Partial-Neurons-Based State Estimation for Delayed Neural Networks with State-Dependent Noises under Redundant Channels

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Abstract

In this paper, the partial-neurons-based state estimation problem is studied for a class of delayed neural networks with state-dependent noises under redundant channels. For the purpose of improving the success rate of the data transmission from the sensor to the estimator, the redundant-channel-based transmission mechanism is considered. The main aim of the addressed problem is to design a state estimator to estimate the neurons' state by use of a small fraction of the sensor measurements. With the help of the Lyapunov stability theory, a sufficient condition is provided to ensure that the estimation error dynamics is exponentially mean-square bounded. The desired estimator gain is acquired by minimizing an asymptotic upper bound of the estimation error. Finally, a numerical simulation is carried out to demonstrate the usefulness of the presented estimator design scheme.

Keywords: Neural networks, state estimation, partially measurable nodes, redundant channels, state-dependent noises.

1. Introduction

It is well known that an artificial neural network (ANN) is an information response network composed of a large number of neurons stored in the network through the node connection weight, which has the capacity of fast processing speed and strong fault tolerance in a parallel and distributed mechanism. The past few decades have seen a popularity surge with ANNs for their wide range of application potentials in engineering practice including image processing, pattern recognition, associative memory, and optimization problems [3, 4, 9, 17, 28, 29, 34, 39, 42, 43]. In recent years, as the technology improvement in the storage and computing power, the ANN has shown its fast growing in large scale, strong coupling and nonlinearities that have contributed greatly to the complexities in the analysis of the dynamical behaviors for ANNs. Accordingly, a rich body of related results has been reported on the stability, synchronization and estimation problems for various ANNs (see [6, 8, 11, 16, 25, 38, 41, 44, 48] and the references therein).

For a large-scale ANN, due to the environment changing, limited sensing capacities or the complicated topology structure, it is difficult to get access to all the neural's state information. In this case, a more realistic situation is to estimate the neurons' states by using the noisy measurement signals measured by sensing devices. To this end, the state estimation problem of ANNs has been a research hotspot in the industrial filed and the artificial intelligence filed and much effort has been devoted to the design of easy-to-implement state estimators for ANNs [46, 47]. On the other hand, given the long-distance transmission of signals and the finite speed of information processing, the time-delay phenomenon ubiquitously exists in ANNs and could guide the system dynamics to an undesirable behavior (e.g instability and chaos). In this context, some in-depth discussions have been conducted on the analysis and synthesis of various delayed ANNs (e.g., constant, time-varying, random, or distributed), see [1, 12, 14, 23, 24, 33, 47] and the references therein.

In most existing literature related to the state estimation issue of ANNs, it implies a prerequisite that the measurement signals are accessible to each neuron node. Nevertheless, such a condition is against reality when considering the harsh measurement environment and the limited measurement technology [27, 30, 31]. This means that, in engineering practice, it is mostly the case that only partial measurement signals can be collected by sensing devices, which results in the so-called *partial-neurons-based state estimation problem*. Lately, the partial-neurons-based state estimation problem has received some initial research attention and the relevant results have been reported, see [13, 21]. For example, in [13], the passivity-guaranteed state estimation problem has been studied for the ANN with randomly occurring time delays and the partial availability of the neurons' measurement information. Moreover, for the sensor networks, the related results have been investigated in [7] for scalable H_∞-consensus filtering issue with censored measurements.

It should be emphasized that, due to the limited bandwidths of the communication channel, the packet dropout phenomenon inevitably occurs and widely exists in networked systems [2, 10, 15, 45, 36], which would deteriorate the system performance to an extent. Up to now, many data compensation schemes have been put forward with the intention to decrease the negative effect caused by the missing phenomenon including the one-step delay strategy and the prediction compensation [26]. On the other hand, from the viewpoint of reducing the occurrence probability of the packet dropout, the redundant channel mechanism has recently proven to be an effective tool to enhance the transmission reliability, whose main idea is to introduce additional channels for the signal transmission in case the current channel fails to work. Special attention has been paid to the redundant channel mechanism and fruitful achievements have appeared in the literature [32, 49, 50]. However, a thorough literature search has displayed that the relevant research work has not been extended to the partial-neurons-based state estimation problem for delayed neural networks owing mainly to the mathematical complexities and this constitutes the main motivation of our current investigation.

Concluding the above discussion, our research focus is on the partial-neurons-based state estimator design and performance analysis issues for delayed neural networks under the redundant channel mechanism. In doing so, the following two major challenges should be overcome: 1) how to compensate the estimation performance when only partial neurons can obtain the measurement signals; and 2) how to design a suitable partial-neurons-based estimation scheme to achieve the exponential boundedness of estimation errors for the overall neural network. To deal with the above two challenges, in this paper, by means of the redundant-channel-based communication scheme, a partial-neurons-based state estimator is first designed based on the partially available measurement signals. Then, with the help of the Lyapunov stability theory, some criteria are derived to ensure the overall exponential boundedness of estimation errors in the mean square sense.

The main contribution of this paper is highlighted as threefold: 1) a partial-neurons-based state estimator is proposed for delayed ANNs with state-dependent noises under redundant channels; 2) a criterion is established to guarantee the overall exponential boundedness of estimation errors in the mean square sense; and 3) the expected estimator gain is acquired in terms of the solution to a convex optimization problem with certain matrix inequality constraints.

The remainder of this paper is organized as follows. Section 2 gives the problem formulation and some preliminaries. Section 3 establishes the main results where some sufficient conditions are provided to ensure the exponentially ultimate boundedness of the estimation error. A numerical simulation is used to show the effectiveness of the proposed estimator design scheme in Section 4 and Section 5 summarizes the paper.

Notation. The notation used here is fairly standard except where otherwise stated. The occurrence probability of the event "·" is denoted by $\mathrm{Prob}\{\cdot\}$. $\mathrm{diag}\{X_1, X_2, \cdots, X_n\}$ stands for a block diagonal matrix whose diagonal blocks are matrices $X_1, X_2, \cdots, X_n$. $\mathbb{I}_p$ denotes the p-dimensional column vector with all entries 1. $\lambda_{\min}\{P\}$ denotes the minimum eigenvalue of the matrix P. In symmetric block matrices, the symbol * is used as an ellipsis for terms induced by symmetry.

2. Problem Formulation

The neural network considered in this paper is described by n coupled discrete time-delayed nonlinear systems where the dynamics of individual nodes is characterized by:

$$\begin{cases} x_{i,k+1} = a_i x_{i,k} + \sum_{j=1}^{n} b_{ij} f_j(x_{j,k}) + \sum_{l=k-d}^{k-1} \sum_{j=1}^{n} c_{ij} g_j(x_{j,l}) + \bar{a}_i x_{i,k} w_k \\ x_{i,k} = \varphi_{i,s}, \ \ s = -d, -d+1, \ldots, 0 \end{cases} \tag{1}$$

where $i \in \mathcal{N} \triangleq \{1, 2, \ldots, n\}$; $x_{i,k} \in \mathbb{R}$ denotes the state of the neuron i; $g_i(\cdot) : \mathbb{R} \mapsto \mathbb{R}$ and $f_i(\cdot) : \mathbb{R} \mapsto \mathbb{R}$ are the activation functions; a_i stands for the state feedback coefficient; $\bar{a}_i$ is the weight coefficient of the state-dependent noise; The constants b_{ij} and c_{ij} represent, respectively, the connection weights of the activation functions $f_i(\cdot)$ and $g_i(\cdot)$; $w_k \in \mathbb{R}$ is the state-dependent Gaussian white noise satisfying $\mathbb{E}\{w_k\} = 0$ and $\mathbb{E}\{w_k^2\} = 1$; $d \geq 1$ denotes the constant distributed time delay; $\varphi_{i,s}$ $(s = -d, -d+1, \ldots, 0)$ are given initial conditions; The scalar nonlinear functions $f_i(\cdot)$ and $g_i(\cdot)$ satisfy $f_i(0) = g_i(0) = 0$ and the following sector-bounded conditions:

$$\begin{aligned} &(f_i(u) - f_i(v) - \phi_1(u-v))(f_i(u) - f_i(v) - \phi_2(u-v)) \leq 0 \\ &(g_i(u) - g_i(v) - \psi_1(u-v))(g_i(u) - g_i(v) - \psi_2(u-v)) \leq 0 \end{aligned} \tag{2}$$

for $\forall\ u,\ v \in \mathbb{R}$, where ϕ_1, ϕ_2, ψ_1 and ψ_2 are known scalars.

