delta\bar{p}_0\rvert$ corresponding to $\lvert\Delta\bar{p}_0\rvert/\sigma_0$

$\frac{\lvert\Delta\bar{P}_0\rvert}{\sigma_0}$	$\bar{p}_0$	μ .	$\lvert\Delta\bar{p}_0\rvert$
0.5	0.02	1000	0.0022
	0.20	1000	0.0063
1.0	0.02	1000	0.0044
	0,20	1000	0.013
2.0	0.02	1000	0.0089
	0.20	1000	0.025
3.0	0.02	1000	0.013
	0.20	1000	0.038

3.2 The Cusum Chart

For this chart the cumulative sum of the T's,

$$C_t = \sum_{j=1}^{t} T_j ,$$

is plotted against t.

While $\bar{p}$ has the value $\bar{p}_0$, the T's have zero expectations, and the expected value of C_t does not change with t; hence the path of the cusum tends to be roughly horizontal. However, if $\bar{p}$ increases, the T's have positive expectations, and the expected value of C_t begins to increase with t; hence the path of the cusum then tends to slope upwards. Similarly if $\bar{p}$ decreases, the path of the cusum then tends to slope downwards.

Thus for the cusum chart, changes in the mean of T_t (and hence in p) are indicated by changes in the slope of the cusum path, the magnitude of the slope indicating the value of the mean of the T's at the corresponding location.

Changes in the slope of the cusum path may occur by chance even if $\bar{p}$ does not change, but a marked upward or downward slope in the path which persists for a sufficiently long period is evidence that the T's corresponding to that period have non-zero means,and hence that $\bar{p}$ has changed from the value $\bar{p}_0$. To assess the significance of the slope of the cusum path the following decision rule is used.

To detect an increase in the mean of the T's, and hence an increase in $\bar{p}$, a so-called 'reference value' $k > 0$ and a 'decision interval' $h > 0$ are chosen and a modified 'cusum' U_t of the T's is formed as follows: U_t is defined to be zero until a value of T_t occurs, say in week t_1, which exceeds k; from this week onwards U_t is then defined to be the cusum

$$U_t = \sum_{j=t_1}^{t} (T_j - k) , \; t = t_1, t_1 + 1, ...,$$

provided the cusum is positive; if the cusum returns to zero or becomes negative, then U_t is again defined to be zero until a further value of T_t occurs which exceeds k; the above cycle is then repeated. Thus for any value of $t \geq 1$, U_t is given by

$$U_t = \max(U_{t-1} + T_t - k\ ,\ 0)\ (U_0 = 0)$$

An increase in the mean of the T's, and hence in $\bar{p}$, is then signalled if the modified cusum U_t exceeds the limit h. We shall call U_t the upper 'decision interval' (d.i.) cusum.

The values of k and h are chosen to give an acceptable sensitivity in detecting an increase in the mean of the T's. This sensitivity is expressed in terms of ARL's. If the mean of the T's remains at zero, the detection of an increase is an error, and so we require a large ARL between such false detections. On the other hand, if a significant increase in the mean of the T's occurs, we would want to detect this quickly, and so require the ARL to detection to be small. There is a trade-off between these two requirements.

For the changes in $\bar{p}$ that we consider the (unconditional or conditional) distribution of T_t becomes approximately $N(\frac{\Delta \bar{p}_0}{\sigma_0}, 1)$ The required sensitivity in detecting the increase $\Delta\bar{p}_0/\sigma_0$ in the mean of the T's may be expressed by specifying a suitably small ARL for a particular 'critical' increase for which quick detection is desired, together with a suitably large ARL for the case of no change in the mean. Values of k and h may then be found, using the nomograms in BS5703 Part 3 (1982), which meet the above ARL specifications as closely as possible.

An alternative approach is to set k equal to half the 'critical' increase in the mean of T's and then to determine h, using the nomogram, to give the required ARL in the case of no change in the mean. This usually also gives a satisfactory ARL in the case of the 'critical' increase, but if this ARL is found to be unsatisfactory, h can be varied until an acceptable compromise between the two ARL's is achieved.

We suggest that we regard an increase of $\frac{\Delta \bar{p}_0}{\sigma_0} = 1$ in the mean of the T's as a 'critical' increase and take the values of k and h to be

$$k = 0.5\ , \qquad h = 3.5\ .$$

These values give an ARL of 200 in the case of no change in the mean and 7.4 in the case of a 'critical' increase of 1.

A corresponding lower d.i. cusum L_t of the T's is used to detect a decrease in their mean, and hence a decrease in $\bar{p}$. Thus, if k and h denote positive quantities, we calculate

$$L_t = \min(L_{t-1} + T_t + k, 0), \; t \geq 1 \; (L_0 = 0).$$

If L_t goes below - h, this is taken as evidence that the mean has become negative and hence that p has decreased. The values of k and h are chosen in the same way as before and the above suggested values for k and h apply also for the detection of a decrease in $\bar{p}$.

Estimates of the probability of the detection of an increase or decrease in p, as appropriate, by the rth week following a change, and of the corresponding ARL, are given in Table 3 for values of $|\Delta \bar{p}_0| /\sigma_0$ =0,0.5, 1, 2, 3. The probability estimates were obtained by simulation using 1000 runs and a 'worst possible' starting value of 0 for the d.i. cusum in each case; the ARL's are given in BS5703 Part 3 (1982). (See Table 2 for changes $|\Delta \bar{p}_0|$ in $\bar{p}$ corresponding the above values of $|\Delta \bar{p}_0|/\sigma_0$.)

If $\bar{p}$ remains unchanged, the ARL both between consecutive detections of an increase and consecutive detections of a decrease is 200 weeks, and hence the ARL between false alarms is 100 weeks; this is the same as for the Shewhart chart discussed earlier.

TABLE 3

Detection Probabilities and ARL's for the Cusum Chart

$\frac{\lvert \Delta \bar{p}_0 \rvert}{\sigma_0}$	P(R ≤ r): r = 2	4	6	8	10	15	20	ARL
0	0.0010	0.013	0.023	0.044	0.056	0.13	0.17	200
0.5	0.0050	0.062	0.14	0.22	0.32	0.51	0.64	22
1	0.044	0.26	0.51	0.72	0.83	0.94	0.98	7.4
2	0.36	0.90	0.99	1.0	1.0	1.0	1.0	3.0
3	0.86	1.0	1.0	1.0	1.0	1.0	1.0	2.0

Except for relatively very large changes, the cusum chart has greater sensitivity in detecting a change in $\bar{p}$ than the Shewhart chart, as can be seen by comparing Tables 1 and 3.

When a change is detected, an estimate of the week in which the change occurred can be obtained from the ordinary cusum chart of the T's which makes the cusum especially useful. This is provided by the point at which the slope of the cusum path changes in the weeks prior to the point of detection.

3.3 An Illustration

The data shown in Table 4 concern purchases of packets of tea bags by a consumer panel over a period of 52 weeks. The weeky total number of purchases of such packets is given together with the number, and proportion, of purchases of a particular brand B. The panel's overall preference for this brand is to be monitored.

For illustration purposes, we shall take the proportion of purchases of B for the first 10 weeks as the established value $\overline{p}$ of $\overline{p}$. Over this period there is no trend in the weekly estimates Y of $\overline{p}$, and so the value of $\overline{p}$ appears to be stable.

Thus we take $\overline{p}_0$ = 0.1933, and monitor the data from week 11 onwards for a change from this values.

Figure 1 shows a time plot of the weekly proportions of B purchased by the panel. There is no clear indication of a change in the level of $\overline{p}$; any such changes are obscured by the variation in the data.

Figure 2 shows the Shewhart chart - a time plot of statistics T - from week 11 onwards. Neither decision limit is reached, and so no change in the value of $\overline{p}$ is detected by this chart.

3. $\overline{p}$ is clearly signalled at week 37.

Referring to the ordinary cusum chart (figure 3) we see that, in the weeks immediately preceding week 37, the slope changes from roughly horizontal to increasing in week 31. This, then, is our estimate of the week in which the change in $\overline{p}$ first occurred.

Further, the slope of the cusum path remains approximately constant from

TABLE 4

Week	Total Number of purchases (X)	Number of purchases of brand B (Z)	Proportion of purchases of B (Y)
1	1384	275	0.1987
2	1265	225	0.1779
3	1370	237	0.1730
4	1371	284	0.2071
5	1362	269	0.1975
6	1318	256	0.1942
7	1308	255	0.1950
8	1354	262	0.1935
9	1348	268	0.1988
10	1339	263	0.1964
11	1320	254	0.1924
12	1466	285	0.1944
13	1335	238	0.1783
14	1405	273	0.1943
15	1395	268	0.1921
16	1396	255	0.1827
17	1469	278	0.1892
18	1487	284	0.1910
19	1438	276	0.1919
20	858	167	0.1946
21	1278	250	0.1956
22	1138	210	0.1845
23	1498	302	0.2016
24	1421	267	0.1879
25	1463	273	0.1866
26	1409	276	0.1959
27	1371	275	02006
28	1417	265	0.1870
29	1447	294	0.2032
30	1414	265	0.1874
31	1401	290	0.2070
32	1391	288	0.2070
33	1369	275	0.2009
34	1448	296	0.2044
35	1332	274	0.2057
36	1150	225	0.1957
37	1384	299	0.2160
38	1386	273	0.1970
39	1361	296	0.2175
40	1437	277	0.1928
41	1227	230	0.1874
42	1413	272	0.1925
43	1360	246	0.1809
44	1349	239	0.1772
45	1366	283	0.2072
46	1299	244	0.1878
47	1321	236	0.1787
48	1242	228	0.1836
49	1262	246	0.1949
50	1308	279	0.2133
51	1248	244	0.1955
52	1150	213	0.1852

