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A Trifurcated Waveguide Problem

A D Rawlins

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PROBLEM

by

A D RAWLINS

Department of Mathematics and Statistics
Brunel University, Uxbridge, Middlesex, UB8 3PH, England

Summary

We consider the diffraction of the dominant wave mode which propagates out of the mouth of a semi-infinite waveguide made of a soft and hard half plane. This semi-infinite waveguide is symmetrically located inside an infinite waveguide whose infinite plates are soft and hard. The whole system constitutes a trifurcated waveguide. A closed form solution of the resulting matrix Wiener-Hopf equation is obtained.

1. Introduction

The emission of unwanted noise from the exhaust systems of various vehicle engines is an environmental nuisance that pollutes our everyday existence. The exhausts of modern engines are design to try to reduce the noise emitted as much as possible. One way of achieving this is novel geometrical designs and the use of various sound absorbent materials for the exhaust system, Rawlins (1978). The present problem was conceived as a possible design aid that would help in the design of practical exhaust systems. In order that mathematical techniques will yield useful acoustic field distribution within such exhaust systems, various simplifications need to be made. In particular practical exhaust systems are of finite length, however, in order to apply the powerful Wiener-Hopf technique it is convenient to consider semi-infinite geometries. However this is not as severe as one might think if one considers the rapid attenuation of the sound field along the absorbent duct system. Also the use of absorbent materials lining the exhaust system requires a mixed boundary condition which involves an impedance parameter related to the material lining. However a simplification that considerably reduces the complexity of the mathematical formulae is to replace these boundary conditions by a soft boundary. Although a perfectly soft surface is a somewhat ideal concept it can be used to give realistic design data for absorbent linings, Butler (1974), Rawlins (1975). These mathematical model simplifications can be regarded as first approximations to a more sophisticated mathematical model that involves finite length structures and absorbent type boundary conditions. The practical numerical calculations which will use extensions of ideas of Rawlins (1975) are not appropriate here because they are of a speculative nature and need to be compared with experimental work. It is hoped that these calculations will be carried out in a separate paper. Here we are directly interested in deriving an exact closed-form solution to a new waveguide trifurcation problem. This could be used to compare approximate techniques applied to practical problems where the matrix cannot be factorized.

The trifurcated waveguide problem under consideration is shown in Figure 1. The plates which make up the waveguide are alternatively soft and hard, and are symmetrically positioned relative to the centre line of the system. A fundamental mode is assumed to propagate out of the mouth of the semi-infinite waveguide. The trifurcated waveguide problem where the plates and the boundary conditions are symmetric with respect to the centre-line of the system can be reduced by symmetry to the solution of a bifuraction problem. Such a bifurcation problem can be solved exactly by the normal scalar Wiener-Hopf technique Pace and Mittra (1966), Mittra and Lee (1971) (p.200 and 212-213), Hurd and Meister (1988). For the trifurcated waveguide problem we shall consider the symmetry with respect to the boundary conditions is lost. This results in a nontrivial matrix Wiener-Hopf problem. Fortunately we are able to solve this matrix Wiener-Hopf problem explicitly. We remark that the present problem does not fall within the set of trifurcation problems solved by Lüneburg and Hurd (1982). In the problems considered in Lüneburg & Hurd (1982) the spacing between the plates was equidistant and the direction of the incident wave was different.

In Section 2 we shall formulate the mathematical problem that we intend to solve. In Section 3 we shall solve the problem formulated in Section 2. The solution will be expressed as complex contour integrals. In Section 4 we shall analytically manipulate these integrals to reduce them to infinite series of modes which propagate in the waveguide system. In order not to disrupt the flow of the solution in the main text, various appendices have been included at the end of the paper. These include analytical details required in the main text.

2. Formulation of the boundary value problem

We shall consider the acoustic diffraction of a plane wave mode propagating out of the open end of a semi-infinite duct; this semi-infinite duct consists of one plate which is rigid and the other which is soft. The semi-infinite duct is situated symmetrically between two infinite plates one of which is soft and the other rigid. The geometry of the problem is shown in Figure 1.

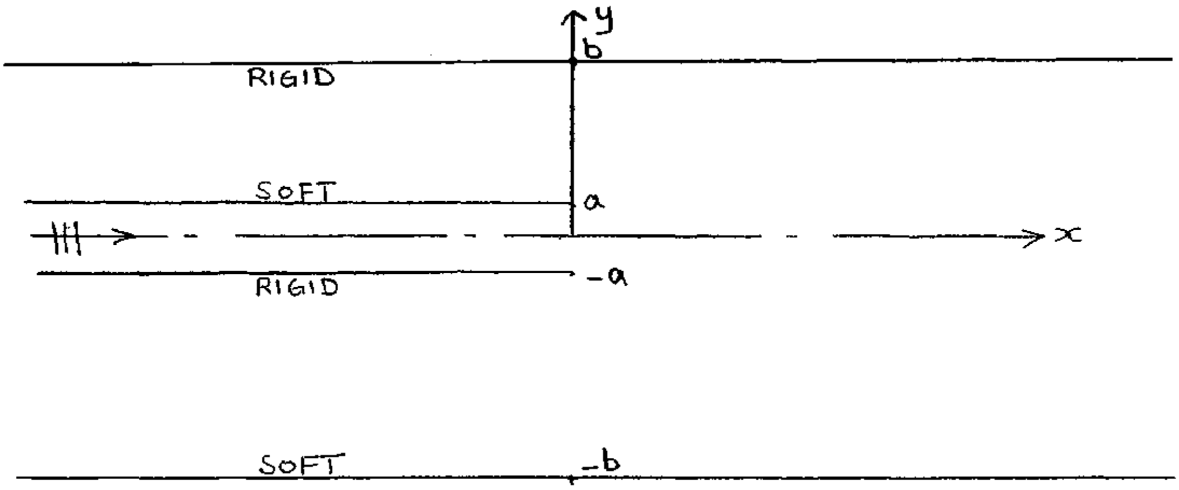

Fig. 1.

The sound source field is located at $x = x_0$ ($x_0 < 0$) and propagates modes along the semi-infinite duct. We shall introduce a scalar potential function $\phi(x,y,t)$ which defines the acoustic pressure and velocity by $p = -\rho_0 \partial\phi/\partial t$, and $u = \mathrm{grad}\,\phi$ respectively, where ρ_0 is the density of the undisturbed medium.