In this paper, the measurement output is only available for a small fraction of the n neuron nodes. Without lose of generality, it is assumed that $\mathcal{N}_a = \{1, 2, \ldots, p\}$ $(p \leq n)$ denotes the set of those nodes whose outputs are measurable. Moreover, due to the bandwidth constraint and the unreliable transmission of communication network, the packet loss may inevitably occur in the network between the sensor and the receiver. To reduce the rate of packet loss, an effective method is to employ additional transmission channels instead of the traditional single transmission channel. In this paper, $N-1$ redundant channels are extra introduced to transmit the measurement outputs, whose model is described by the following equation:

$$y_{i,k} = \alpha_{1,k} d_1 x_{i,k} + m_i v_{i,k} + \sum_{l=2}^{N} \left\{ \prod_{s=1}^{l-1} ((1 - \alpha_{s,k}) \alpha_{l,k} d_l) x_{i,k} \right\} \tag{3}$$

for $i \in \mathcal{N}_a$, where $y_{i,k} \in \mathbb{R}$ represents the output received by the receiver; $v_{i,k} \in \mathbb{R}$ are the Gaussian white noises satisfying $\mathbb{E}\{v_{i,k}\} = 0$, $\mathbb{E}\{v_{i,k}v_{i,k}\} = 1$ and $\mathbb{E}\{v_{i,k}v_{j,k}\} = 0$ $(i \neq j)$; d_l $(l = 1, \ldots, N)$ and m_i are known scalars; $\alpha_{l,k}$ is a random Bernoulli distributed variable with the following statistical properties:

$$\text{Prob}\{\alpha_{l,k} = 1\} = \bar{\alpha}_l, \quad \text{Prob}\{\alpha_{l,k} = 0\} = 1 - \bar{\alpha}_l \tag{4}$$

with $\bar{\alpha}_l \in [0, 1]$ $(l = 1, \ldots, N)$ being the given constants. $\alpha_{l,k}$ $(l = 1, \ldots, N)$ are assumed to be uncorrelated with $w_{i,k}$, $v_{i,k}$ and $\varphi_{i,s}$ $(s = -d, -d+1, \ldots, 0)$.

Remark 1. *We have the following observations from the measurement model (3). First, considering the environment changing and the limited measurement capacity of the sensing device, certain measurement data may be difficult to be obtained. As such, in this paper, it is assumed that the measurement outputs are accessible for only a small fraction of the n neuron nodes to better reflect the engineering practice. Second, it is well known that the phenomenon of the packet dropout would inevitably degrade the system performance in a network environment. For the sake of decreasing the occurrence rate of packet loss, the redundant channel mechanism is employed in this paper, i.e. the communication network is equipped with additional $N - 1$ channels to enhance the transmission reliability. All N channels are sequentially detected whether the data packet is delivered to the receiver side until one of channels successfully transmits the data packet. Third, under the N-channel-based communication scheme, the probability of packet dropouts is $\Pi_{i=1}^{N}\bar{\alpha}_i$, which is far less than $\bar{\alpha}_i$ $(i = 1, 2, \ldots, N)$. To this end, an alternative and feasible approach is to increase the number of redundant channels to improve the reliability of the data transmission. However, it should be pointed out that the implementation of such a strategy needs some additional detection devices at each channel terminal to check whether the packet loss occurs. Therefore, one would like to fine-tune the number of redundant channels so as to play the balance between the cost of equipment/energy and the estimation performance.*

Based on (3), the following estimator is designed for the neuron i:

$$\begin{cases} \hat{x}_{i,k+1} = a_i\hat{x}_{i,k} + \sum_{j=1}^{n} b_{ij} f_j(\hat{x}_{j,k}) + \sum_{l=k-d}^{k-1}\sum_{j=1}^{n} c_{ij} g_j(\hat{x}_{j,l}) + l_i(y_{i,k} - \bar{\gamma}\hat{x}_{i,k}), \ i \in \mathcal{N}_a \\ \hat{x}_{i,k+1} = a_i\hat{x}_{i,k} + \sum_{j=1}^{n} b_{ij} f_j(\hat{x}_{j,k}) + \sum_{l=k-d}^{k-1}\sum_{j=1}^{n} c_{ij} g_j(\hat{x}_{j,l}), \ i \in \mathcal{N}/\mathcal{N}_a \\ \hat{x}_{i,k} = \hat{\varphi}_{i,s}, \ s = -d, -d+1, \ldots, n \end{cases} \tag{5}$$

where $\bar{\gamma} \triangleq \bar{\alpha}_1 d_1 + \sum_{l=2}^{N}\left\{\prod_{s=1}^{l-1}(1 - \bar{\alpha}_s)\bar{\alpha}_l d_l\right\}$ and l_i is the estimator gain to be determined.

Denote by $e_{i,k} \triangleq x_{i,k} - \hat{x}_{i,k}$ the estimation error whose dynamics can be written as

$$\begin{cases} e_{i,k+1} = a_i e_{i,k} + \sum_{j=1}^{n} b_{ij}\tilde{f}_j(e_{j,k}) + \bar{a}_i x_{i,k} w_k + \sum_{l=k-d}^{k-1}\sum_{j=1}^{n} c_{ij}\tilde{g}_j(e_{j,l}) \\ \qquad\qquad - l_i m_i v_{i,k} - l_i(\tilde{\gamma}_k x_{i,k} + \bar{\gamma} e_{i,k}), \ i \in \mathcal{N}_a \\ e_{i,k+1} = a_i e_{i,k} + \sum_{j=1}^{n} b_{ij}\tilde{f}_j(e_{j,k}) + \bar{a}_i x_{i,k} w_k + \sum_{l=k-d}^{k-1}\sum_{j=1}^{n} c_{ij}\tilde{g}_j(e_{j,l}), \ i \in \mathcal{N}/\mathcal{N}_a \end{cases} \tag{6}$$

where

$$\tilde{f}_j(e_{j,k}) \triangleq f_j(x_{j,k}) - f_j(\hat{x}_{j,k}),\ \tilde{g}_j(e_{j,k}) \triangleq g_j(x_{j,k}) - g_j(\hat{x}_{j,k})$$
$$\tilde{\gamma}_k \triangleq \gamma_k - \bar{\gamma},\ \gamma_k \triangleq \alpha_{1,k}d_1 + \sum_{l=2}^{N}\left\{\prod_{s=1}^{l-1}(1-\alpha_{s,k})\alpha_{l,k}d_l\right\}.$$