FIGURE 1.
PROPORTION OF PURCHASES OF BRAND B

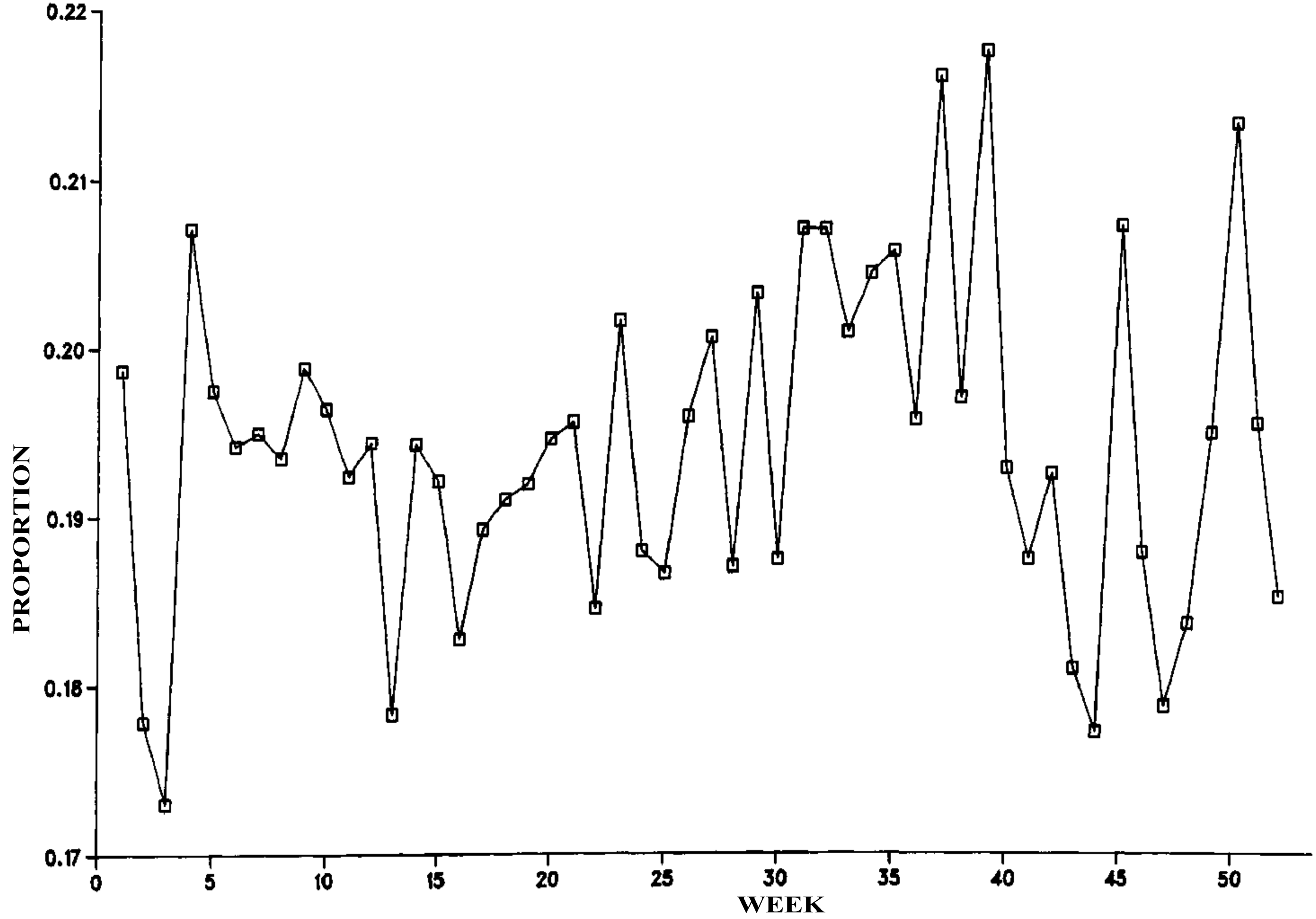


FIGURE 2.
SHEWHART CHART

DECISION LIMIT

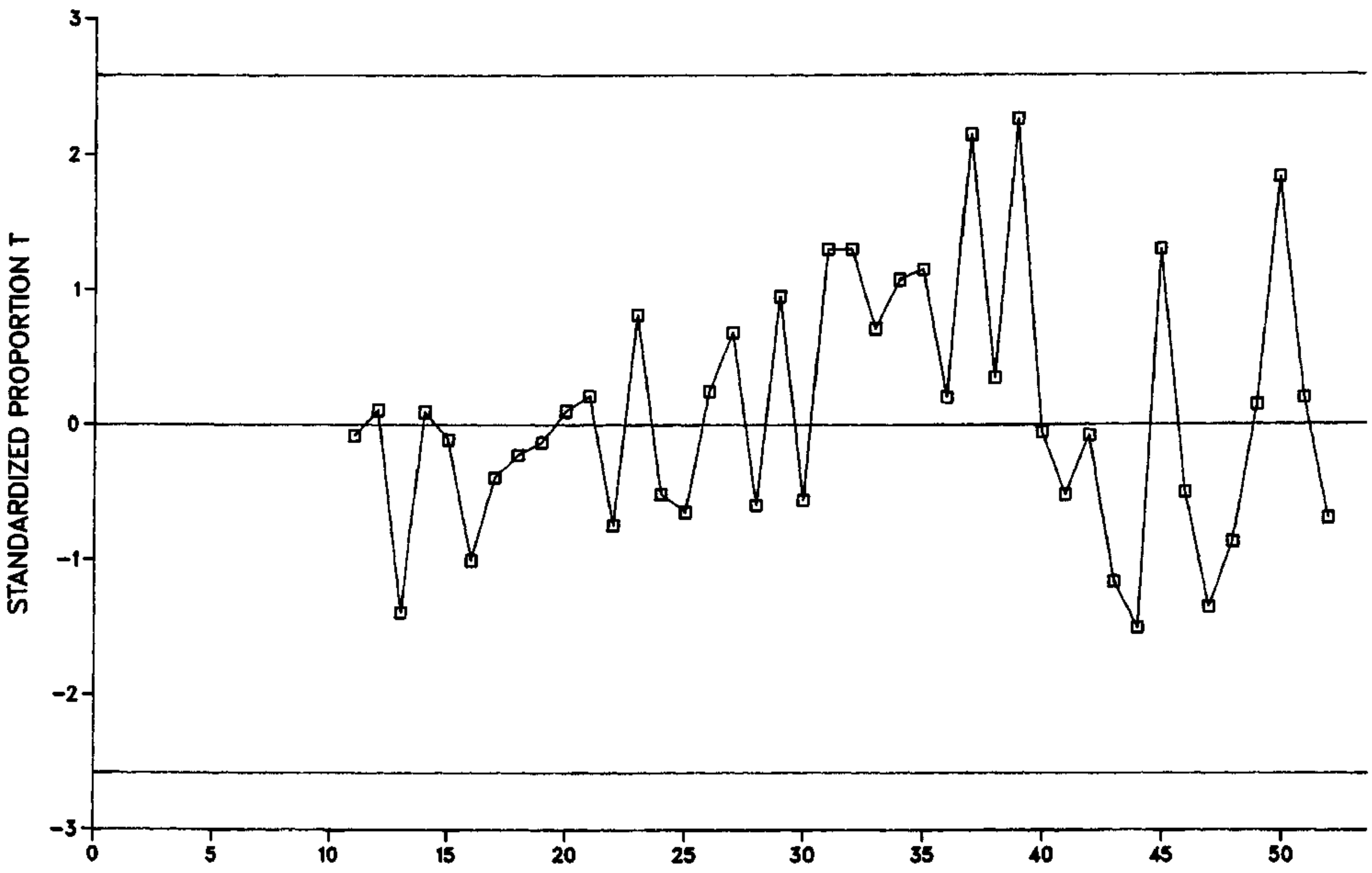


FIGURE 3.
ORDINARY CUSUM CHART

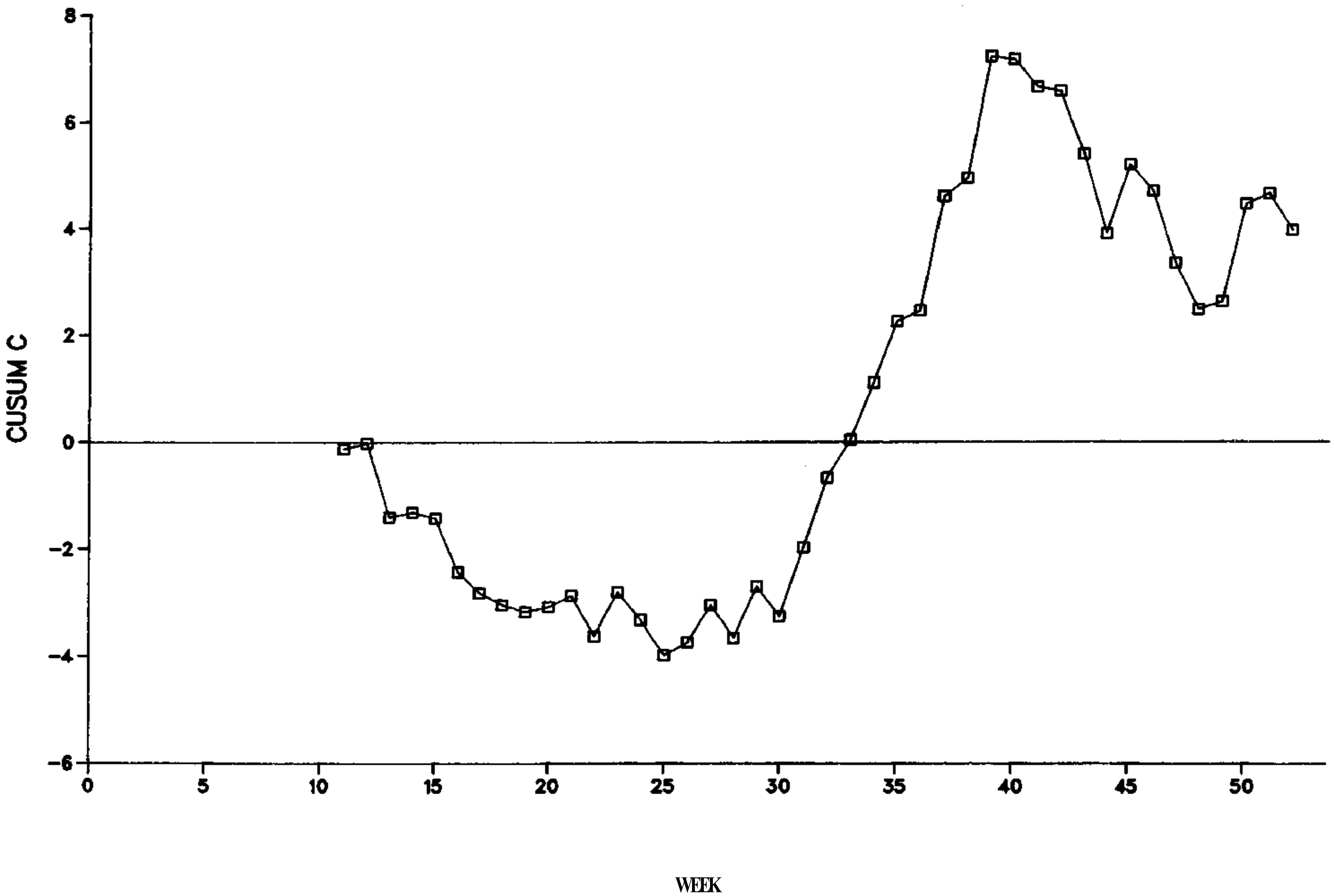
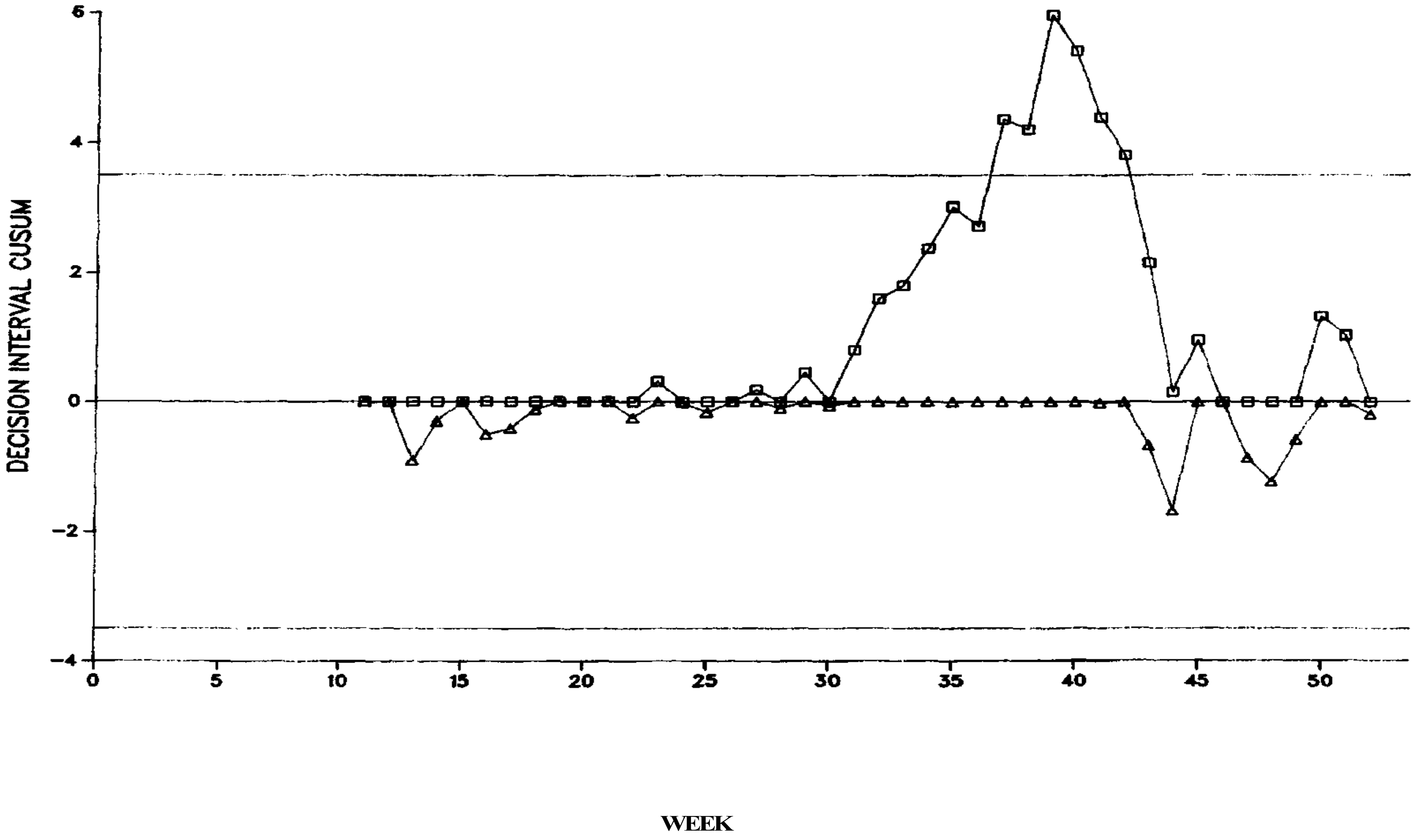


FIGURE 4.
DECISION INTERVAL CUSUM CHART



wek 31 to week 39, after which it begins to decrease. Hence, the increas;-ed level of $\overline{P}$ appears to be held over this period, after which it begins to fall off. The proportion of B purchased over this period is 0.2059, which provides an estimate of the increased level of $\overline{P}$.

4. Adapting to a **Gradual Change in the Panel's Preference for Brand B**

(Here we denote the values of μ and $\overline{P}$ in week t by μ_t and p_t, respect-ively.)

Apart from rapid changes in the value of $\overline{P}$ that may occur during promotion-al activity, the value of $\overline{P}$ may change slowly over a comparatively long period of time due to many small background influences on the panel's overall preference for B, or because the membership of the panel gradually changes.

The above procedures for detecting a change in the value of $\overline{P}$ from an est-ablished value p will eventually detect a gradual drift away from $\overline{P}_0$ when it becomes sufficiently large. If we wish to detect only rapid, local local changes in p, then we shall need to compare the estimate of p in a given week with an estimate of the current value of $\overline{P}$ just prior to the given week.

An estimate of the current value of $\overline{p}$ at a given time, which adapts to gradually changing $\overline{p}$, is provided by an exponential smoothing of the week-ly estimates. At week t, the exponentially smoothed value $\tilde{Y}_t$ of the avail-able weekly estimate $Y_1, \ldots, Y_t$ is given by

$$\tilde{Y}_t = (1-\alpha)Y_t + \tilde{Y}_{t-1}$$

where $0 < a < 1$ is a chosen smoothing constant; the value of a controls the rate at which $\tilde{Y}$ adapts to changing p, although the more rapidly it is made to adapt the greater its variance becomes. To start up the smoothing process, the initial smoothed value $\tilde{Y}_0$ is taken to be the 'established' value $\overline{p}_0$ of $\overline{p}$ at that time, as given, say, by the mean of the Y's over preceding weeks.