The incident sound field is assumed to have time harmonic variation $e^{-i\omega t}$, where the wave number $k = \omega/c$, and c is the speed of sounds. In the rest of this paper we shall drop the dependence of time, it being tacitly understood that $\phi(x,y,t) = e^{-i\omega t}\phi(x, y)$. To this end we require a representation for the solution $\phi(x,y)$ of the two-dimensional Helmholtz equation

$$\phi_{xx} + \phi_{yy} + k^2\phi = 0 \qquad (1)$$

in the trifurcated duct system which satisfies the following boundary conditions (see Fig.l)

$$\begin{aligned} &\phi_y = 0, && y = b && -\infty < x < \infty, \\ &\phi = 0, && y = a && -\infty < x < 0, \\ &\phi_y = 0, && y = -a && -\infty < x < 0, \\ &\phi = 0, && y = -b && -\infty < x < \infty, \end{aligned} \qquad \text{(2a,b,c,d)}$$

where it is assumed that b > a.

To these boundary conditions we add those conditions at infinity which are relevant to the nature of the propagating modes which the various duct regions can sustain. From appendix A we have

For $x \to -\infty$, $-a \le y \le a$

$$\phi(x,y) = e^{i\alpha_1 x} \cos\left((y+a)\frac{\pi}{4a}\right) + R_2 e^{-i\alpha_1 x} \cos((y+a)\frac{\pi}{4a}) + 0(e^{-i\alpha_2 x}) \quad (3)$$

where $\alpha_1 = \sqrt{k^2 - \pi^2/16a^2}$, $\alpha_2 = \sqrt{k^2 - 9\pi^2/16a^2}$, $\alpha_{2n-1} = \sqrt{k^2 - ((2n-1)\pi/4a)^2}$, $n = 2,\dots$.

If we restrict $\pi/4<ka<3\pi/4$ then $\alpha_1>0$ and $\alpha_2 = i\sqrt{9\pi^2/16a^2 - k^2}$ so that $\mathrm{Im}\alpha_2>0$, $\mathrm{Re}\alpha_2 = 0$. Thus the semi-infinite duct region $-\infty < x < 0$, $-a \le y \le a$ can only sustain the lowest incident and reflected modes.

For $x \to +\infty$, $-b \le y \le b$

$$\phi(x,y) = T e^{i\hat{\alpha}_1 x} \sin((y+b)\frac{\pi}{4b}) + 0(e^{i\hat{\alpha}_2 x}) \quad (4)$$

Where $\hat{\alpha}_1 = \sqrt{k^2 - \pi^2/16b^2}$, $\hat{\alpha}_2 = \sqrt{k^2 - 9\pi^2/16b^2}$, $\hat{\alpha}_{2n-1} = \sqrt{k^2 - ((2n-1)\pi/4b)^2}$, $n = 1,2,\dots$.

For $x \to -\infty$, $a \le y \le b$

$$\phi(x,y) = T_3 e^{-i\hat{\alpha}_1 x} \sin\left((y-a)\frac{\pi}{2(b-a)}\right) + 0(e^{-i\hat{\alpha}_2 x}) \quad (5)$$

Where $\hat{\alpha}_1 = \sqrt{k^2 - \frac{\pi^2}{4(b-a)^2}}$, $\hat{\alpha}_2 = \sqrt{k^2 - \frac{9\pi^2}{4(b-a)^2}}$, $\hat{\alpha}_{2n-1} = \sqrt{k^2 - ((2n-1)\pi/2(4b-a))^2}$, $n = 1, 2,\dots$.

For $x \rightarrow -\infty$, $-b < y \leq -a$

$$\phi(x,y) = T_1 e^{i\hat{\alpha}_1 x} \sin((y+b)\pi/(2(b-a))) + 0(e^{-i\hat{\alpha}_2 x}) \tag{6}$$

Finally in order to ensure the uniqueness of the solution to the problem we need to specify the "edge condition" at the end of the semi-infinite planes, that is

$$\phi(x,\pm a) = 0(1) \text{ and } \phi_y(x,\pm a) = 0(x^{-1/2}) \quad \text{as} \quad x \rightarrow 0 . \tag{7}$$

3. Solution of the boundary value problem

For analytic convenience we shall assume that k = Rek + ilmk (Rek > Imk $\geq$ 0). A suitable representation for the total field ϕ (x,y) in all space $-\infty < x < \infty$, $|y| < b$, which satisfies (1) and (2) is given by an application of the Fourier transform approach as:

$$\phi(x,y) = \frac{1}{2\pi i}\int_{-\infty+i\tau}^{\infty+i\tau} e^{i\alpha x} \frac{\sin k(y+b)}{k\cos k(b-a)} \phi_1^-(\alpha(-b \leq y \leq -a, -\infty < x < \infty); \tag{8}$$

$$\phi(x,y) = e^{i\alpha_1 x}\cos[(y+a)\pi/4a]$$
$$+\frac{1}{2\pi\pi}\int_{-\infty+i\tau}^{\infty+i\tau} \frac{e^{i\alpha x}}{\kappa\cos 2\kappa a}\left\{\sin\kappa(y-a)\phi_1^-(\alpha)+\kappa\cos\kappa(y+a)\phi_2^-(\alpha)\right\}d\alpha , \tag{9}$$
$$(-a \leq y \leq a, -\infty < x < \infty);$$

$$\phi(x,y) = \frac{1}{2\pi i}\int_{-\infty+i\tau}^{\infty+i\tau} \frac{e^{i\alpha x}}{\cos\kappa(b-a)}\cos\kappa(y-b)\phi_2^-(\alpha)d\alpha , \tag{10}$$
$$(a \leq y \leq b, -\infty < x < \infty).$$

In the expressions (8) to (10), $\kappa = (k^2-\alpha^2)^{1/2}$ and the branch cuts are taken to be from k to $i\infty$ and from -k to $-i\infty$. The cut sheet on which we shall work is defined by $0 \leq \arg\kappa \leq \pi$, see fig. 2. The parameter τ is restricted by requiring the asymptotic behaviour (3), (4), (5) and (6) to be achieved. This necessitates that the contour of integration in (8) to (9) lies in the strip : $\text{Max}\{-\text{Im}\kappa, -\text{Im}\alpha_1, -\text{Im}\,\bar{\alpha}_1\} < \tau < \text{Min}\{\text{Im}\kappa, \text{Im}\hat{\alpha}_1\}$.