For notational simplicity, we denote

$$e_k \triangleq [e_{1,k}, e_{2,k}, \dots, e_{n,k}]^T,\ v_k \triangleq [v_{1,k}, v_{2,k}, \dots, v_{p,k}]^T,\ x_k \triangleq [x_{1,k}, x_{2,k}, \dots, x_{n,k}]^T$$
$$\tilde{f}(e_k) \triangleq [\tilde{f}_1(e_{1,k}), \tilde{f}_2(e_{2,k}), \dots, \tilde{f}_n(e_{n,k})]^T,\ \tilde{g}(e_k) \triangleq [\tilde{g}_1(e_{1,k}), \tilde{g}_2(e_{2,k}), \dots, \tilde{g}_n(e_{n,k})]^T$$
$$f(x_k) \triangleq [f_1(x_{1,k}), f_2(x_{2,k}), \dots, f_n(x_{n,k})]^T,\ g(x_k) \triangleq [g_1(x_{1,k}), g_2(x_{2,k}), \dots, g_n(x_{n,k})]^T$$
$$A \triangleq \mathrm{diag}\{a_1, a_2, \dots, a_n\}, B \triangleq [b_{ij}]_{n\times n},\ \bar{A} \triangleq \mathrm{diag}\{\bar{a}_1, \bar{a}_2, \dots, \bar{a}_n\},\ C \triangleq [c_{ij}]_{n\times n}$$
$$M \triangleq \mathrm{diag}\{m_1, m_2, \dots, m_p\},\ L \triangleq \mathrm{diag}\{l_1, l_2, \dots, l_p\},\ \bar{L} \triangleq [L^T\ 0]^T_{n\times p}$$
$$\bar{\Gamma} \triangleq [I_p \otimes \bar{\gamma}\quad 0]_{p\times n},\ \tilde{\Gamma}_k \triangleq [(I_p \otimes \tilde{\gamma}_k)\quad 0]_{p\times n},\ \theta_{i,k} \triangleq \begin{cases} \alpha_{1,k}, & i = 1 \\ \prod_{l=1}^{i-1}(1-\alpha_{l,k})\alpha_{i,k}, & 2 \le i \le N \end{cases}$$
$$\theta_k \triangleq [\theta_{1,k}, \theta_{2,k}, \dots, \theta_{N,k}]^T,\ \bar{\theta}_i \triangleq \mathbb{E}\{\theta_{i,k}\},\ \bar{\theta} \triangleq [\bar{\theta}_1, \bar{\theta}_2, \dots, \bar{\theta}_N]^T,\ \tilde{\theta}_k \triangleq \theta_k - \bar{\theta}_k$$
$$\bar{\phi}_1 \triangleq \mathrm{diag}_{2n}\{\phi_1, \dots, \phi_1\},\ \bar{\psi}_1 \triangleq \mathrm{diag}_{2n}\{\psi_1, \dots, \psi_1\},\ \bar{\phi}_2 \triangleq \mathrm{diag}_{2n}\{\phi_2, \dots, \phi_2\}$$
$$\bar{\psi}_2 \triangleq \mathrm{diag}_{2n}\{\psi_2, \dots, \psi_2\},\ \mathbb{H} \triangleq \begin{bmatrix} 0_{p\times n} & I_p & 0_{p\times(n-p)} \end{bmatrix},\ \bar{d} \triangleq [d_1, d_2, \dots, d_N]^T$$
$$\Sigma \triangleq \mathbb{E}\{(\tilde{\theta}_k\tilde{\theta}_k^T)\},\ \tilde{d} \triangleq \bar{d}^T\bar{d},\ \bar{\sigma} \triangleq \bar{d}^T\Sigma\bar{d}.$$

Then, the error dynamics (6) can be written in a more compact from:

$$e_{k+1} = (A - \bar{L}\bar{\Gamma})e_k + B\tilde{f}(e_k) + \sum_{l=k-d}^{k-1} C\tilde{g}(e_l) + \bar{A}x_k w_k - \bar{L}Mv_k - \bar{L}\tilde{\Gamma}_k x_k. \tag{7}$$

Furthermore, the plant (1) can be compactly denoted as

$$x_{k+1} = Ax_k + Bf(x_k) + \sum_{l=k-d}^{k-1} Cg(x_l) + \bar{A}x_k w_k. \tag{8}$$

Letting $\xi_k \triangleq \begin{bmatrix} e_k^T & x_k^T \end{bmatrix}^T$, one has

$$\xi_{k+1} = \mathcal{A}\xi_k + \mathcal{B}\mathcal{F}(\xi_k) + \sum_{l=k-d}^{k-1} \mathcal{C}\mathcal{G}(\xi_l) + \bar{\mathcal{A}}\xi_k w_k - \bar{\mathcal{L}}_1 M v_k - \bar{\mathcal{L}}_1\tilde{\Gamma}_{1,k}\xi_k \tag{9}$$

where

$$\mathcal{A} \triangleq \begin{bmatrix} A - \bar{L}\bar{\Gamma} & 0 \\ 0 & A \end{bmatrix},\ \bar{\mathcal{A}} \triangleq \begin{bmatrix} 0 & \bar{A} \\ 0 & \bar{A} \end{bmatrix},\ \mathcal{G}(\xi_k) \triangleq \begin{bmatrix} \tilde{g}(e_k) \\ g(x_k) \end{bmatrix},\ \mathcal{F}(\xi_k) \triangleq \begin{bmatrix} \tilde{f}(e_k) \\ f(x_k) \end{bmatrix}$$
$$\bar{\mathcal{L}}_1 \triangleq \begin{bmatrix} \bar{L} \\ 0 \end{bmatrix}_{2n\times p},\ \tilde{\Gamma}_{1,k} \triangleq \begin{bmatrix} 0 & \tilde{\Gamma}_k \end{bmatrix}_{p\times 2n},\ \mathcal{B} \triangleq \mathrm{diag}\{B, B\},\ \mathcal{C} \triangleq \mathrm{diag}\{C, C\}.$$

For the sake of sequel developments, we provide the following definition and lemma.

Definition 1. *The estimation error dynamics (9) is said to be exponentially mean-square bounded if there exist constants $\vartheta > 0$, $\pi > 0$ and $\varrho \in (0, 1)$ such that the inequality*

$$\mathbb{E}\{\|\xi_k\|^2\} \leq \vartheta \varrho^k \sup_{k=-d,-d+1,\ldots,0} \mathbb{E}\{\|\xi_k\|^2\} + \pi \tag{10}$$

holds for all k.

Lemma 1. *[22] Given a positive semidefinite matrix $Y \in \mathbb{R}^{n\times n}$, a vector $z_i \in \mathbb{R}^n$ and constants $\zeta_i \geq 0$ $(i = 1, 2, \ldots, s)$. The following inequality holds*

$$\left(\sum_{i=1}^{s} \zeta_i z_i\right)^T Y \left(\sum_{i=1}^{s} \zeta_i z_i\right) \leq \left(\sum_{i=1}^{s} \zeta_i\right) \sum_{i=1}^{s} \zeta_i z_i^T Y z_i. \tag{11}$$

3. Main results

In this section, a sufficient condition is established to ensure the exponential boundedness of the estimation error in the mean square sense. Then, the estimator gain is obtained in the sense of the minimum of an asymptotic upper bound of the estimation error.