The statistics T may then be modified to measure only local changes in the weekly estimates Y of $\overline{p}$: we replace $\overline{p}_0$ in the statistic T_t corresponding to week t by the estimate $\tilde{Y}_{t-1}$ of the current value of $\overline{p}$ at week t - 1, giving

$$\tilde{T}_t = \frac{Y_t - \tilde{Y}_{t-1}}{\sqrt{\dfrac{\tilde{Y}_{t-1}(1-\tilde{Y}_{t-1})}{x_t}}}$$

We now consider weekly changes in $\bar{p}$ which are small compared with the value of $\bar{p}$, but comparable in magnitude to $\sqrt{\frac{\bar{p}(1-\bar{p})}{\mu}}$, the approximate unconditional standard deviation of Y. We shall express the changes as multiples of the approximate standard deviation of Y in week 0, and write

$$\bar{p}_j - \bar{p}_{j-1} = \lambda_j \sqrt{\frac{\bar{p}_o(1-\bar{p}_o)}{\mu_{.0}}}\ ,\ j = 1,2,...$$

We show in Appendix 2 that, for large values of the μ 's, both unconditionally and conditionally on the realised values of the X's, the statistic $\tilde{T}_t^1 = \frac{\tilde{T}_t}{\sqrt{1+C_t^2}}$ is distributed approximation $N\left(\frac{\delta_t}{\sqrt{1+\gamma_t^2}}, 1\right)$, where

$$C_t^2 = \frac{1-\alpha}{1+\alpha} X_t \ S_t\left(\frac{1}{X}; \alpha^2\right),$$

$$Y_t^2 = \frac{1-\alpha}{1+\alpha} \mu_{.t} \ S_t\left(\frac{1}{\mu}; \alpha^2\right),$$

$$\delta_t = \left(\frac{\mu_{.t}}{\mu_{.0}}\right)^{\frac{1}{2}} \sum_{j=0}^{t-1} \alpha^j \lambda_{t-j},$$

and where the S's, which denote the exponentially smoothed values of the quantities shown in brackets together with the smoothing constants, are given by

$$S_t\left(\frac{1}{x};\alpha^2\right) = (1-\alpha^2)\frac{1}{x_{t2}} + \alpha^2 S_{t-1}\left(\frac{1}{x};\alpha^2\right),\ S_0\left(\frac{1}{x};\alpha^2\right) = 0,$$

$$S_t\left(\frac{1}{\mu};\alpha^2\right) = (1-\alpha^2)\frac{1}{\mu_t} + \alpha^2 S_{t-1}\left(\frac{1}{\mu};\alpha^2\right),\ S_0\left(\frac{1}{\mu};\alpha^2\right) = 0.$$

4.1 Case where a is close to 1

If a is close to 1, then c_t^2 and y_t^2 are both close to zero, and hence the statistic $\tilde{T}_t$ is distributed approximately $N(\delta_t, 1)$ for large μ 's, both unconditionally and conditionally on the realised values of the X's.