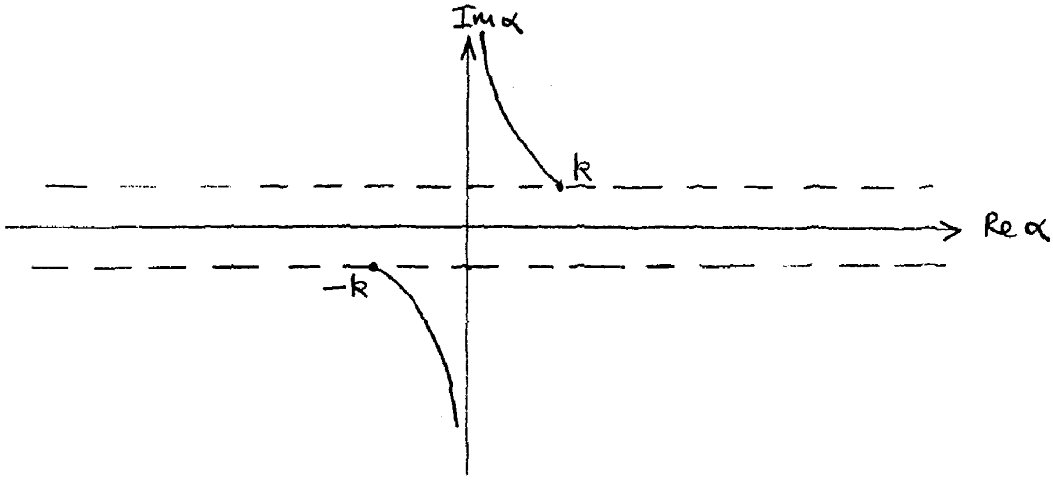

Fig. 2

It is shown in appendix B that this inequality is satisfied by -Imk < τ < Imk. No singularities of the integrands of (8) to (10) lie within the strip -Imk < Imα < Imk. The unknown functions $\phi_1^-(\alpha)$ and $\phi_2^-(\alpha)$ are functions which are analytic and regular in the region Imα < Imk. We must now ensure that two remaining boundary conditions are satisfied, namely that the field and its normal derivative are continuous across y = -a and y = a, (x>0) respectively, that is

$$\phi(x,-a^-) = \phi(x,-a^+), \frac{\partial\phi}{\partial y}(x,a^-) = \frac{\partial\phi}{\partial y}(x,a^+) \quad x > 0. \tag{11}$$

Substituting (8) to (10) into (11) gives

$$\frac{1}{2\pi i}\int_{-\infty+i\tau}^{\infty+i\tau} e^{i\alpha x}\left\{\frac{-\sin\kappa\,(a+b)\phi_1^-(\alpha)}{\kappa\cos\kappa\,(b-a)\cos 2\kappa a} + \frac{\phi_2^-(\alpha)}{\cos 2\kappa a} + \frac{1}{\alpha-\alpha_1}\right\}d\alpha = 0, (x > 0); \tag{12}$$

$$\frac{1}{2\pi i}\int_{\infty+i\tau}^{\infty+i\tau} e^{i\alpha x}\left\{\frac{\phi_1^-(\alpha)}{\cos 2\kappa a} - \frac{\kappa\sin\kappa\,(b+a)\phi_2^-(\alpha)}{\cos 2\kappa a\cos\kappa\,(b-a)} - \frac{\kappa_1}{\alpha-\alpha_1}\right\}d\alpha = 0, (x > 0), \tag{13}$$

where $\kappa_1 = \pi/4a$.

A solution of (12) and (13) is given by

$$\frac{-\sin\kappa(a+b)}{\kappa\cos 2\kappa\cos\kappa(b-a)}\phi_1^-(\alpha)+\frac{\phi_1^-(\alpha)}{\cos 2\kappa a}+\frac{1}{\alpha-\alpha_1}=\phi_2^+(\alpha), \tag{14}$$

$$\frac{\phi_1^-(\alpha)}{\cos 2\kappa a}-\frac{\kappa\sin\kappa(b+a)}{\cos 2\kappa a\cos\kappa(b-a)}\phi_2^-(\alpha)+\frac{\kappa_1}{\alpha-\alpha_1}=\phi_1^+(\alpha), \tag{15}$$

where $\phi_1^+ \phi_2^+$ are functions that are analytic and regular in the region Imα > Imk.

The equations (14) and (15) constitute a coupled system of Wiener-Hopf equations which we can write in matrix form as:

$$\psi^+(\alpha)=K(\alpha)\psi^-(\alpha)+D/(\alpha-\alpha_1) \tag{16}$$

where

$$\psi^\pm(\alpha)=\begin{pmatrix}\phi_1^\pm(\alpha)\\ \phi_2^\pm(\alpha)\end{pmatrix}, \tag{17}$$

$$D=\begin{pmatrix}-\kappa_1\\ 1\end{pmatrix}, \tag{18}$$

$$K(\alpha)=f(\alpha)G(\alpha) \tag{19}$$

$$f(\alpha)=\sec 2\kappa a, \tag{20}$$

$$G(\alpha)=\begin{pmatrix}1 & \dfrac{-\kappa\sin\kappa(b+a)}{\cos\kappa(b-a)}\\ \dfrac{-\sin\kappa(b+a)}{\kappa\cos\kappa(b-a)} & 1\end{pmatrix} \tag{21}$$

The standard Wiener-Hopf technique can be applied in a straightforward manner if the system (16) can be uncoupled into separate Wiener-Hopf equations. However, this problem requires that the matrix function K(α) be factorized. This is a non-trivial operation and it is not always obvious how to carry it out explicitly, at least in the classical sense, Noble (1958) (where the factors have algebraic behaviour at infinity). The matrix G(a) given by (21) is of a special form:

$$G(\alpha)=\begin{pmatrix}1 & a(\alpha)\\ b(\alpha) & 1\end{pmatrix}$$

with $a(\alpha)/b(\alpha)$ = ratio of polynomials in α. In the present problem we have $a(\alpha)/b(\alpha) = k^2-\alpha^2$ so that it can be factorized immediately (see Daniele (1978), and Rawlins (1980)). What is more since $a(\alpha)/b(\alpha)$ is a polynomial of degree 2 the factors will have algebraic growth at infinity. Without going into the details, we obtain

$$K(\alpha) = K_+(\alpha)\, K_-(\alpha) = K_-(\alpha)\, K_+(\alpha) \tag{22}$$

with

$$K_\pm(\alpha) = \sqrt{K_\pm(\alpha)}\begin{pmatrix}\cosh[\tfrac{1}{2}\kappa t_\pm(\alpha\alpha) & \kappa\sinh[\tfrac{1}{2}\kappa t_\pm(\alpha\alpha)\\ \tfrac{1}{\kappa}\sinh[\tfrac{1}{2}\kappa t_\pm(\alpha\alpha) & \cosh[\tfrac{1}{2}\kappa t_\pm(\alpha\alpha)\end{pmatrix}, \tag{23}$$

where

$$K(\alpha)=\det(K(\alpha))=K_+(\alpha)K_-(\alpha)=\frac{\cos 2\kappa b}{\cos^2\kappa(b-a)\cos 2\kappa a}, \tag{24}$$

$$t(\alpha)=\frac{1}{\kappa}\ell n\left(\frac{\cos\left(\frac{\pi}{4}+\kappa b\right)\cos\left(\frac{\pi}{4}+\kappa a\right)}{\cos\left(\frac{\pi}{4}-\kappa b\right)\cos\left(\frac{\pi}{4}-\kappa a\right)}\right)=t_+(\alpha)+t_-(\alpha). \tag{25}$$