Theorem 1. *Given the scalar $0 < \mu < 1$. Considering the system (1) under redundant channels (3), assume that there exist positive definite matrices P_1 and P_2, positive scalars ϵ_1 and ϵ_2 such that the following inequalities*

$$\Omega_2 \triangleq \begin{bmatrix} \bar{\Omega}_{11} & \bar{\Omega}_{12} & \bar{\Omega}_{13} & \Omega_{14} \\ * & \bar{\Omega}_{22} & 0 & \Omega_{24} \\ * & * & \bar{\Omega}_{33} & 0 \\ * & * & * & \Omega_{44} \end{bmatrix} < 0 \tag{12}$$

hold, where

$$\begin{aligned}
&\bar{\Omega}_{11} \triangleq \Omega_{11} - \epsilon_1 \frac{\bar{\phi}_1^T \bar{\phi}_2 + \bar{\phi}_2^T \bar{\phi}_1}{2} - \epsilon_2 \frac{\bar{\psi}_1^T \bar{\psi}_2 + \bar{\psi}_2^T \bar{\psi}_1}{2} \\
&\bar{\Omega}_{12} \triangleq \Omega_{12} + \epsilon_1 \frac{\bar{\phi}_1^T + \bar{\phi}_2^T}{2}, \ \bar{\Omega}_{13} \triangleq \epsilon_2 \frac{\bar{\psi}_1^T + \bar{\psi}_2^T}{2} \\
&\bar{\Omega}_{22} \triangleq \Omega_{22} - \epsilon_1 I, \ \bar{\Omega}_{33} \triangleq \Omega_{33} - \epsilon_2 I \\
&\Omega_{11} \triangleq \mathcal{A}^T P_1 \mathcal{A} + \bar{\mathcal{A}}^T P_1 \bar{\mathcal{A}} + (\mu - 1) P_1 + \bar{\sigma} \mathbb{H}^T \bar{\mathcal{L}}_1^T P_1 \bar{\mathcal{L}}_1 \mathbb{H} \\
&\Omega_{12} \triangleq \mathcal{A}^T P_1 \mathcal{B}, \ \Omega_{14} \triangleq \mathcal{A}^T P_1 \mathcal{C}, \ \Omega_{22} \triangleq \mathcal{B}^T P_1 \mathcal{B} \\
&\Omega_{24} \triangleq \mathcal{B}^T P_1 \mathcal{C}, \ \Omega_{33} \triangleq d P_2, \ \Omega_{44} \triangleq \mathcal{C}^T P_1 \mathcal{C} + \frac{(\mu - 1)^d}{d} P_2.
\end{aligned}$$

Then, the estimation error satisfies the exponential boundedness in the mean square sense. Moreover, an asymptotic bound is $\frac{\delta}{\rho\mu}$, i.e. $\lim_{k\to\infty} \mathbb{E}\{\|e_k\|^2\} = \frac{\delta}{\lambda_{\min}\{P_1\}\mu}$ where $\delta \triangleq \mathrm{tr}\{M^T \bar{\mathcal{L}}_1^T P_1 \bar{\mathcal{L}}_1 M\}$.

Proof. Construct the Lyapunov-Krasovskii functional as

$$V_k = V_{1,k} + V_{2,k} \tag{13}$$

where

$$V_{1,k} \triangleq \xi_k^T P_1 \xi_k,\ V_{2,k} \triangleq \sum_{j=k-d}^{k-1} \sum_{l=j}^{k-1} (1-\mu)^{k-l-1} \mathcal{G}^T(\xi_l) P_2 \mathcal{G}(\xi_l).$$

Calculating the term $\Delta V_k + \mu V_k$ and taking the mathematical expectation result in

$$\begin{aligned}
&\mathbb{E}\{\Delta V_{1,k} + \mu V_{1,k}\} \\
=&\mathbb{E}\{V_{1,k+1} + (\mu-1)V_{1,k}\} \\
=&\mathbb{E}\left\{\left(\mathcal{A}\xi_k + \mathcal{B}\mathcal{F}(\xi_k) + \sum_{l=k-d}^{k-1} \mathcal{C}\mathcal{G}(\xi_l) + \bar{\mathcal{A}}\xi_k w_k \right.\right. \\
&\left. -\bar{\mathcal{L}}_1 M v_k - \bar{\mathcal{L}}_1 \tilde{\Gamma}_{1,k}\xi_k \right)^T P_1 \left(\mathcal{A}\xi_k + \mathcal{B}\mathcal{F}(\xi_k)\right. \\
&\left.\left. + \sum_{l=k-d}^{k-1} \mathcal{C}\mathcal{G}(\xi_l) + \bar{\mathcal{A}}\xi_k w_k - \bar{\mathcal{L}}_1 M v_k - \bar{\mathcal{L}}_1 \tilde{\Gamma}_{1,k}\xi_k \right)\right\} \\
=&\mathbb{E}\left\{\xi_k^T(\mathcal{A}^T P_1 \mathcal{A} + \bar{\mathcal{A}}^T P_1 \bar{\mathcal{A}})\xi_k + \mathcal{F}^T(\xi_k)\mathcal{B}^T P_1 \mathcal{B}\mathcal{F}(\xi_k)\right. \\
&+ \left(\sum_{l=k-d}^{k-1} \mathcal{G}(\xi_l)\right)^T \mathcal{C}^T P_1 \mathcal{C} \left(\sum_{l=k-d}^{k-1} \mathcal{G}(\xi_l)\right) \\
&+ v_k^T M^T \bar{\mathcal{L}}_1^T P_1 \bar{\mathcal{L}}_1 M v_k + \xi_k^T \tilde{\Gamma}_{1,k}^T \bar{\mathcal{L}}_1^T P_1 \bar{\mathcal{L}}_1 \tilde{\Gamma}_{1,k}\xi_k \\
&+ 2\xi_k^T \mathcal{A}^T P_1 \mathcal{B}\mathcal{F}(\xi_k) + 2\xi^T \mathcal{A}^T P_1 \mathcal{C}\left(\sum_{l=k-d}^{k-1} \mathcal{G}(\xi_l)\right) \\
&\left. + 2\mathcal{F}^T(\xi_k)\mathcal{B}^T P_1 \mathcal{C}\left(\sum_{l=k-d}^{k-1} \mathcal{G}(\xi_l)\right) + (\mu-1)\xi_k^T P_1 \xi_k \right\}
\end{aligned} \tag{14}$$

and

$$\begin{aligned}
&\mathbb{E}\{\Delta V_{2,k} + \mu V_{2,k}\} \\
=&\mathbb{E}\left\{\sum_{j=k-d+1}^{k} \sum_{l=j}^{k} (1-\mu)^{k-l} \mathcal{G}^T(\xi_l) P_2 \mathcal{G}(\xi_l) \right. \\
&\left. - \sum_{j=k-d}^{k-1} \sum_{l=j}^{k-1} (1-\mu)^{k-l} \mathcal{G}^T(\xi_l) P_2 \mathcal{G}(\xi_l) \right\} \\
=&\mathbb{E}\left\{\sum_{j=k-d}^{k-1} \left(\sum_{l=j+1}^{k} - \sum_{l=j}^{k-1}\right)(1-\mu)^{k-l} \mathcal{G}^T(\xi_l) P_2 \mathcal{G}(\xi_l)\right\} \\
=&\mathbb{E}\left\{d\mathcal{G}^T(\xi_k) P_2 \mathcal{G}(\xi_k) - (1-\mu)^d \sum_{j=k-d}^{k-1} \mathcal{G}^T(\xi_j) P_2 \mathcal{G}(\xi_j)\right\}.
\end{aligned} \tag{15}$$

Next, we pay our attention to the term $\tilde{\Gamma}_{1,k}^T \bar{\mathcal{L}}_1^T P_1 \bar{\mathcal{L}}_1 \tilde{\Gamma}_{1,k}$. First, noting that $\tilde{\gamma}_k = \sum_{i=1}^N \tilde{\theta}_{i,k} d_i$, we derive

$$\tilde{\Gamma}_{1,k} = \left(\sum_{i=1}^{N} \tilde{\theta}_{i,k} d_i \right) \otimes \mathbb{H} = (\tilde{\theta}_k^T \bar{d}) \otimes \mathbb{H}. \tag{16}$$

Then, setting $\mathcal{P} \triangleq \bar{\mathcal{L}}_1^T P_1 \bar{\mathcal{L}}_1$, one has

$$\begin{aligned} \mathbb{E}\{\tilde{\Gamma}_{1,k}^T \mathcal{P} \tilde{\Gamma}_{1,k}\} =& \mathbb{E}\{((\tilde{\theta}_k^T \bar{d}) \otimes \mathbb{H})^T \mathcal{P}((\tilde{\theta}_k^T \bar{d}) \otimes \mathbb{H})\} \\ =& \mathbb{E}\left\{((\tilde{\theta}_k^T \bar{d})^T (\tilde{\theta}_k^T \bar{d})) \otimes (\mathbb{H}^T \mathcal{P} \mathbb{H})\right\} \\ =& (\bar{d}^T \Sigma \bar{d}) \otimes (\mathbb{H}^T \mathcal{P} \mathbb{H}) \\ =& \bar{\sigma} \mathbb{H}^T \mathcal{P} \mathbb{H}. \end{aligned} \tag{17}$$