Further, is $\overline{p}$ is changing so slowly that $\lambda_j \doteq 0, j = 1, 2, \ldots,$ and $\delta_{t.}$ is of negligible size, then, under the above conditions, the distribution of $\tilde{T}_t$ is approximately $N(0,1)$.

Suppose now that,in week s, $\overline{p}$ changes by a 'large' amount, and thereafter continues to change very slowly, so that $\lambda_j = 0$ except for j = s. Then for $t \geq s$, $\tilde{T}_t$ is distributed approximately $N\left(\alpha^{t-s}\lambda_s\left(\frac{\mu.t}{\mu.0}\right)^{\frac{1}{2}}, 1\right)$, under the above

Conditions. Thus the mean of $\tilde{T}_t$ suddenly changes to $\lambda_s\left(\frac{\mu.t}{\mu.0}\right)^{\frac{1}{2}}$ in the week of the change, and then gradually decays back to zero as the estimator $\tilde{Y}$ gradually adapts to the new 'level' of $\overline{p}$.

Thus to detect only rapid, local changes in the value of $\overline{p}$, we may monitor the panel's weekly purchasing behaviour using Shewhart and cusum charts of
the modified statistics $\tilde{T}$.

Unlike the T's the $\tilde{T}$'s are autocorrelated. We show in Appendix 2 that if a is close to 1, then for large $\mu.$'s, both unconditionally and conditionally on the realised X's, the correlation between $\tilde{T}_{t+t}$ and $\tilde{T}_t$ is approximately

$$-\left(\frac{\mu_{\cdot t+\tau}}{\mu_{.t}}\right)^{\frac{1}{2}}(1-\alpha)\alpha^{r-1} + (\mu_{.t}\,\mu_{.t+\tau})^{\frac{1}{2}}\; S_{t-1}\left(\frac{1}{\mu};\alpha^2\right)\left(\frac{1-\alpha}{1+\alpha}\right)\alpha^{\tau}$$

$$= -\left(\frac{1-\alpha}{1+\alpha}\right)\alpha^{\tau-1}(1+\alpha^{2t-1}) \text{, if the } \mu\text{'s are approximately equal.}$$

These autocorrelations are in fact quite small. For example, if a = 0.9, and if the μ 's are approximately equal, the autocorrelation at lag 1 (the largest autocorrelation) ranges from about -0.1 for t = 1 to about - 0.05 for large t. Hence, for large a, the presence of autocorrelations amongst the T's should have only a slight effect on the behaviour of the Shewhart and cusum charts of these statistics.

4.2 Choice of α

The choice of a depends on what magnitude of gradual change in the value of $\overline{p}$ we do not wish to detect.

Suppose that we do not wish to detect a change in $\overline{p}$ if the weekly amounts by which it chanqes are such that $|\lambda_j| \leq \lambda_m$, j = 1,2,... . For such changes, the mean of T_t is bounded (for large μ 's) as follows

$$\left|E(\tilde{T}_t)\right| = \left|\frac{\delta_t}{\sqrt{1+\gamma_t^2}}\right| \leq |\delta_t|$$

$$= \left|\left(\frac{\mu_{.t}}{\mu_{.0}}\right)^{\frac{1}{2}} \sum_{j=0}^{t-1} \alpha^j \lambda_{t-j}\right| \leq \frac{\lambda_m}{1-\alpha}\left(\frac{\max \mu_{.j}}{\mu_{.0}}\right)^{\frac{1}{2}} .$$

Now, if $\bar{p}$ does not change, the means of the $\tilde{T}$'s are zero. Hence, if p is changing as above, to keep the risk of detecting a change with a cusum chart close to what it would be for unchanging $\bar{p}$, we need to choose a so that the means of the $\tilde{T}$'s are close to zero, say less than 0.1 in absolute value. Thus we should choose a so that

$$\frac{\lambda_m}{1-\alpha} \cdot \left(\frac{\max \mu_{.j}}{\mu_{.0}}\right)^{\frac{1}{2}} < 0.1 ,$$

$$\Rightarrow \alpha < 1 - 10\lambda_m\left(\frac{\max \mu_{.j}}{\mu_{.0}}\right)^{\frac{1}{2}}$$

If $|\nabla \bar{p}|_m$ denotes the maximum weekly change in p that we do not wish to detect, then

$$\lambda_m = \left|\nabla \bar{P}\right|_m \Big/ \sqrt{\frac{\bar{p}_0(1-\bar{p}_0)}{\mu_{\cdot 0}}} ,$$

and hence we require

$$\alpha < 1 - \frac{10\left|\nabla \bar{P}\right|_m}{\sqrt{\frac{\bar{p}_0(1-\bar{p}_0)}{\mu_{\cdot 0}}}} ,$$

where $\mu_m = \max \mu$.

As an illustration, suppose $\bar{p} \backsimeq 0.2$ and μ . 1000. Then, if we do not wish to detect a change if $\bar{p}$ gradually changes by 0.005 over 50 weeks, we should take a < 0.92; whereas, if we do not wish to detect a change if $\bar{p}$ gradually changes by 0.015 over 50 weeks, we should take a < 0.76.

If the required value of a is not sufficiently close to 1 to justify taking

$C_t^2 = 0$, then we should have to use the statistics $\tilde{T}_t^1 = \frac{\tilde{T}_t}{\sqrt{1+C_t^2}}$ instead of $\tilde{T}_t$

4.3 Modified Decision Rules for Cusum Charts of the $\tilde{T}$'s

The sensitivity of the cusum of the $\tilde{T}$'s in detecting a change in their means is slightly different to that for the T's.

Thus, if $\bar{p}$ is constant or is only slowly changing, and if μ is approximately constant, the small negative autocorrelations amongst the $\tilde{T}$'s (for large a) slightly reduce the variances of their upper and lower d.i. cusums compared with the independent T's. Hence each of these cusums reaches its decision limit slightly less frequently than those for independent T 's, and hence the ARL between false alarms is slightly greater for the $\tilde{T}$'s than for the T's.

On the other hand, if $\bar{p}$ suddenly changes to a new level, the appropriate d.i. cusum moves towards the decision limit slightly less rapidly for the $\tilde{T}$'s than for the T's. This is because the statistics $\tilde{Y}$ gradually adapt to the new level of $\bar{p}$ and so gradually reduce the magnitude of the $\tilde{T}$'s, whereas the T's are not so affected. Hence the sensitivity in detecting a sudden change in $\bar{p}$ is slightly less for the cusum of the $\tilde{T}$'s than for the T's.

We can increase the sensitivity of the cusum of the $\tilde{T}$'s in detecting a sudden change in p by reducing the value of the decision interval h. A preliminary simulation study indicates that for $a = 0.9$, and taking $k = 0.5$, h should be reduced to about 3.2 to maintain an ARL of about 100 weeks between false alarms when $\bar{p}$ remains unchanged and y is constant.

A more extensive simulation study needs to be carried out to obtain estimates, as in Table 3, of the ARL's to detection, and detection probabilities, following a sudden change in the value of $\bar{p}$ of various magnitudes.

4.4 An Illustration

We use the data of the previous illustration (section 3.3) to demonstrate use of Shewhart and cusum charts of the modified statistics $\tilde{T}$.