Explicit expressions for $K_\pm(a)$ and $t_\pm((x)$ are given in the appendix C. They are:

$$K_+(\alpha)=K_-(-\alpha),$$

(26)

$$K_-(\alpha)=\frac{K_-(\alpha,b)}{K_-(\alpha,a)}\cdot\frac{1}{K_-^2(\alpha,(b-a)/2)} \tag{27}$$

with

$$K\_(\alpha,a) = \prod_{n=1}^{\infty}\left(\frac{a}{(2n-1)\pi k}\right)^2 (l_{4n-1}\,u - u_{4n-1})\,(l_{4n-1}\,u + u_{4n-1}l)\,(l_{4n-3}\,u - u_{4n-3}l)\,(l_{4n-3}\,u + u_{4n-3}l) \quad \textbf{(28)}$$

$$t_+(\alpha) = t\_(-\alpha)\ ;$$

and

$$t\_(\alpha) = \frac{1}{\kappa}\,\ell n \prod_{n=1} \frac{(u_{4n-1}l + l_{4n-1}u)(u_{4n-3}l - l_{4n-3}u)}{(u_{4n-1}l - l_{4n-1}u)(u_{4n-3}l + l_{4n-3}u)}\ . \qquad (29)$$

$$(\hat{u}_{4n-1}l + \hat{l}_{4n-1}u)(\hat{u}_{4n-3}l + \hat{l}_{4n-3}u)/\{(\hat{u}_{4n-1}l - \hat{l}_{4n-1}u)(\hat{u}_{4n-3}l + \hat{l}_{4n-3}u)\}$$

with

$$u = (k+\alpha)^{1/2},\ u_n = (k+\alpha_n)^{1/2},\ l = (k-\alpha)^{1/2},\ l = (k-\alpha_n)^{1/2}$$

$$\alpha_n = \left(k^2 - \left(\frac{n\pi}{4a}\right)^2\right)^{1/2},\ \hat{u}_n = (k+\hat{\alpha}_n)^{1/2},\ \hat{l}_n = (k-\hat{\alpha}_n)^{1/2},\ \hat{\alpha}_n = \left(k^2 - \left(\frac{n\pi}{4b}\right)^2\right)^{1/2}, \qquad (30)$$

$$\tilde{u}_n = (k+\tilde{\alpha}_n)^{1/2},\ \tilde{l} = (k-\tilde{\alpha}_n)^{1/2},\ \tilde{\alpha}_n = \left(k^2 - \left(\frac{n\pi}{2(b-a)}\right)^2\right)^{1/2}\ .$$

The Wiener-Hopf equation (16) may now be solved by using (22) to give

$$K\_(\alpha)\Psi^-(\alpha) + \frac{K_+^{-1}(\alpha_1)}{\alpha-\alpha_1}\,D = K_+^{-1}(\alpha)\Psi^+(\alpha) - \frac{[\ _+^{-1}(\alpha) - \ _+^{-1}(\alpha_1)]}{(\alpha-\alpha_1)}\,D\ . \quad (31)$$

The left hand side of the above equation is analytic in Imα < Imk; the right hand side is analytic in Imα > -Imk. Consequently each side of equation (31) is equal to an entire function E(α), that is, a matrix with polynomial entries. Hence setting

$$C = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = K_+^{-1}\ (\alpha_1)\ D \qquad (32)$$

we have

$$\Psi^-(\alpha) = K_-^{-1}(\alpha)[-\,C(\alpha-\alpha_1)^{-1} + E(\alpha)]\ . \qquad (33)$$

The form of E(α) is determined from the asymptotic behaviour of various functions.

From Appendix C we have that as $|\alpha| \to \infty$ in $\mathrm{Im}\alpha < \mathrm{Im}k$ that:

$$K\_(\alpha) = 2 + 0(\alpha^{-1}), \quad t\_(\alpha) = \frac{i}{\alpha}\ell n\alpha + 0(\alpha^{-1}) . \tag{34}$$

Hence

$$K\_(\alpha) = \frac{1}{\sqrt{2}}\begin{pmatrix} \alpha^{1/2} + 0(1) & -\alpha^{3/2} + 0(\alpha) \\ \alpha^{-1/2} + 0(\alpha^{-1}) & \alpha^{1/2} + 0(1) \end{pmatrix} . \tag{35}$$

The asymptotic behaviour of $\Psi^-(\alpha)$ may be found from the edge conditions (7), it is not difficult to show that as $|\alpha| \to \infty$ in $\mathrm{Im}\alpha < \mathrm{Im}k$,

$$\phi_1^-(\alpha) = 0(\alpha^{-1/2}), \qquad \phi_2^-(\alpha) = 0(\alpha^{-1}) . \tag{36}$$

It is now straightforward to show, by substituting (35), (36) into (33), that the entire matrix $E(\alpha)$ is given by

$$E(\alpha) = ic_2\begin{pmatrix}1\\0\end{pmatrix} \tag{37}$$

in order that the asymptotic behaviour (36) is obtained from (33).

The solution can now be written in a very succinct form by introducing the following functions:

$$F(u,l) = ul_1 + lu_1 \tag{38}$$

$$P(l,u) = \frac{e^{-i\pi/4}}{2\sqrt{k}} \cdot \frac{(1+iu)e^{\frac{1}{2}\kappa t\_(\alpha)}}{\sqrt{K\_(\alpha)}}$$

$$= \frac{e^{-ix/4}}{2\sqrt{k}}(1+iu)\prod_{n=1}^{\infty}\left(\frac{(b-a)}{(2n-1)\pi}\left(\frac{a}{b}\right)^{1/2}\right)^2 \frac{(u_{4n-1}l+l_{4n-1}u)(u_{4n-3}l-l_{4n-3}u)}{(\hat{u}_{4n-1}l-\hat{l}_{4n-1}u)(\hat{u}_{4n-3}l+\hat{l}_{4n-3}u)} \tag{39}$$

$$\times\ (\tilde{u}_{4n-1}^2 l^2 - \tilde{l}_{4n-1}^2 u^2)(\tilde{u}_{4n-3}^2 l^2 - \tilde{l}_{4n-3}^2 u^2) ;$$

then the expression (33) can be written explicitly as:

$$\Psi^{-}(\alpha) = \frac{P(u_1,l_1)}{(\alpha-\alpha_1)}\begin{pmatrix} P(l,u)F(u,l) & + & P(l,-u)F(-u,l) \\ [P(l,u)F(-u,l) & - & P(l,u)F(u,l)]/\kappa \end{pmatrix}. \quad (40)$$