According to Lemma 1, it is not difficult to see that

$$-\sum_{l=k-d}^{k-1} \mathcal{G}^T(\xi_l) P_2 \mathcal{G}(\xi_l) \leq -\frac{1}{d} \left(\sum_{l=k-d}^{k-1} \mathcal{G}(\xi_l) \right)^T P_2 \left(\sum_{l=k-d}^{k-1} \mathcal{G}(\xi_l) \right). \tag{18}$$

Therefore, we conclude from (14)–(18) that

$$\begin{aligned} & \mathbb{E}\{\Delta V_k + \mu V_k\} \\ =& \mathbb{E}\{\Delta V_{1,k} + \mu V_{1,k} + \Delta V_{2,k} + \mu V_{2,k}\} \\ \leq& \mathbb{E}\left\{\xi_k^T (\mathcal{A}^T P_1 \mathcal{A} + \bar{\mathcal{A}}^T P_1 \bar{\mathcal{A}} + (\mu - 1) P_1 + \bar{\sigma} \mathbb{H}^T \bar{\mathcal{L}}_1^T P_1 \bar{\mathcal{L}}_1 \mathbb{H}) \xi_k \right. \\ & + \mathcal{F}^T(\xi_k) \mathcal{B}^T P_1 \mathcal{B} \mathcal{F}(\xi_k) + d \mathcal{G}^T(\xi_k) P_2 \mathcal{G}(\xi_k) \\ & + v_k^T M^T \bar{\mathcal{L}}_1^T P_1 \bar{\mathcal{L}}_1 M v_k + 2 \xi_k^T \mathcal{A}^T P_1 \mathcal{B} \mathcal{F}(\xi_k) \\ & + \left(\sum_{l=k-d}^{k-1} \mathcal{G}(\xi_l) \right)^T \left(\mathcal{C}^T P_1 \mathcal{C} + \frac{(\mu - 1)^d}{d} P_2 \right) \left(\sum_{l=k-d}^{k-1} \mathcal{G}(\xi_l) \right) \\ & \left. + 2 \mathcal{F}^T(\xi_k) \mathcal{B}^T P_1 \mathcal{C} \left(\sum_{l=k-d}^{k-1} \mathcal{G}(\xi_l) \right) + 2 \xi_k^T \mathcal{A}^T P_1 \mathcal{C} \left(\sum_{l=k-d}^{k-1} \mathcal{G}(\xi_l) \right) \right\} \\ \triangleq& \mathbb{E}\{\eta_k^T \Omega_1 \eta_k\} + \operatorname{tr}\{M^T \bar{\mathcal{L}}_1^T P_1 \bar{\mathcal{L}}_1 M\} \end{aligned} \tag{19}$$

where

$$\begin{aligned} \eta_k \triangleq& \begin{bmatrix} \xi_k^T & \mathcal{F}^T(\xi_k) & \mathcal{G}^T(\xi_k) & \sum_{l=k-d}^{k-1} \mathcal{G}^T(\xi_l) \end{bmatrix}^T \\ \Omega_1 \triangleq& \begin{bmatrix} \Omega_{11} & \Omega_{12} & 0 & \Omega_{14} \\ * & \Omega_{22} & 0 & \Omega_{24} \\ * & * & \Omega_{33} & 0 \\ * & * & * & \Omega_{44} \end{bmatrix}. \end{aligned}$$

It follows readily from (2) that

$$(\mathcal{F}(\xi_k) - \bar{\phi}_1 \xi_k)^T (\mathcal{F}(\xi_k) - \bar{\phi}_2 \xi_k) \leq 0, \tag{20}$$

which is equivalent to

$$\frac{(\mathcal{F}(\xi_k)-\bar{\phi}_1\xi_k)^T(\mathcal{F}(\xi_k)-\bar{\phi}_2\xi_k)}{2}+\frac{(\mathcal{F}(\xi_k)-\bar{\phi}_2\xi_k)^T(\mathcal{F}(\xi_k)-\bar{\phi}_1\xi_k)}{2}\leq 0. \tag{21}$$

As a result, we have

$$\begin{aligned}\Theta_{1,k}\triangleq&\ \mathcal{F}^T(\xi_k)\mathcal{F}(\xi_k)-\frac{\mathcal{F}^T(\xi_k)\bar{\phi}_2\xi_k}{2}-\frac{\mathcal{F}^T(\xi_k)\bar{\phi}_1\xi_k}{2}\\&-\frac{\xi_k^T\bar{\phi}_2^T\mathcal{F}(\xi_k)}{2}-\frac{\xi_k^T\bar{\phi}_1^T\mathcal{F}(\xi_k)}{2}+\frac{\xi_k^T\bar{\phi}_1^T\bar{\phi}_2\xi_k}{2}+\frac{\xi_k^T\bar{\phi}_2^T\bar{\phi}_1\xi_k}{2}\\\leq&0.\end{aligned} \tag{22}$$

Similarly, one has

$$\begin{aligned}\Theta_{2,k}\triangleq&\ \mathcal{G}^T(\xi_k)\mathcal{G}(\xi_k)-\frac{\mathcal{G}^T(\xi_k)\bar{\psi}_2\xi_k}{2}-\frac{\mathcal{G}^T(\xi_k)\bar{\psi}_1\xi_k}{2}\\&-\frac{\xi_k^T\bar{\psi}_2^T\mathcal{G}(\xi_k)}{2}-\frac{\xi_k^T\bar{\psi}_1^T\mathcal{G}(\xi_k)}{2}+\frac{\xi_k^T\bar{\psi}_1^T\bar{\psi}_2\xi_k}{2}+\frac{\xi_k^T\bar{\psi}_2^T\bar{\psi}_1\xi_k}{2}\\\leq&0.\end{aligned} \tag{23}$$

Therefore, it is known from (19)–(23) that the following inequality holds

$$\begin{aligned}\mathbb{E}\{\Delta V_k+\mu V_k\}\leq&\mathbb{E}\{\eta_k^T\Omega_1\eta_k-\epsilon_1\Theta_{1,k}-\epsilon_2\Theta_{2,k}\}+\mathrm{tr}\{M^T\bar{\mathcal{L}}_1^TP_1\bar{\mathcal{L}}_1M\}\\\triangleq&\mathbb{E}\{\eta_k^T\Omega_2\eta_k\}+\delta.\end{aligned} \tag{24}$$

Then, it follows from (12) that

$$\mathbb{E}\{\Delta V_k+\mu V_k\}\leq\delta. \tag{25}$$

Through straightforward algebraic manipulations, we have

$$\begin{aligned}\mathbb{E}\{V_k\}\leq&(1-\mu)^k\left(\mathbb{E}\{V_0\}-\frac{\delta}{\mu}\right)+\frac{\delta}{\mu}\\\leq&(1-\mu)^k\mathbb{E}\{V_0\}+\frac{\delta}{\mu}.\end{aligned} \tag{26}$$

It is easily seen from (26) that

$$\begin{aligned}&\lambda_{\min}\{P_1\}\mathbb{E}\{\|e_k\|^2\}\leq\lambda_{\min}\{P_1\}\mathbb{E}\{\|\xi_k\|^2\}\\\leq&\lambda_{\min}\{P_1\}\mathbb{E}\{\|\eta_k\|^2\}\leq\mathbb{E}\{V_k\}\\\leq&(1-\mu)^k\mathbb{E}\{V_0\}+\frac{\delta}{\mu}\end{aligned} \tag{27}$$

and one further has

$$\mathbb{E}\{\|e_k\|^2\}\leq\frac{(1-\mu)^k}{\lambda_{\min}\{P_1\}}\mathbb{E}\{V_0\}+\frac{\delta}{\lambda_{\min}\{P_1\}\mu}. \tag{28}$$

Then, letting $k \to \infty$, one has

$$\lim_{k\to\infty} \mathbb{E}\{\|e_k\|^2\} = \frac{\delta}{\lambda_{\min}\{P_1\}\mu}. \tag{29}$$

As a consequence, the proof of this theorem is complete.