FIGURE 5.
ADAPTIVE SHEWHART CHART

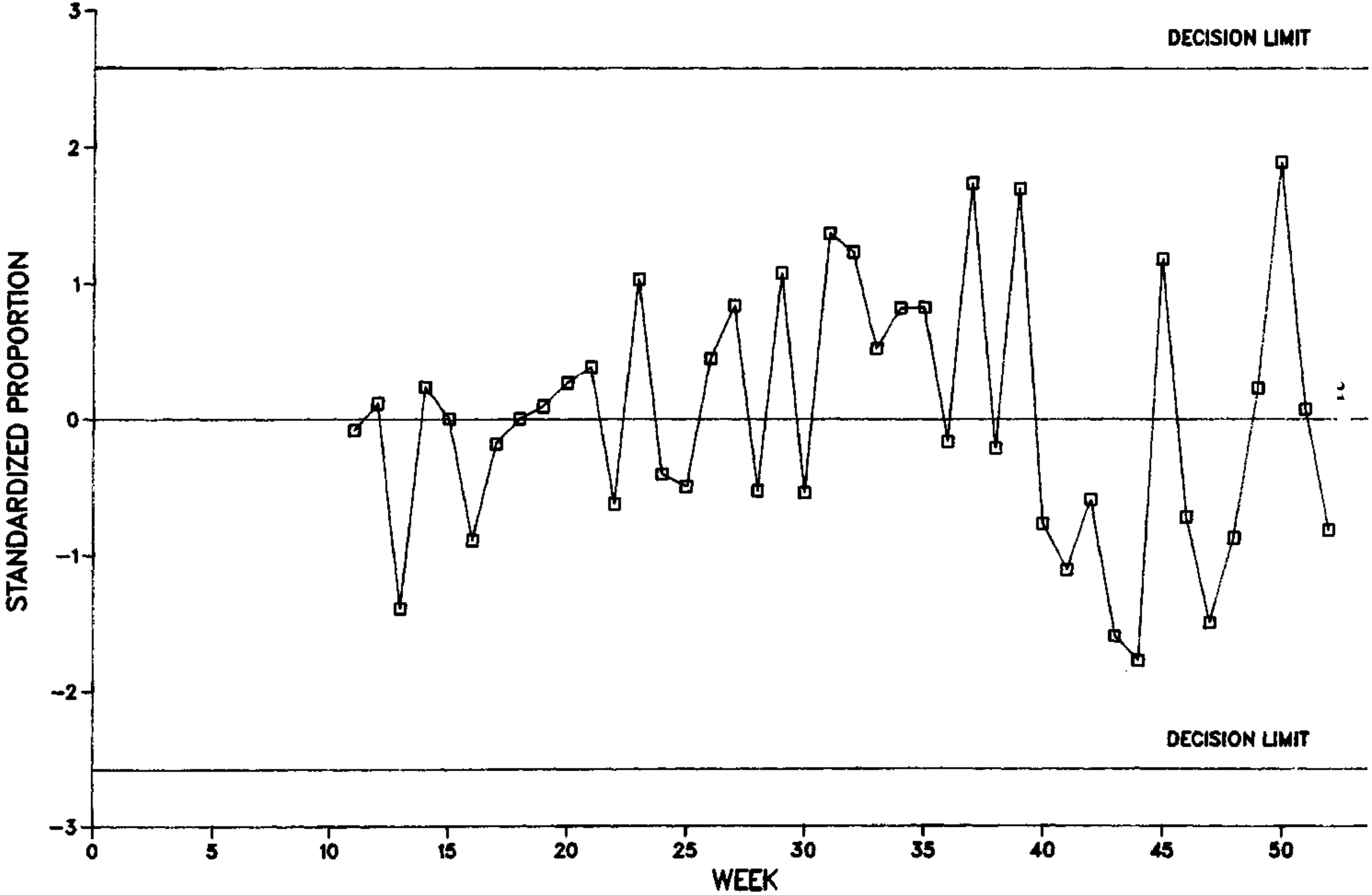


FIGURE 6.
ADAPTIVE ORDINARY CUSUM CHART

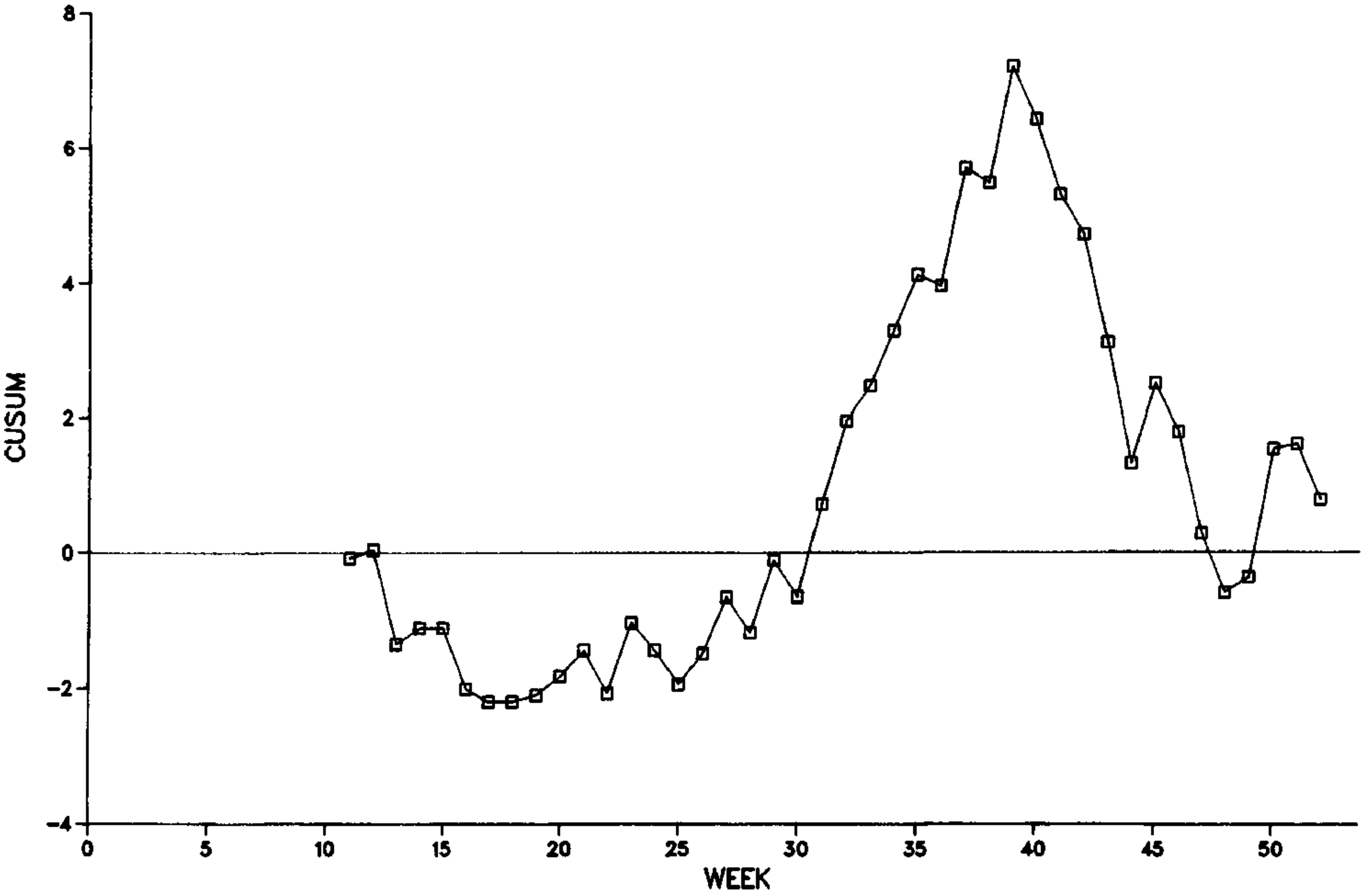
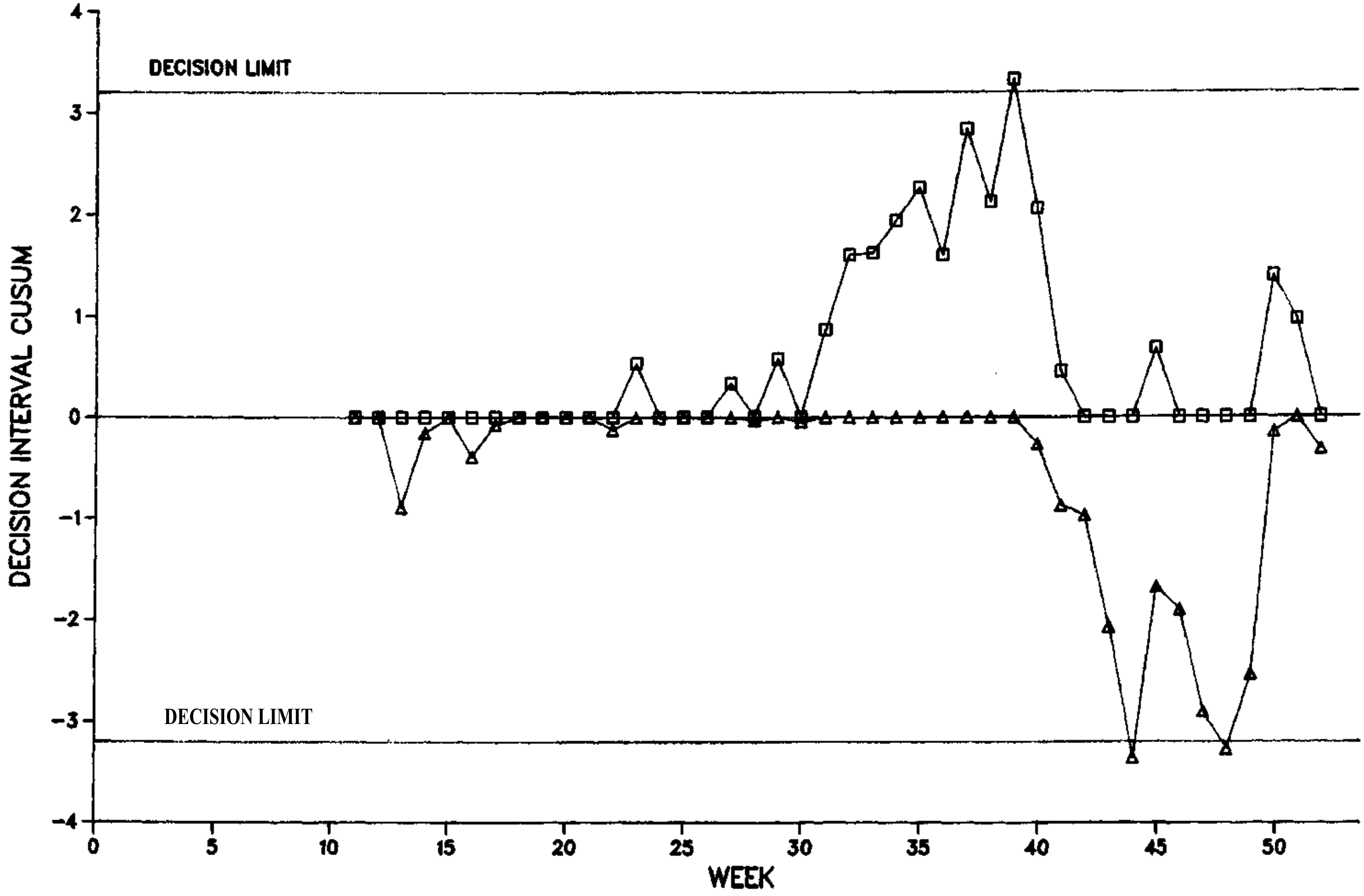


FIGURE 7.
ADAPTIVE DECISION INTERVAL CUSUM CHART



We take as the value of the smoothing constant $\alpha = 0.9$, which will allow for a gradual change in $\bar{p}$ of about 0.5% over a period of a year.

We take as the established value of $\bar{p}$ at the start of the smoothing process $\bar{p}_0 = 0.1933$ (the proportion of purchases of B for the first 10 weeks), and monitor the data for a rapid change in $\bar{p}$ from week 11 onwards.

The results are very similar to those for the T's. No change in p is detected by the Shewhart chart (figure 5), but the d.i. cusum chart (figure 7) clearly detects an increase at week 39. Also from the ordinary cusum chart (figure 6), as before, we estimate that the sudden increase in $\bar{p}$ first occurred in week 31, and that the increased level was pproximately held until week 39.

5. **Monitoring by Weight of Product Purchased**

Suppose that the product is available in various packet sizes. We may regard the different packet sizes of a given brand as brands in their own right; suppose that there are g such brands and that brands 1 to b correspond to different packet sizes of the particular brand B of interest. Let w_r be the weight of a packet of the r^{th} brand, r = 1,. . .,g.