It is easy to verify that P(l,u), P(l,-u) have neither zeros nor poles in Imα < Imk. They do have branch points at α = -k, but the particular combinations in (40) are invariant under the transformation u $\rightarrow$-u, so that Ψ^{-}(or) is indeed analytic in Imα < Imk. We now substitute (40) into the integral representations (8) to (10) and obtain the field representations for the different regions as:

Region A (-b < y < -a, x < 0)

$$\phi(x,y) = \frac{P(u_1,l_1)}{2\pi i}\int_{-\infty+i\tau}^{\infty+i\tau} e^{i\alpha x}\frac{\sin\kappa(y+b)}{\kappa\cos\kappa(b-a)}\{P(l,u)F(u,l) + P(l,-u)F(-u,l)\}\frac{d\alpha}{(\alpha-\alpha_1)}. \quad (41)$$

Region B (-a < v < a, x < 0)

$$\phi(x,y) = e^{i\alpha_1 x}\cos\left[(y+a)\frac{\pi}{4a}\right]$$

$$+ \frac{P(u_1,l_1)}{2\pi i}\int_{-\infty+i\tau}^{\infty+i\tau}\frac{e^{i\alpha x}}{\cos 2\kappa a}\{[\sin\kappa(y-a) - \cos\kappa(y+a))P(l,u)F(u,l) \quad (42)$$

$$+ (\sin\kappa(y-a) + \cos\kappa(y-a))\ P(l-u)F(-u,l)]/\kappa\}\frac{d\alpha}{\alpha-\alpha_1}$$

Region C (a < y < b, x < 0)

$$\phi(x,y) = \frac{P(u_1,l_1)}{2\pi i}\int_{-\infty+i\tau}^{\infty+i\tau}\frac{e^{i\alpha x}\cos\kappa(y-b)}{\cos\kappa(y-b)}\{[P(l-u)F(-u,l) - P(l,u)F(u,l)]/\kappa\}\frac{d\alpha}{(\alpha-\alpha_1)}. \quad (43)$$

Region D (-b < y < b, x > 0)

$$\phi(x,y) = \frac{P(u_1,l_1)}{2\pi i}\int_{-\infty+i\tau}^{\infty+i\tau}\frac{e^{i\alpha x}\cos\kappa(y-b)}{\cos 2\kappa b}\{[\cos(\kappa a-\pi/4)\cos(\kappa b-\pi/4)F(u,-1)/P(u,-1)$$

$$-\cos(\kappa a+\pi/4)\cos(\kappa b+\pi/4)F(u,1)/P(u,1)]/\kappa\}\frac{d\alpha}{(\alpha-\alpha_1)}\ . \tag{44}$$

In the expressions (41) to (44) the pole $\alpha = \alpha_1$ lies above the contour of integration, that is $\mathcal{T} < \mathrm{Im}\alpha_1$. We also note that the term in the curly bracket { } of (41) to (43) has no singularities in $\mathrm{Im}\alpha < \mathrm{Im}k$, and in (44) has no singularities in $\mathrm{Im}\alpha > -\mathrm{Im}k$. Thus the only singularities in $\mathrm{Im}\alpha < \mathrm{Im}k$ of the integrands of (41) and (43) occur at the zeros of $\cos\kappa\ (b-a) = 0$ that is $\alpha = -\ \tilde{\alpha}_{2n-1} = -\sqrt{(k^2-((2n-1)\pi/2(b-a))^2)},\ n = 1,2,...$. The only singularities in $\mathrm{Im}\alpha < \mathrm{Im}k$ of the integrand of (42) occur at the zeros of $\cos 2\kappa a = 0$, that is $\alpha = -\alpha_{2n-1} = -\sqrt{(k^2-((2n-1)\pi/4a^2)}$. The only singularities in $\mathrm{Im}\alpha > -\mathrm{Im}k$ of the integrand (44) occur at the zeros of $\cos 2\kappa b = 0$, that is, $\alpha = \hat{\alpha}_{2n-1} = \sqrt{(k^2 - ((2n-1)\pi/4b)^2)}$, n = 1,2,... and also the pole $\alpha = \alpha_1$.

4 Mode field structure

An application of Cauchy's residue theorem to the complex integrals (41) to (44) then gives the field in the various regions as a sum of waveguide modes.

Region A (-b < y < -a, x < 0)

$$\phi(x,y) = \frac{P(u_1,l_1)}{(b-a)}\sum_{n=1}^{\infty}\frac{(-)^n\sin\tilde{\kappa}_m(y+b)}{\tilde{\alpha}_m(\tilde{\alpha}_m+\alpha_1)}\left\{P(\tilde{u}_m,\tilde{l}_m)F(\tilde{l}_m,\tilde{u}_m)+P(\tilde{u}_m,-\tilde{l}_m)F(-\tilde{l}_m,\tilde{u}_m)\right\}e^{-i\tilde{\alpha}_m x}\ ,$$

$$(m = 2n-1)\ . \tag{45}$$

Region B (-a < y < a. x < 0)

$$\phi(x,y) = e^{i\alpha_1 x}\cos\left[(y+a)\kappa_1\right]$$

$$+\frac{P(u_1,l_1)}{2a}\sum_{n=1}^{\infty}\frac{e^{-i\alpha_m x}(-)^n}{\alpha_m(\alpha_m+\alpha_1)}\{[(\sin\kappa_m(y-a)-\cos\kappa_m(y+a))\,P(u_m,l_m)F(l_m,u_m) \quad (46)$$

$$+(\sin\kappa_m(y-a)+\cos\kappa_m(y+a))P(u_m,-l_m)F(-l_m,u_m)]\},\ (m=2n-1)\ .$$

Region C (a < y < b, x < 0)

$$\phi(x,y)=\frac{P(u_1,l_1)}{(b-a)}\sum_{n=1}^{\infty}\frac{(-)^n e^{i\tilde{\alpha}_m x}\cos\tilde{\kappa}_{2n-1}(y-b)}{\alpha_m(\alpha_m+\alpha_1)}\{[P((\tilde{u}_m,-\tilde{l}_m)F(\tilde{l}_m,-\tilde{u}_m)-P((\tilde{u}_m,\tilde{l}_m)F(\tilde{l}_m,\tilde{u}_m)]\}\ ,$$

$$(m = 2n - 1)\ . \tag{47}$$

Region D (-b < y < b. x > 0)

$$\phi(x,y)=\frac{P(u_1,l_1)}{2b}\sum_{n=1}^{\infty}\frac{(-)^{[n/2]}\cos\hat{\kappa}_m(y-b)F(\hat{u}_m,(-)^n\hat{l}_m)}{\hat{\alpha}_m(\hat{\alpha}_m-\alpha_1)P(\hat{u}_m,(-)^n\hat{l}_m)}\cos(\hat{\kappa}_m a+(-)^n\pi/4)e^{i\alpha_m x}\ ,$$

$$(m=2n-1)\ , \tag{48}$$

where [x] denotes the largest integer <x.