Having established a sufficient condition to ensure the exponential boundedness of the estimation error dynamics (9), next, we are interested in providing an analytical solution of the desired estimator gain and developing an optimization problem to obtain an asymptotic upper bound of the estimation error.

Theorem 2. *Given the scalars $0 < \mu < 1$ and $\rho > 0$. Considering the system (1) under redundant channels (3), suppose that there exist a positive definite diagonal matrix $P_1 \triangleq \mathrm{diag}\{p_{11}, \ldots, p_{2n,2n}\}$, a positive definite matrix P_2, a real-value matrix $X \triangleq [X_1\ 0]^T \in \mathbb{R}^{2n\times p}$ with $X_1 \triangleq \mathrm{diag}\{\chi_{11}, \chi_{22}, \ldots, \chi_{pp}\} \in \mathbb{R}^{p\times p}$, positive scalars ϵ_1, ϵ_2 and ϱ such that the following inequalities*

$$\begin{cases} \Omega_4 \triangleq \begin{bmatrix} \tilde{\Omega}_{11} & \tilde{\Omega}_{12} & \tilde{\Omega}_{13} & 0 & \vec{\Omega}_{15} & \tilde{\Omega}_{16} & \vec{\Omega}_{17} \\ * & \tilde{\Omega}_{22} & 0 & 0 & \tilde{\Omega}_{25} & 0 & 0 \\ * & * & \bar{\Omega}_{33} & 0 & 0 & 0 & 0 \\ * & * & * & \Omega_{44} & \tilde{\Omega}_{45} & 0 & 0 \\ * & * & * & * & \tilde{\Omega}_{55} & 0 & 0 \\ * & * & * & * & * & \tilde{\Omega}_{66} & 0 \\ * & * & * & * & * & * & \tilde{\Omega}_{77} \end{bmatrix} < 0 \\ \rho I - P_1 \le 0 \\ \begin{bmatrix} -\varrho I & M^T X^T \\ & -P_1 \end{bmatrix} < 0 \end{cases} \tag{30}$$

hold, where

$$\begin{aligned}
&\tilde{\Omega}_{11} \triangleq (\mu - 1)P_1 - \epsilon_1 \frac{\bar{\phi}_1^T \bar{\phi}_2 + \bar{\phi}_2^T \bar{\phi}_1}{2} - \epsilon_2 \frac{\bar{\psi}_1^T \bar{\psi}_2 + \bar{\psi}_2^T \bar{\psi}_1}{2} \\
&\tilde{\Omega}_{12} \triangleq \epsilon_1 \frac{\bar{\phi}_1^T + \bar{\phi}_2^T}{2},\ \bar{\Omega}_{13} \triangleq \epsilon_2 \frac{\bar{\psi}_1^T + \bar{\psi}_2^T}{2},\ \tilde{\Omega}_{77} \triangleq -P_1 \\
&\tilde{\Omega}_{22} \triangleq -\epsilon_1 I,\ \tilde{\Omega}_{33} \triangleq dP_2 - \epsilon_2 I,\ \tilde{\Omega}_{44} \triangleq \frac{(\mu-1)^d}{d} P_2 \\
&\vec{\Omega}_{15} \triangleq \begin{bmatrix} A^T P_{11} - \bar{\Gamma}^T [X_1^T\ \ 0] & 0 \\ 0 & A^T P_{22} \end{bmatrix},\ \tilde{\Omega}_{25} \triangleq \mathcal{B}^T P_1 \\
&\tilde{\Omega}_{45} \triangleq \mathcal{C}^T P_1,\ \tilde{\Omega}_{55} \triangleq -P_1,\ \tilde{\Omega}_{16} \triangleq \bar{\mathcal{A}}^T P_1,\ \tilde{\Omega}_{66} \triangleq -P_1 \\
&\vec{\Omega}_{17} \triangleq \sqrt{\bar{\sigma}} \mathbb{H}^T [X_1^T\ \ 0],\ P_{11} \triangleq \mathrm{diag}\{p_{11}, \ldots, p_{pp}\} \\
&P_{22} \triangleq \mathrm{diag}\{p_{(p+1),(p+1)}, \ldots, p_{2n,2n}\}.
\end{aligned}$$

Then, the estimation error is exponentially bounded in the mean square sense, whose asymptotic upper bound is $\lim_{k\to\infty} \mathbb{E}\{\|e_k\|^2\} = \frac{\varrho}{\rho\mu}$. Furthermore, the minimum of this asymptotic upper bound can be derived by solving the following minimization problem

$$\min\{\varrho\} \tag{31}$$

subject to the LMI constraints (30). The estimator gain can be given as

$$l_i = p_{ii}^{-1}\chi_{ii}, \ i = 1, 2, \ldots, p. \tag{32}$$

PROOF. Substituting $X_1 = P_1\bar{L}$ into the first and second inequalities in (30) yields

$$\Omega_3 \triangleq \begin{bmatrix} \tilde{\Omega}_{11} & \tilde{\Omega}_{12} & \tilde{\Omega}_{13} & 0 & \tilde{\Omega}_{15} & \tilde{\Omega}_{16} & \tilde{\Omega}_{17} \\ * & \tilde{\Omega}_{22} & 0 & 0 & \tilde{\Omega}_{25} & 0 & 0 \\ * & * & \bar{\Omega}_{33} & 0 & 0 & 0 & 0 \\ * & * & * & \Omega_{44} & \tilde{\Omega}_{45} & 0 & 0 \\ * & * & * & * & \tilde{\Omega}_{55} & 0 & 0 \\ * & * & * & * & * & \tilde{\Omega}_{66} & 0 \\ * & * & * & * & * & * & \tilde{\Omega}_{77} \end{bmatrix} < 0 \tag{33}$$

where $\tilde{\Omega}_{15} \triangleq \mathcal{A}^T P_1$ and $\tilde{\Omega}_{17} \triangleq \sqrt{\bar{\sigma}}\mathbb{H}^T\bar{\mathcal{L}}_1^T P_1$. Then, applying the Schur Complement Lemma to (33), one easily obtains the first inequality of (30).

Then, according to the second inequality of (30), we have

$$\mathbb{E}\{\|e_k\|^2\} \leq \frac{(1-\mu)^k}{\rho}\mathbb{E}\{V_0\} + \frac{\delta}{\rho\mu}. \tag{34}$$

Furthermore, with the help of the Schur Complement Lemma, it is easily seen from the third inequality of (30) that

$$\delta \triangleq \mathrm{tr}\{M^T\bar{\mathcal{L}}_1^T P_1\bar{\mathcal{L}}_1 M\} < \varrho I. \tag{35}$$

Combining (34) and (35) results in

$$\mathbb{E}\{\|e_k\|^2\} \leq \frac{(1-\mu)^k}{\rho}\mathbb{E}\{V_0\} + \frac{\varrho}{\rho\mu}. \tag{36}$$

Consequently, it can be concluded that an asymptotic bound of $\mathbb{E}\{\|e_k\|^2\}$ is $\frac{\varrho}{\rho\mu}$ and the minimum of this asymptotic bound can be derived by minimizing ϱ, which is equivalent to the optimization problem (31). The proof of this theorem is ended.