Let μ_i, $i = 1, \ldots, n$, and $\mu.$ be as previously defined, but here let p_{rj}, $i = 1,\ldots,n$, $r = 1, \ldots, g$, denote the probability of member i choosing the r^{th} brand on a given purchase occasion, $\sum_{r=1}^{g} p_{rj} = 1$. The probability of the panel choosing the r^{th} brand on a given occasion is then

$$\bar{P}_r = \sum_{i=1}^{n} \frac{\mu_i}{\mu.} P_{ri}, \quad r = 1, \ldots, g, \quad \sum_{r=1}^{g} \bar{P}_r = 1 .$$

Let X_t, $t = 1, 2, \ldots$, be as previously defined but here let

Z_{rt} = overall number of occasions on which the r^{th} brand is purchase by the panel in week t .

$Y_{rt} = \frac{Z_{rt}}{X_t}$ overall proportion of occasions on which the r^{th} brand is purchased by the panel in week t.

$$and \quad W_{rt} = \frac{W_r Z_{rt}}{\sum_{s=1}^{g} W_s Z_{st}} = \frac{W_r Y_{rt}}{\sum_{s=1}^{g} W_s Y_{st}} = \text{overall proportion by weight of the rth brand purchased in week t}$$

The overall proportion by weight of brand B purchased in week t is then $W_t^B = \sum_{r=1}^{b} W_{rt}$.

We show in Appendix 3 that for sufficiently large realised X , the conditional distribution of W_t^B given X_t is approximately normal with

$$E(W_t^B \mid X_t) = \sum_{r=1}^{b} P_r^w = \frac{\sum_{r=1}^{b} W_r \bar{P}_r}{\sum_{s=1}^{g} W_s \bar{P}_s} = P^B \text{ , say}$$

and

$$\text{var } (W^B{}_t | X_t) = x \frac{1}{X_t \sum_{s=1}^{g} W_s \bar{P}_s} \left\{ \sum_{r=1}^{b} W_r P_r^w - P^B \left(2 \sum_{r=1}^{b} W_s P_s^w \right) \right\} = (\sigma^B)^2 \text{ , say}$$

where

$$P_r^w = \frac{W_r \bar{P}_r}{\sum_{s=1}^{g} W_s \bar{P}_s} .$$

p^B is thus the approximate expected proportion by weight of brand B purchased by the panel in a given week t, and does not depend on X_t .

The value of p^B may be monitored from week to week in the same way as $\bar{p}$. Thus to detect a change from an established value $p^B{}_o$ (corresponding to established values p_{r0} of $\bar{p}_r$, $r = 1, \ldots, g$), the monitoring would be based on the statistics

$$T_t^w = \frac{W_t^B - P_O^B}{\sigma_O^B} ,$$

where $u^{P}{}_{o}$ ' is the value of o^{R} corresponding to the established values of the $\bar{p}_r$.

If the situation remains stable, then for large μ 's, both unconditionally and conditionally on the realised X's, the T^{W} 's are independent and each is

distributed approximately $N(0,1)$; and if there is a change in p^{B} the mean of the T^{W} 's changes accordingly.

To adapt to a gradually changing p^{B} in order to detect only rapid, local changes in its value, the monitoring would be based on the statistics

$$\tilde{T}_t^{W} = \frac{W_t^{B} - \tilde{P}_{t-1}^{B}}{\tilde{\sigma}_{t-1}^{B}},$$

where $\tilde{P}^{B}{}_{t-1}$ and $\tilde{\sigma}^{B}{}_{t-1}$ are estimates of the current values of p^{B} and σ^{B} at week t-1. The estimates of p^{B} and σ^{B} at week t are obtained by replacing the $\tilde{P}_r$, r = l,...,g, in the expressions for these quantities by the corresponding exponentially smoothed estimates $\tilde{Y}_{rt}$, given by

$$\tilde{Y}_{rt} = (1-\alpha)Y_{rt} + \alpha\, \tilde{Y}_{r,t-1} \quad r = 1,...,g .$$

If p^{B} is constant or slowly changing, and if a is close to 1, then for large μ 's, both unconditionally and conditionally on the realised X's, the $\tilde{T}^{\sim W}$'s are distributed approximately $N(0,1)$; and if p^{B} changes rapidly the means of the $\tilde{T}^{\sim W'}$ s change accordingly.

Note that, apart from the b 'pseudo-brands' which make up the particular brand B of interest, the other 'pseudo-brands' may be regrouped in terms of packet size, which may be computationally more convenient.

The behaviour of these procedures for monitoring by weight (which can also be applied to expenditure) have not yet been fully investigated.

Reference

BS 5703 Part 3 (1982). Data analysis and quality control using cusum techniques. British Standards Institution.

APPENDIX 1

Distribution of T_t when $\bar{p} = \bar{p}_1$

Now T_t can be written

$$T_t = \frac{Y_t - \bar{P}_0}{\sqrt{\dfrac{\bar{P}_0(1-\bar{P}_0)}{X_t}}} = \sqrt{\frac{\bar{P}_1(1-\bar{P}_1)}{\bar{P}_0(1-\bar{P}_0)}}\left[\frac{Y_t - \bar{P}_1}{\sqrt{\dfrac{\bar{P}_1(1-\bar{P}_1)}{X_t}}}\right] + \sqrt{\frac{X_t}{\mu.}}\left[\frac{\bar{P}_1 - \bar{P}_0}{\sqrt{\dfrac{\bar{P}_0(1-\bar{P}_0)}{\mu.}}}\right].$$

We consider a change in $\bar{p}$ of magnitude

$$\Delta\bar{P}_0 = \bar{P}_1 - \bar{P}_0 = \lambda\sqrt{\frac{\bar{P}_0(1-\bar{P}_0)}{\mu.}} \text{ , where } \lambda \text{ is fixed .}$$

We have the following limit results

(i) If $\bar{p} = \bar{p}_1$, then as $\mu \to \infty$

$$\frac{Y_t - \bar{P}_1}{\sqrt{\dfrac{\bar{P}_1(1-\bar{P}_1)}{X_t}}} \xrightarrow{d} Z \sim N(0,1)$$ unconditionally, or conditionally on the realised X_t

(ii) Now $X_t \sim Po(\mu)$.

Hence $E\left[\frac{X_t}{\mu.}\right] = 1$ for all u. and

$$\text{Var } E\left[\frac{X_t}{\mu.}\right] = \frac{1}{\mu.} \to 0 \text{ as } \mu \to \infty$$

$$\Rightarrow \frac{X_t}{\mu} \xrightarrow{P} 1 \text{ as } \mu \to \infty .$$

(iii) As $\mu \to \infty$, $\Delta p_0 \to 0$ and hence $\dfrac{\bar{P}_1(1-\bar{P}_1)}{\bar{P}_0(1-\bar{P}_0)} - 1$.

Hence as $\mu \to \infty$

$$T_t \xrightarrow{d} Z + \lambda \sim N(\lambda,1) \ .$$

Hence for large μ , both unconditonally and conditionally on the realised value of X_t, T_t is distributed approximately $N(\lambda, 1)$, where $\lambda = \dfrac{\Delta\bar{P}_0}{\sqrt{\dfrac{\bar{P}_0(1-\bar{P}_0)}{\mu}}}$.

For the typical values $\mu = 1000$, $\lambda = 1$, $\bar{p}_0 = 0.02$ and 0.2, we note that:

1) if $\bar{p} = \bar{p}_1$, the distribution of $\dfrac{Y_t - \bar{P}_1}{\sqrt{\dfrac{\bar{P}_1(1-\bar{P}_1)}{X_t}}}$ is well approximated by the N(0,1) distribution;

2) since X_t is approximately normally distributed, X_t/μ will almost certainly fall in the range $1 \pm 3/\sqrt{\mu} = 1 \pm 0.095$, and so $\sqrt{x_t/2\mu}$ will almost certainly fall in the range $1 + 0.046$ and so be close to the above limiting value;

3) $\sqrt{\dfrac{\bar{P}_1(1-\bar{P}_1)}{\bar{P}_0(1-\bar{P}_0)}} = \begin{cases} 1.10 & \text{if } \bar{P}_0 = 0.02 \\ 1.02 & \text{if } \bar{P}_0 = 0.2 \end{cases}$; these values are reasonably close to the above limiting value for this quantity.

Thus the above approximation to the distribution of T_t should be adequate for the values of μ , $\bar{p}_0$ and λ which we consider.

APPENDIX 2

2.1 Asymptotic Distribution of $\tilde{T}_t$

Here we denote the values of μ and p in week t by μ_t and $\bar{p}_t$, respectively.

Now $\tilde{Y}_t$ is given by

$$Y_t = (1-\alpha)(Y_t + \alpha Y_{t-1} + \alpha^2 / Y_{t-2} + \ldots\ldots + \alpha\ t\text{-}1Y1 + a^{t-}Y_0 .$$

We shall take $\tilde{Y}_0$ to be a constant 'starting' value p_0, representing the 'established' value of $\bar{p}$ in week 0. Hence

$$E(Y_t \ IX_1,\ldots,X_t) = (1\text{-}a)(\bar{p}_t + \alpha\ \bar{p}_{t\_1} + a^2\ \bar{p}_{t\_2} + \ldots + a^{t-1}\ \bar{p}_1) + \alpha^t \text{-} \bar{p}_0$$

$$= S_t(\bar{p};\alpha) ,$$

the exponentially smoothed value of p at week t with smoothing constant a, where $S_0(\bar{p};\alpha) = \bar{p}_0$.