5. Conclusion

We have solved a new diffraction problem in closed-form by using matrix factorization. This solution can be used as a benchmark for the comparison of approximate techniques applied to more practical problems that also result in matrix Wiener-Hopf problems that cannot be solved exactly. It can be regarded as a possible model for an exhaust system. If the boundary conditions on ±b are interchanged a new diffraction problem is obtained. This new problem can be solved by the method in this paper. The present approach would not be substantially changed if we included a flow in the region $-a<y<a$. This would of course be a more realistic model of an exhaust, where exhaust gases flow out of the system. It is hoped to address these problems in the near future.

We also remark that if we considered the same duct geometry and boundary conditions, but with the incident wave propagating in the region $x > 0$, $|y| < b$ from $z = \infty$, see fig.3, then we would obtain a similar matrix Wiener-Hopf problem which can be solved exactly. As a special case of this latter problem by letting $a \to 0$ one would obtain the solution to the problem considered by Lüneburg and Hurd (1985).

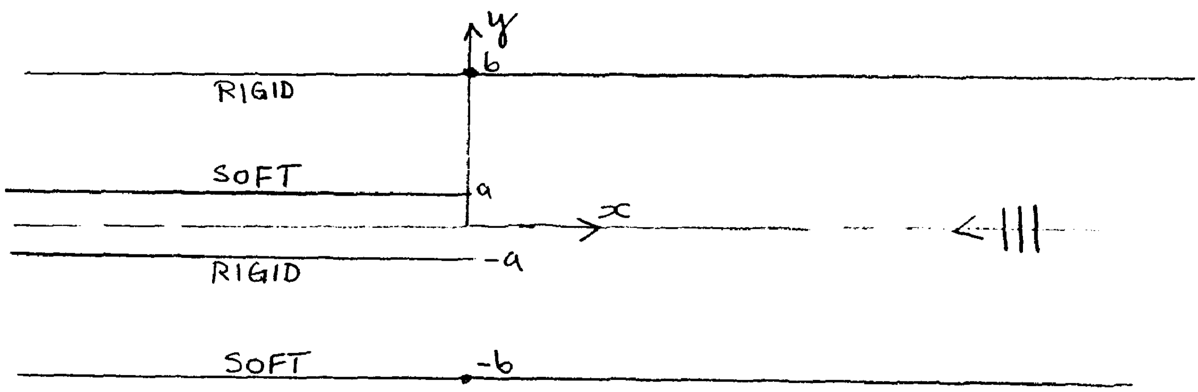

Fig. 3

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Appendix A

Here we shall derive the possible waveguide modes in an infinite soft and hard waveguide. We shall, in this appendix, assume k is real.

Solutions of the wave equation $(\nabla^2 + k^2)\phi = 0$ in $-\infty < x < \infty$, $a_1 \le y \le a_2$ are given by

$$\phi = e^{\pm ik\cos\theta_0 x}\begin{Bmatrix}\cos\\ \sin\end{Bmatrix}(k\sin\theta_0(y-a_1)),\ \ \theta_0 \in R.$$

Now $\partial\phi/\partial y = 0$ on $y = a_1$ means that we must choose cos in the curly bracket. Thus

$$\phi = e^{\pm ik\cos\theta_0 x}\cos(k\sin\theta_0(y-a)).$$

$\phi = 0$ on $y = a_2$ requires that

$$\cos(k\sin\theta_0(a_2 - a_1)) = 0$$

$$\Rightarrow \qquad k\sin\theta_0 = \frac{(2n-1)\pi/2}{(a_2 - a_1)},\ n = 1,2,\ldots\ .$$

Hence the possible modes are given, with $d = a_2 - a_1$, by

$$\phi_n = e^{\pm i\sqrt{(k^2-(2n-1)^2\pi^2/4d^2)}}\cos\left(\frac{2n-1)\pi}{2d}(y-a_1)\right), n = 1,2,\ldots\ . \qquad \text{(1a)}$$

From the way we have defined the cutsheet then for $k > 0$, $\sqrt{(k^2-(2n-1)^2\pi^2/4d^2)} > 0$ or $\mathrm{Im}\sqrt{(k^2-(2n-1)^2\pi^2/4d^2)} > 0$. Hence the upper and lower sign in (la) corresponds to outgoing or evanescent wave modes travelling towards $x = \pm\infty$ respectively.

Similarly, if we interchange the boundary conditions we obtain the solutions of the problem:

$$(\nabla^2 + k^2)\psi = 0 \quad -\infty < x < \infty,\ \ a_1 \le y \le a_2,\ a_2 - a_1 = d,$$
$$\partial\psi/\partial y = 0 \text{ on } y = a_2;\ \psi = 0 \text{ on } y = a_1,$$

as:

$$\psi_n = e^{\pm i\sqrt{(k^2-(2n-1)^2\pi^2/(4d^2))}} \sin\left(\frac{2n-1)\pi}{2d}(y-a_1)\right), n = 1,2,\dots . \qquad (2a)$$

where the upper and lower sign corresponds to outgoing or evanescent wave modes travelling towards $x \rightarrow \pm\infty$ respectively. Thus it is seen that when $\pm\sqrt{(k^2 - (2n-1)^2\pi^2/(4d^2))}x$ is positive real, that is $k^2 - (2n-1)^2\pi^2/(4d^2) > 0$, the waves are outgoing at infinity. When $\pm\sqrt{(k^2-(2n-1)^2\pi^2/(4d^2))}x$ is purely imaginary positive, that is $k^2 - (2n-1)^2\pi^2/(4d^2) < 0$, the waves are attenuated at infinity.