Remark 2. *So far, the analysis and synthesis problem has been successfully solved by resorting to the Lyapunov stability theory, the stochastic analysis technique and the linear matrix inequality method. To be more specific, Theorem 1 has established a sufficient condition to guarantee the resulting estimation error systems (9) to achieve the overall exponential mean-square boundedness. However, notice that these conditions are non-convex with respect to the estimator gain matrix* $\bar{\mathcal{L}}_1$*. To this end, by means of the Schur Complement Lemma, Theorem 2 has been established where the required estimator gain matrix can be acquired by solving an optimization problem with certain matrix inequality constraints. It should be particularly emphasized that all important elements affecting the estimation performance have been involved in the matrix inequalities in Theorem 2, which includes the system parameters, time-delay length, damping exponent and probabilities of the packet dropout.*

Remark 3. *It should be stressed that the computational complexity of a standard LMI system is polynomial-time, which is bounded by* $O(\mathcal{M}\mathcal{N}^3 \log(\mathcal{V}/\epsilon))$ *where* $\mathcal{M}$ *is the total row size of the LMI,* $\mathcal{N}$ *is the total number of scalar decision variables,* $\mathcal{V}$ *is a data-dependent scaling factor, and* ϵ *is the relative accuracy set for the algorithm. In view of this, we focus our attention on the state estimation problem for delayed neural networks (1) where the number of neuron nodes is* n*, and the dimensions of network variables are, respectively,* $x_k \in \mathbb{R}^n$*,* $w_k \in \mathbb{R}$ *and* $v_k \in \mathbb{R}^p$*. From Theorem 1, we can easily observe that* $\mathcal{M} = 8n$ *and* $\mathcal{N} = n^2 + n + 2$*, and therefore the computational complexity of the LMI-based stability criterion of the estimation error is* $O(8n^3)$*. Similarly, from Theorem 2, the computational complexity of the LMI-based state estimator design problem for delayed neural networks (1) is* $O(7n^3 + 35n^2)$*. Obviously, the computational complexity of the LMI-based estimation algorithm is dependent polynomially on the number of neurons. Fortunately, with the accelerated development of the computer technology, the computing speed has been not a main concern, which greatly facilitates the application of the LMI-based technology in the large-scale network.*

4. An illustrative example

In this section, a numerical simulation is presented to illustrate the effectiveness of the developed estimator design scheme. A neural network consisting of four neurons is considered with the following parameters:

$$A = \begin{bmatrix} 0.4 & 0 & 0 & 0 \\ 0 & 0.2 & 0 & 0 \\ 0 & 0 & 0.3 & 0 \\ 0 & 0 & 0 & 0.1 \end{bmatrix}, \ \bar{A} = \begin{bmatrix} 0.31 & 0 & 0 & 0 \\ 0 & 0.3 & 0 & 0 \\ 0 & 0 & 0.3 & 0 \\ 0 & 0 & 0 & 0.2 \end{bmatrix}, \ B = C = \begin{bmatrix} 0.5 & 0.2 & 0.3 & 0.1 \\ 0.2 & 0.2 & 0 & 0 \\ 0.2 & 0 & 0.2 & 0 \\ 0.1 & 0 & 0 & 0.1 \end{bmatrix}.$$

The activation functions $f_i(\cdot)$ and $g_i(\cdot)$ $(i = 1, \dots, 4)$ are selected as

$$\begin{aligned} f_1(x_{1,k}) &= \tanh(0.2x_{1,k}), \ f_2(x_{2,k}) = \tanh(0.5x_{2,k}) \\ f_3(x_{3,k}) &= \tanh(0.4x_{3,k}), \ f_4(x_{4,k}) = \tanh(0.2x_{4,k}) \\ g_1(x_{1,k}) &= \tanh(0.5x_{1,k}), \ g_2(x_{2,k}) = \tanh(0.35x_{2,k}) \\ g_3(x_{3,k}) &= \tanh(0.24x_{3,k}), \ g_4(x_{4,k}) = \tanh(0.2x_{4,k}), \end{aligned}$$

which satisfy the sector-bounded condition (2) with

$$\begin{aligned} \phi_1 &= \mathrm{diag}\{-0.2, -0.5, -0.4, -0.2\}, \ \phi_2 = \mathrm{diag}\{0.2, 0.5, 0.4, 0.2\} \\ \psi_1 &= \mathrm{diag}\{-0.5, -0.35, -0.24, -0.2\}, \ \psi_2 = \mathrm{diag}\{0.5, 0.35, 0.24, 0.2\} \end{aligned}$$

where $x_k = [x_{1,k} \ x_{2,k} \ x_{3,k} \ x_{4,k}]^T$.

The outputs are measured by four sensors. Assume that, during the transmission of the measurement outputs, there exist one redundant channel for the sake of improving the success rate of signal transmission. The corresponding parameters are set as $d_1 = d_2 = 0.5$, $m_1 = 0.16$, $m_2 = 0.18$, $m_3 = 0.2$, $m4 = 0.25$, $\bar{\alpha}_1 = 0.9$ and $\bar{\alpha}_2 = 0.8$. The length of the time-delay is $d = 1$. The initial values are given as $x_0 = \begin{bmatrix} 0.25 & 0.3 & -0.2 & 0.1 \end{bmatrix}^T$ and $\hat{x}_0 = \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}^T$.

Due to the harsh measurement environment and the limited measurement technology, only partial measurements can be available, which is divided into three cases in this simulation. **C**ase I: only the measurement from node 1 can be obtained; **C**ase II: only the measurements from nodes 1 and 2 can be obtained; and **C**ase III: all measurements from nodes 1–4 can be obtained.

For **C**ase II, according to Theorem 2, the state estimation problem can be solved by using LMI Toolbox. The corresponding parameters are computed as

$$p_{11} = 1.2329,\ p_{22} = 2.1213,\ \chi_{11} = 0.1966,\ \chi_{22} = 0.1627.$$

Then, based on the relationship $l_i = p_{ii}^{-1}\chi_{ii}$ $(i = 1, 2)$, the estimator gains are $l_{11} = 0.1594$ and $l_{22} = 0.0767$. The simulation results are shown in Figures 1–5 where Figures 1–4 plot the trajectories of actual states and their estimates and Figure 5 plots the estimation errors.

Figure 6 and Table 1 reveal the influence of the number of available nodes on the estimation performance. Obviously, with the increase of the obtained measurement information, the mean-square error defined by MSE=$\frac{1}{T}\sum_{k=1}^{T}(e_k^T e_k)$ and the optimized upper $\frac{\varrho}{\rho\mu}$ accordingly decrease, which implies that the more nodes' information is used and better estimation performance is achieved. The simulation has well confirmed the effectiveness of our theoretical results.

Table 1: The asymptotic upper bound $\frac{\varrho}{\rho\mu}$ for different cases

Case I	Case II	Case III
5.1797	3.8505	2.3922

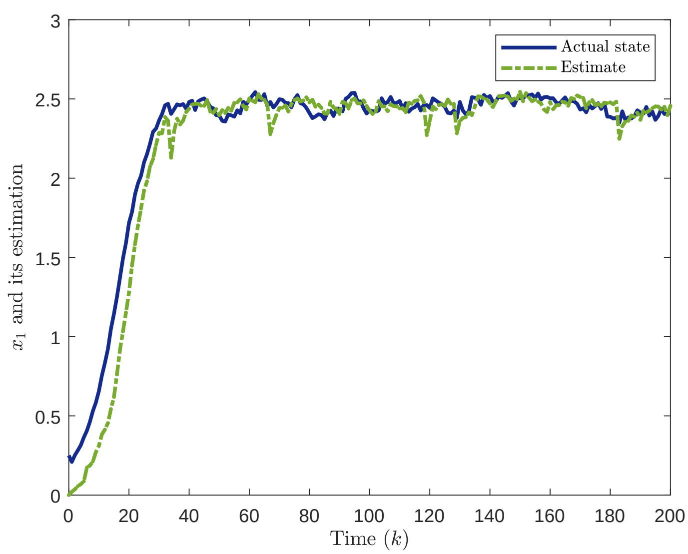

Figure 1: State trajectory x_1 and its estimate.