And var $(\tilde{Y}_t \mid X_1,\ldots X_t) = (1-\alpha)^2 \left\{ \frac{\bar{P}_t(1-\bar{P}_t)}{X_t} + \alpha^2 \frac{\bar{P}_{t-1}(1-\bar{P}_{t-1})}{X_{t-1}} + \alpha^4 \frac{\bar{P}_{t-2}\ (1-\bar{P}_{t-2})}{X_{t-2}} \right.$

$$\left. + \ldots + \alpha^{2t-2} \frac{\bar{P}_t(1-\bar{P}_t)}{X_t} \right\} = \frac{1-\alpha}{1+\alpha} S_t\left(\frac{\bar{P}(1-\bar{P}\ )}{X} ; \alpha^2 \right),$$

where

$$S_t\left(\frac{\bar{P}(1-\bar{P}\ )}{X} ; \alpha^2 \right) = (1-\alpha^2) \frac{\bar{P}_t(1-\bar{P}_t)}{X_t} + \alpha^2 S_{t-1}\left(\frac{\bar{P}(1-\bar{P}\ )}{X} ; \alpha^2 \right)$$

is the exponentially smoothed value of $\frac{\bar{P}(1-\bar{P}\ )}{X}$ at week t with smoothing constant α^2, where $S_0\left(\frac{\bar{P}(1-\bar{P}\ )}{X} ; \alpha^2 \right)$

Now $\tilde{T}_t$ can be written

$$\tilde{T}_t = \frac{Y_t - \tilde{Y}_{t-1}}{\sqrt{\frac{\tilde{Y}_{t-1}(1-\tilde{Y}_{t-1})}{X_t}}} = b_t\,(U_t' - C_t' V_t' + r_t \delta_t')$$

where
$$U'_t = \frac{Y_t - \bar{P}_t}{\left\{ \frac{\bar{P}_t(1-\bar{P}_t)}{X_t} \right\}},$$

$$V'_t = \frac{\tilde{Y}_{t-1} - \tilde{S}_{t-1}(\bar{P};\alpha)}{\left\{ \frac{1-\alpha}{1+\alpha} S_{t-1}\left(\frac{\bar{P}(1-\bar{P})}{X};\alpha^2 \right) \right\}^{\frac{1}{2}}},$$

$$b_t = \left\{ \frac{\bar{P}_t(1-\bar{P}_t)}{\tilde{Y}_{t-1}(1-\tilde{Y}_{t-1})} \right\}^{\frac{1}{2}},$$

$$C'_t = \left\{ \frac{(1-\alpha)S_{t-1}\left(\frac{\bar{P}(1-\bar{P})}{X};\alpha^2 \right)}{1+\alpha \frac{\bar{P}_t(1-\bar{P}_t)}{X_t}} \right\}^{\frac{1}{2}}$$

$$r_t = \left\{ \frac{X_t}{\mu._t} \right\}^{\frac{1}{2}},$$

$$\delta_t = \frac{\bar{P}_t - S_{t-1}(\bar{P};\alpha)}{\left\{ \frac{\bar{P}_t(1-\bar{P}_t)}{\mu._t} \right\}^{\frac{1}{2}}}.$$

Note that U'_t and V'_t are independent both unconditionally and conditionally on $X_1, \ldots\ldots, X_t$.

We have the following limit results.

(i) As $\mu_t \to \infty$, $U'_t \xrightarrow{d} N(0,1)$, both unconditionally and conditionally on the realised value of X_t.

(ii) As $X_1, \ldots, X_{t-1} \to \infty$, $V'_t \xrightarrow{d} N(0,1)$ conditionally on $X_1, \ldots\ldots, X_{t-1}$. Also, since $X_1, \ldots X_{t-1} \xrightarrow{p} \infty$ as $\mu_1, \ldots \ldots \mu_{t-1} \to \infty$, it follows that $V'_t \xrightarrow{d} N(0,1)$ unconditionally as $\mu_1, \ldots, u_{t-1} \to \infty$.

(iii) As $X_1, \ldots, X_{t-1} \to \infty$, $\mathrm{var}(Y_{t-1} | X_1, \ldots\ldots, X_{t-1}) \to 0$, and hence

$Y_{t-1} \xrightarrow{p} S_{t-1}(\bar{p};a)$ conditionally on $X_1, \ldots, X_{t-1}$. And since

$X_1, \ldots X_{t-1} \xrightarrow{p} \infty$ as $\mu_1, \ldots \mu_{t-1} \longrightarrow \infty$, it follows that $\tilde{Y}_{t-1} \xrightarrow{p} S_{t-1}(\bar{p};a)$ unconditionally on $\mu_1, \ldots, \mu_{t-1} \longrightarrow \infty$.(The convergence is uniform in the values of $\bar{p}_1, \ldots, \bar{p}_{t-1}$.)

Hence as $\mu_1, \ldots, \mu_{t-1} \longrightarrow \infty$

$$b_t \xrightarrow{p} \left\{ \frac{\bar{P}_t(1-\bar{P}_t)}{S_{t-1}(\bar{P};\alpha)(1-S_{t-1}(\bar{P};\alpha))} \right\}^{\frac{1}{2}} = \beta_t \text{, say ,}$$

both unconditionally and conditionally on the realised values of $X_1, \ldots X_{t-1}$ (and uniformly in the $\bar{p}$'s).

(iv) As $\mu_t \longrightarrow \infty$, $r_t \xrightarrow{p} 1$ (see Appendix 1).

(v) Let $\mu_0, \ldots, \mu_t \longrightarrow \infty$ with fixed ratios $\theta_j = \mu_j / \mu_0$, $0 < \theta_j < \infty$, j= i,....... t.

$$\text{Then } C_t' = \left\{ \frac{(1-\alpha)\frac{\mu_{.t}}{\mu_0} S_{t-1}\left(\frac{\mu}{X}\frac{\mu_0}{\mu}\bar{p}(1-\bar{p});\alpha^2\right)}{(1+\alpha)\frac{\mu_t}{X_t}\bar{p}_t(1-\bar{p}_t)} \right\}^{\frac{1}{2}}$$

$$\xrightarrow{p} \left\{ \frac{(1-\alpha)\theta_t S_{t-1}\left(\frac{1}{\theta}\bar{p}(1-\bar{p})\,;\alpha^2\right)}{(1+\alpha)\,\bar{p}_t(1-\bar{p}_t)} \right\}^{\frac{1}{2}} = \gamma_t' \text{, say}$$

(The convergence is uniform in the p's.)

(vi) Consider now weekly changes in p of magnitude

$$V\bar{P}_j = \bar{P}_j - \bar{P}_{j-1} = \lambda_j \sqrt{\frac{\bar{P}_0(1-\bar{P}_0)}{\mu_{.0}}} \,,$$

j = 1,2,..., where the λ's are fixed.

Then as $\mu_{.0}, \mu_{.1}, \ldots \longrightarrow \infty$, $V\bar{p}_j \longrightarrow 0$ and hence $\bar{p}_j \longrightarrow \bar{p}_0$ j = 1,2,.... ; and as $\bar{p}_j \longrightarrow \bar{p}_0$, $j = 1, \ldots, t$, $\beta_t \longrightarrow 1$, since $S_{t-1}(\bar{p};\alpha) \longrightarrow \bar{p}_0$, and

$$\gamma_t' \rightarrow \left\{ \frac{(1-\alpha)\theta_t S_{t-1}\left(\frac{1}{\theta};\alpha^2\right)}{1+\alpha} \right\}^{\frac{1}{2}} = \gamma_t \text{ , say .}$$

Since, for fixed $\bar{p}$'s, the convergence of b_t to β_t, and of c_t to γ'_t is uniform in the $\bar{p}$'s, it follows that if $\mu_0, \mu_1, \ldots, \mu_t \rightarrow \infty$ with

fixed ratios $\theta_j = \mu_{.j} / \mu_{.0}$ and with $\bar{p}_j - \bar{p}_{j-1}, = \lambda_j. \frac{\bar{P}_0(1-\bar{P}_0)}{\mu_{.0}}$, where

λ_j. is fixed, j = l,......,t, then

$$b_t \xrightarrow{p} 1 \quad \text{and} \quad C'_t \xrightarrow{p} \gamma_t ,$$

both unconditionally and conditionally on the realised values of $X_1, \ldots\ldots X_t$.

(vii) For weekly changes in $\bar{p}$ as in (vi),

$$\bar{p}_t - S_{t-1}(\bar{p};\alpha) = \bar{p}_t - (1-\alpha)p_{t-1} - aS_{t-2}(\bar{p};\alpha)$$

$$= V\bar{p}_t + \alpha(\bar{p}_{t-1} - S_{t-2}(\bar{p};\alpha))$$

$$= V\bar{p}_t + av\bar{p}_{t-1} + a^2 v\bar{p}_{t-2} + \ldots\ldots + \alpha^{t-1} V\bar{p}_1$$

$$= \sum_{j=0}^{t-1} \alpha^j \lambda_{t-j} \left\{\frac{\bar{P}_0(1-\bar{P}_0)}{\mu_{.0}}\right\}^{\frac{1}{2}}$$

Therefore as $\mu_{.0}, \mu_{.1}, \ldots, \mu_{.t} \rightarrow \infty$ with . and θ_j fixed, λ_j = l,...,t,

$$\delta'_t \rightarrow \theta_t^{\frac{1}{2}} \sum_{j=0}^{t-1} \alpha^j \lambda_{t-j} = \delta_t \text{ , say .}$$

The following asymptotic distribution for T_t follows from the above results.

Theorem

Let $\mu_{.0}, \mu_{.t}, \ldots\ \ldots, \mu_{.t} \rightarrow \infty$ with fixed ratios $\theta_j = \mu_{.j} / \mu_{.0}$, $0 < \theta_j < \infty$,

j = l,...,t, and let the weekly changes in $\bar{p}$ be given by

$$\bar{p}_j - \bar{p}_{j-1} = \lambda_j \sqrt{\frac{\bar{P}_0(1-\bar{P}_0)}{\mu_{.0}}} \text{ where } \lambda_j \text{ is fixed ,}$$

j = 1, . . . ,t.