The previous analysis has assumed that k is real and positive. If we introduce a small positive imaginary part to k, so that k = Rek + iImk(Rek > Imk ≥ 0) then with the square root determination in the cut plane $\mathrm{Im}\sqrt{(k^2-(2n-1)^2\pi^2/(4d^2))} > 0$ and consequently the wave modes (la) or (2a) will be evanescent and bounded for the upper (lower) sign as $x \rightarrow +\infty(-\infty)$.

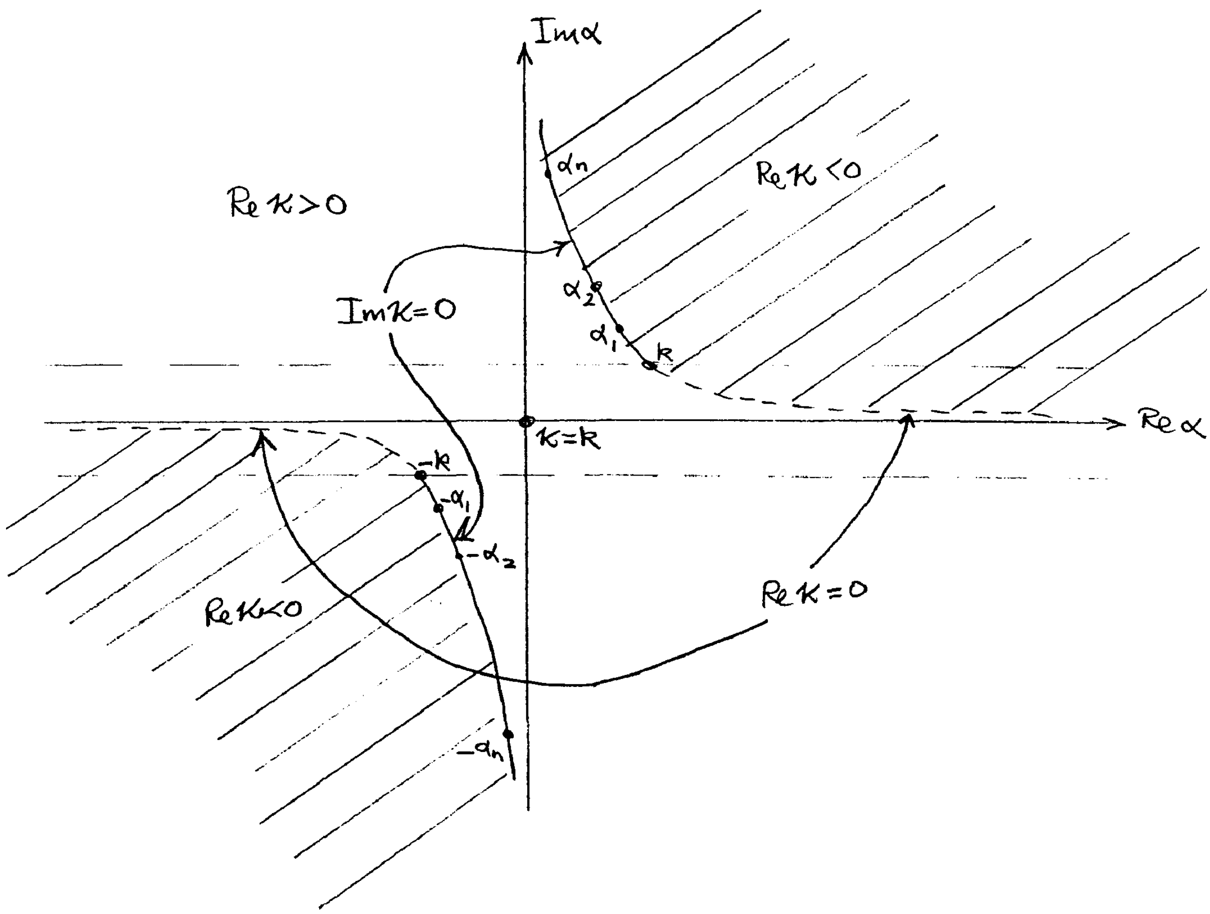
Appendix B

In this appendix we shall locate the disposition of the propagating and evanescent mode singularities that can arise in the complex α-plane with k = Rek + iImk(Rek > Imk ≥ 0). We have seen in appendix A that a typical wave mode travelling to x = +∞ is

$$A(y)e^{i\alpha_n x}\ \text{, where } \mathrm{Im}\alpha_n = 0\ \mathrm{Re}\alpha_n > 0 \text{ or } \mathrm{Im}\alpha_n > 0\ \ \mathrm{Re}\alpha_n = 0 \text{ for K real, or } \mathrm{Im}\alpha_n > 0$$

for k = Rek + iImk.

To determine the position of the α_n for k = Rek + iImk(Rek > Imk ≥ 0) we need to consider the cutα-plane for $\kappa = (k^2-\alpha^2)^{1/2}$, κ = k for α = 0 and Imκ > 0. In this cut plane 0 < argκ < π and the cuts at ±k are as shown in the figure below:



Cut Riemann sheet for which Imκ > 0, κ = k for α = 0.

Fig. 4

Now we have seen from appendix A that the waveguide modes are given by real values of $\kappa = \kappa_{2n-1}$ (= (2n-l)π/2d). This means that the corresponding values α_n in the cut *a*-plane $\kappa_n = (k^2 - \alpha_n^2)^{1/2}$ must lie on the hyperbola Re α_n Imα_n = Reklmk > 0, which is such that Imα_n > 0. This is the solid branch cut from k to i∞. See figure 2.

For a typical wave mode propagating or evanescent at x = -∞, that is $e^{-i\alpha_n x}$, will

correspond to the pole singularity $\alpha = -\alpha_n$ which will lie along the branch cut from –k to -i∞.

Clearly it can be seen that no singularities lie in the strip -Imk < Imα < Rek.

Appendix C

Here we shall carry out the explicit factorization of (24) and (25), and give their asymptotic behaviour.