5. Conclusions

In this paper, the state estimation problem has been discussed for a class of time-delay neural networks subject to the state-dependent noises under the redundant channel transmission mecha-

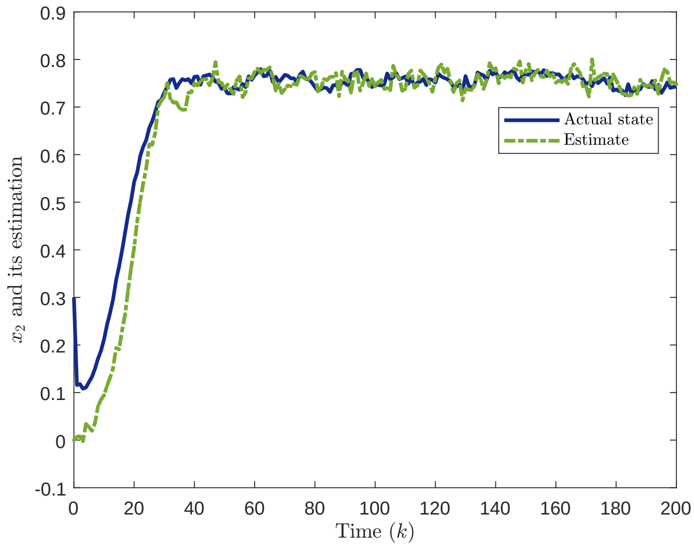

Figure 2: State trajectory x_2 and its estimate.

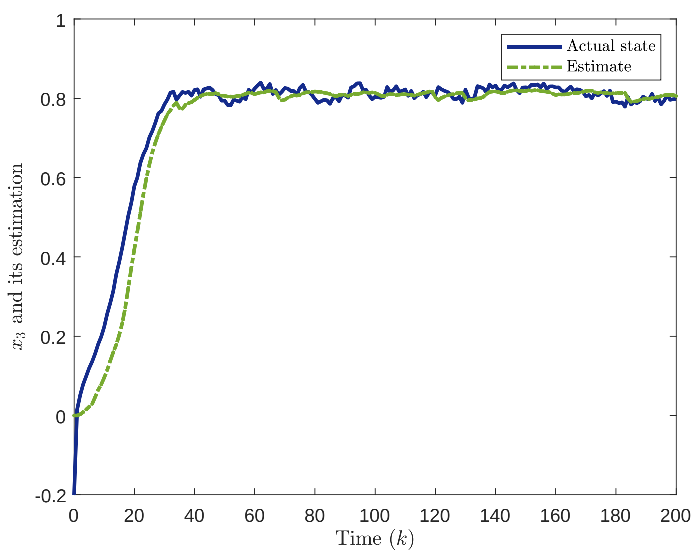

Figure 3: State trajectory x_3 and its estimate.

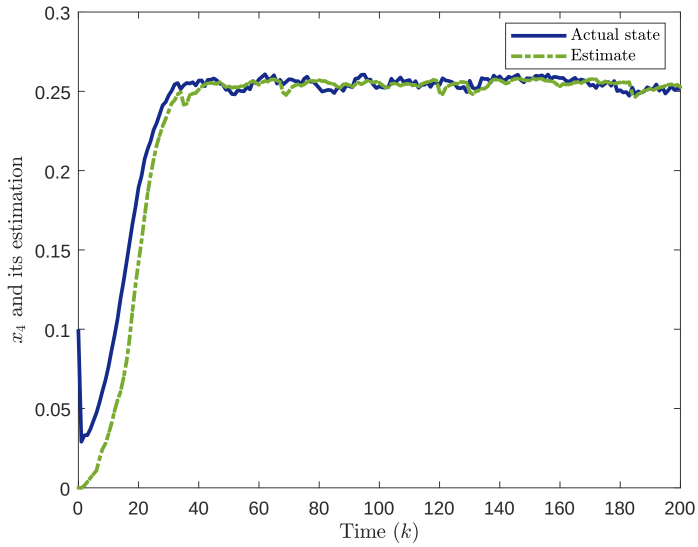

Figure 4: State trajectory x_4 and its estimate.

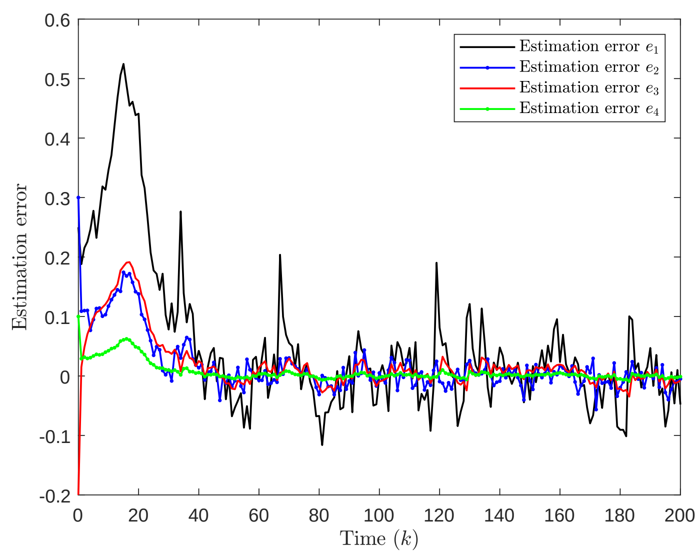

Figure 5: Estimation errors.

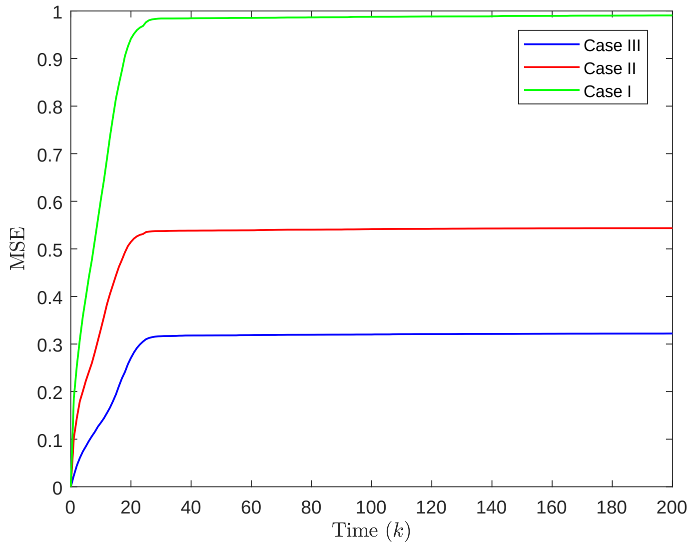

Figure 6: MSE.

nism. Additional transmission channels have been employed in order to decrease the occurrence probability of packet loss. A redundant-channel-based estimator has been designed by using a small fraction of the measurement outputs. In the simultaneous presence of both the redundant channels and the state-dependent noises, a sufficient condition has been established to ensure that the estimation error dynamics achieves the exponential mean-square boundedness. The required estimator gain has been derived by means of the solution to an optimization problem with certain matrix inequality constraints. In the end, the effectiveness of the designed estimator has been illustrated via a numerical simulation. Further research topics would include the extension of our results to 1) the partial-neurons-based state estimator design problem for ANNs with various communication protocols [5, 18, 19, 20, 35] and 2) time-varying probability distributions of packet dropouts [37, 40].

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