Then

$$\tilde{T}_t \xrightarrow{d} N\left(\delta_t , 1+\gamma_t^2\right),$$

both unconditionally and conditionally on the realised values of $X_1 , . . .,X_t$

Proof Under the limiting conditions of the theorem, we have from the above results that, both unconditionally and conditionally on the realised values of the X's,

$$U'_t \xrightarrow{b} U_t , \quad \text{where } U_t \sim N(0,1) ,$$
$$V'_t \xrightarrow{b} V_t , \quad \text{where } V_t \sim N(0,1) ,$$
$$b_t \xrightarrow{p} 1 , \; C'_t \xrightarrow{p} \gamma_t \; \text{ and } \; r_t \xrightarrow{p} 1 .$$

Also $\delta'_t \longrightarrow \delta_t$, and U_t and V_t are independent,
Hence $T_t = b_t(U'_t - c'_t V'_t + r_t \delta'_t)$

$$\xrightarrow{d} U_t - \gamma_t V_t + \delta_t \sim N(\delta_t ,1 + \gamma_t^2$$

Corollary

Under the above conditions

$$\frac{\tilde{T}_t}{(1+\gamma_t^2)^{\frac{1}{2}}} \xrightarrow{b} N\left(\frac{\delta_t}{(1+\gamma_t^2)^{\frac{1}{2}}} , 1\right),$$

and

$$C_t = \left\{\frac{(1-\alpha)X_t S_{t-1}\left(\frac{1}{X};\alpha^2\right)}{1+\alpha}\right\}^{\frac{1}{2}} \xrightarrow{p} \gamma_t .$$

Hence
$$\frac{\tilde{T}_t}{(1+\gamma_t^2)^{\frac{1}{2}}} \xrightarrow{b} N\left(\frac{\delta_t}{(1+\gamma_t^2)^{\frac{1}{2}}} , 1\right),$$

both unconditionally and conditionally on the realised values of $X_1 ,...,X_t$.

It follows from this corollary that for large values of the μ 's, $\frac{\tilde{T}_t}{(1+\gamma_t^2)^{\frac{1}{2}}}$

is distributed approximately N $\left(\dfrac{\delta_t}{(1+\gamma_t^2)^{\frac{1}{2}}}, 1 \right)$ both unconditionally and conditionally on the realised values of $X_1, \ldots, X_t$.

2-2 Correlation Structure of the $\tilde{T}$'s

First note that V'_t can be written

$$V'_t = \frac{(1-\alpha)\sum_{j=0}^{t-2} \alpha^j (Y_{t-1-j}\,\bar{P}_{t-1-j})}{\left\{ \dfrac{1-\alpha}{1+\alpha} S_{t-1}\left(\dfrac{\bar{P}(1-\bar{P}\ )}{X}; \alpha^2 \right) \right\}^{\frac{1}{2}}}$$

$$= \sum_{j=0}^{t-2} f_{t,j}\ U'_{t-1-j}\ ,$$

where

$$f_{t,j} = \left\{ \frac{(1-\alpha^2)\bar{P}_{t-1-j}(1-\bar{P}_{t-1-j})}{X_{t-1-j} S_{t-1}\left(\dfrac{\bar{P}(1 \sim \bar{P})}{x};\ \alpha^2 \right)} \right\}^{\frac{1}{2}} \alpha^j\ .$$

And under the limiting conditions of the above theorem

$$f_{t,j} \xrightarrow{p} \left\{ \frac{1-\alpha^2}{\theta_{t-1-j} S_{t-1}\left(\dfrac{1}{\theta}; \alpha^2 \right)} \right\}^{\frac{1}{2}} \alpha^j = \varphi_{t,j}\ \text{, say.}$$

and hence

$$V'_t \xrightarrow{b} = \sum_{j=0}^{t-2} \varphi_{t,j}\ U'_{t-1-j}\ ,$$

both unconditionally and conditionally on the realised values of the X's.

Hence, for the asymptotic distribution, the covariance between $\tilde{T}_{t+\tau}$ and $\tilde{T}_t$, $\tau \geq 0$, is

$$\text{a.cov}(\tilde{T}_{t+\tau}, \tilde{T}_t) = \text{cov}(U_{t+\tau} - \Upsilon_{t+\tau} V_{t+\tau} + \delta_{t+\tau}\, U_t - \Upsilon_t V_t + \delta_t)$$

$$= \text{cov}\left(U_{t+\tau} - \gamma_{t+\tau} \sum_{j=0}^{t+\tau-2} \varphi_{t+\tau,j} U_{t+\tau-1-j},\ U_t - \gamma_t \sum_{j=0}^{t-2} \varphi_{t,j}\, U_{t-1-j} \right)$$

$$= -\gamma_{t+\tau}\,\varphi_{t+\tau,\tau-1} + \gamma_{t+\tau}\gamma_t \sum_{j=0}^{t-2} \varphi_{t+\tau,\tau+j}\,\varphi_{t,j}$$

$$= -\left(\frac{\theta_{t+\tau}}{\theta_t}\right)^{\frac{1}{2}} (1-\alpha)\,\alpha^{\tau-1} + \left(\theta_t\theta_{t+\tau}\right)^{\frac{1}{2}} (1-\alpha)^2\,\alpha^{\tau} \sum_{j=0}^{t-2} \frac{\alpha^{2j}}{\theta_{t-1-j}}$$

$$= -\left(\frac{\theta_{t+\tau}}{\theta_t}\right)^{\frac{1}{2}} (1-\alpha)\,\alpha^{\tau-1} + \left(\theta_t\theta_{t+\tau}\right)^{\frac{1}{2}} \left[\frac{1-\alpha}{1+\alpha}\right] \alpha^{\tau} S_{t-1}\left[\frac{1}{\theta};\alpha^2\right].$$

The asymptotic correlation coefficient between $\tilde{T}_{t+\tau}$ and $\tilde{T}_t$ is then

$$\text{a. corr}\left[\tilde{T}_{t+\tau},\tilde{T}_t\right] = \frac{\text{a.cov}\left[\tilde{T}_{t+\tau},\tilde{T}_t\right]}{\sqrt{\text{a.var}\left(\tilde{T}_{t+\tau}\right)\text{a.var}\left(\tilde{T}_t\right)}} = \frac{\text{a.cov}\left[\tilde{T}_{t+\tau},\tilde{T}_t\right]}{\sqrt{\left(1+\gamma^2{}_{t+\tau}\right)\left(1+\gamma_t^2\right)}}.$$

For large μ 's, the correlation between $\tilde{T}_{t+\tau}$ and $\tilde{T}_t$ may be approximated by their asymptotic correlation, with θ_j replaced by $\frac{\mu_{.j}}{\mu_{.0}}$.

APPENDIX 3

Conditional Distribution of W^B_t given X_t

For given $X_t > 0$, $Z_{1t}, \dots, Z_{gt}$ have a multinomial distribution with index X_t and probability parameters $p_1,\dots,P_g$

Hence, for $Y_{rt} = Z_{rt}/X_t$,

$$E(Y_{rt}|X_t) = \overline{P}_r, \quad \text{var}(Y_{rt}|X_t) = \frac{\overline{P}_r(1-\overline{P}_r)}{X_t}, \quad \text{cov}(Y_{rt}, Y_{st}|X_t) = -\frac{\overline{P}_r\overline{P}_s}{X_t}, \quad r \neq s.$$

Now
$$W_t^B = \frac{\sum_{r=1}^{b} w_r Y_{rt}}{\sum_{s=1}^{g} W_s Y_{st}} = \frac{U}{V}, \text{ say.}$$

Where
$$E(U|X_t) = \sum_{r=1}^{b} W_r \overline{P}_r,$$

$$\text{var}(U|X_t) = \frac{\sum_{r=1}^{b} W_r^2 \overline{P}_r - \left[\sum_{r=1}^{b} W_r \overline{P}_r\right]^2}{X_t},$$

with corresponding expressions for E(VIX) and var{VlX,), and where

$$\text{cov}(U,V|X_t) = = \frac{\left(\sum_{r=1}^{b} W_r \overline{P}_r\right) - \left(W_r - \sum_{s=1}^{g} W_s \overline{Ps}\right)}{X_t},$$

For sufficiently large realised X , the conditional variances of U and V are small, and hence we may approximate W^B_t by the linear terms of the Taylor expansion of U/V about the conditional expectations of these variables . Thus

$$W_t^B = \frac{E(U|X_t)}{E(V|X_t)} + (U - E(U|X_t))\frac{1}{E(V|X_t)} - (v - E(V|X_t))\frac{E(U|X_t)}{E^2(V|X_t)}.$$

Hence, for large realised X_t,

$$E\left(W_t^B \middle| X_t\right) = \frac{E(U|X_t)}{E\ (v|X_t)}) = \frac{\sum_{r=1}^{b} W_r \bar{P}r}{\sum_{s=1}^{g} W_s \bar{P}s}, \text{and}$$

$$\text{Var}\ \left(W_t^B | X_t\right) = \frac{1}{E^2(v|X_t)}\ \text{var}\ (U|X_t) + \frac{E^2(U|X_t)}{E^2(V|X_t)}\ \text{Var}\ (v|X_t)$$

$$-\frac{2E(U|X_t)}{E^3(v|X_t)}\ \text{COV}\ (U|X_t)$$

$$= \frac{1}{X_t \sum_{s=1}^{g} W_s \bar{P}_s} \left\{ \sum_{r=1}^{b} W_r P_r^W - P^B \left[2 \sum_{r=1}^{b} W_r P_r^W - P^B \sum_{s=1}^{g} W_s\ P_s^W \right] \right\}$$

(after a little algebra),where $p_r^W = \frac{W_r \bar{P}_r}{\sum_{s=1}^{g}}$ and $P^B = \sum_{r=1}^{b} P_r^W$.

Further, for sufficiently large realised X_t , the multinomial conditional distribution of $Z_{1\ L}, \ldots, Z_{\ g\ t}$,. and hence the conditional distribution of Y_{1t}.. , . . . ,$Y_{g\ t}$, may be approximated by multivariate normal distributions. And

since, for large X_t , W^B_t is well approximated by a linear combination of

$Y_{1\ t}$... ,$Y_{g\ t}$ it follows that the conditional distribution of W_t^B given X_t is approximately normal.

Note also that, since $X_t \xrightarrow{p} \infty$ as $\mu \to \infty$, and since the approximate conditional distribution of W_t^B given large realised X_t does not depend on X_t , it follows that, for large μ , this approximate distribution applies unconditionally also.