For K(α) given by (24) we can write it in a more convenient form

$$K(\alpha)=\frac{\cos(\kappa b+\pi/4)\cos(\kappa b-\pi/4)[\cos(\kappa a+\pi/4)\cos(\kappa a-\pi/4)]^{-1}}{4\cos^2(\kappa\frac{(b-a)}{2}+\pi/4)\cos^2(\kappa\frac{(b-a)}{2}-\pi/4).} \qquad (1c)$$

By using the well known product for the cosine function

$$\cos z=\prod_{n=1}^{\infty}\left[1-\left(\frac{2z}{(2n-1)\pi}\right)^2\right],$$

then

$$\cos\left(a\kappa\pm\frac{\pi}{4}\right)=\prod_{n=1}^{\infty}\left(\frac{2a}{(2n-1)\pi}\right)^2\left[(4n-2+1)\frac{\pi}{4a}-\kappa\right]\left[(4n-2\pm1)\frac{\pi}{4a}+\kappa\right].$$

If we let

$$K(\alpha,a)=\cos\left(a\kappa+\frac{\pi}{4}\right)\cos\left(a\kappa-\frac{\pi}{4}\right), \qquad (2c)$$

then we can express K(α,a) as the convergent infinite product

$$K(\alpha,a)=\prod_{n=1}^{\infty}\left(\frac{2a}{(2n-1)\pi}\right)^4\left(\alpha^2-\left(K^2-\left(\frac{\pi(4n-1)}{4a}\right)^2\right)\right)\left(\alpha^2-\left(K^2-\left(\frac{\pi(4n-3)}{4a}\right)^2\right)\right)$$

$$a\neq0.$$

so that if

$$K(\alpha,a) = K_+(\alpha,a)\ K_-(\alpha,a)$$

then

$$K_+(\alpha,a) = K_-(-\alpha,a)$$

and

$$K_-(\alpha,a)=\prod_{n=1}^{\infty}\left(\frac{a}{(2n-1)\pi k}\right)^2(u_{4n-1}l-l_{4n-1}u)(u_{4n-1}l+l_{4n-1}u)(u_{4n-3}l-l_{4n-3}u)(u_{4n-3}l-l_{4n-3}u) \tag{3c}$$

where $l = (k-\alpha)^{1/2}$, $u = (k+\alpha)^{1/2}$, $u_n = (k+\alpha_n)^{1/2}$, $l_n = (k-\alpha_n)^{1/2}$, $\alpha_n = (k^2 - (n\pi/4a)^2)^{1/2}$. We can now express the function $K(\alpha)$ given by (lc) as a product

$$K(\alpha) = K_+(\alpha)K_-(\alpha) ,$$

where

$$K_+(\alpha) = K_-(-\alpha)$$

and

$$K_-(\alpha)=\frac{K_-(\alpha,b)}{K_-(\alpha,a)\{K_-(\alpha,(b-a)/2)\}^2}$$

or more explicitly

$$K_-(\alpha)=\prod_{n=1}^{\infty}\left(\frac{(2n-1)\pi}{(b-a)}\sqrt{\frac{b}{a}}\right)^4 \frac{\left(\tilde{u}_{4n-1}^2l^2-\tilde{l}_{4n-1}^2u^2\right)\left(\hat{u}_{4n-3}^2l^2-\hat{l}_{4n-3}^2u^2\right)}{\left(u_{4n-1}^2l^2-l_{4n-1}^2u^2\right)\left(u_{4n-3}^2l^2-l_{4n-3}^2u^2\right)\left(\tilde{u}_{4n-1}^2l^2-\tilde{l}_{4n-1}^2u^2\right)^2\left(\tilde{u}_{4n-3}^2l^2-\tilde{l}_{4n-3}^2u^2\right)}, \tag{4c}$$

where

$$u_n=(k+\alpha_n)^{1/2}, l_n=(k-\alpha_n)^{1/2}, \alpha_n=\left(k^2-\left(\frac{n\pi}{4a}\right)^2\right)^{1/2} \quad ;$$

$$\hat{u}_n=(k+\hat{\alpha}_n)^{1/2}, \hat{l}_n=(k-\hat{\alpha}_n)^{1/2}, \hat{\alpha}_n=\left(k^2-\left(\frac{n\pi}{4a}\right)^2\right)^{1/2} \quad ;$$

$$\tilde{u}_n=(k+\tilde{\alpha}_n)^{1/2}, \tilde{l}_n=(k-\tilde{\alpha}_n)^{1/2}, \tilde{\alpha}_n=\left(k^2-\left(\frac{n\pi}{4,(b-a)/2}\right)^2\right)^{1/2} .$$

In order to obtain the asymptotic behaviour of the factors $K_\pm(\alpha)$ given by (4c) we need to obtain the asymptotic behaviour of (3c) as $\alpha\rightarrow\infty$ in Imα>- Imk(Imα< Imk).

We note that K(α,a) = K(-α,a) and since $K(\alpha, a) = 0(e^{2a|\alpha|}$ as $\alpha \rightarrow \pm\infty$, then by Theorem 3.2 of Kranzer & Radlow (1965)

$$K_{\pm}(\alpha, a) = 0(e^{a|\alpha|}). \tag{5c}$$

Thus

$$K_{\pm}(\alpha) = 0(e^{-a|\alpha|}e^{b|\alpha|}e^{-(b-a)|\alpha|}) = 0(1) \tag{6c}$$

The factorization of

$$t(\alpha) = \frac{1}{\kappa}\text{In}\left[\frac{\cos(\pi/4+\kappa a)}{\cos(\pi/4-\kappa a)}\cdot\frac{\cos(\pi/4+\kappa b)}{\cos(\pi/4-\kappa b)}\right] = t_{+}(\alpha)+t_{-}(\alpha)$$

follows directly from Lüneburg and Hurd (1988) (when corrected), as

$$t_{-}(\alpha) = \frac{1}{\kappa}\text{In}\left[\prod_{n=1}^{\infty}\frac{(u_{4n-1}l+l_{4n-1}u)(u_{4n-3}l+l_{4n-3}u)(\hat{u}_{4n-1}l+\hat{l}_{4n-1}u)(\hat{u}_{4n-3}l-\hat{l}_{4n-3}u)}{(u_{4n-1}l-l_{4n-1}u)(u_{4n-3}l+l_{4n-3}u)(\hat{u}_{4n-1}l-\hat{l}_{4n-1}u)(\hat{u}_{4n-3}l+\hat{l}_{4n-3}u)}\right],$$

$$t_{+}(\alpha) = t_{-}(-\alpha)\,,$$

where

$$u = (k+\alpha)^{1/2}, u_n = (k+\alpha_n)^{1/2}, l = (K-\alpha)^{1/2}, l_n = (K-\alpha_n)^{1/2}\,,$$

$$\alpha_n = \left(K^2 - \left(\frac{n\pi}{4a}\right)^2\right)^{1/2}, \hat{u}_n = (K+\hat{\alpha}_n)^{1/2}, \hat{l}_n = (K+\hat{\alpha}_n = \left(K^2 - \left(\frac{n\pi}{4b}\right)^2\right)^{1/2}.$$

The behaviour of $t_{\pm}(\alpha)$ as $\alpha\rightarrow\infty$) is easily shown, by using the method of Lüneburg and Hurd (1988), to be

$$t_{-}(\alpha) = \frac{i}{\alpha}\ln\alpha + 0(\alpha^{-1}).$$