TR/10/83 October 1983

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paper submitted to SIAM Journal on Scientific and Statistical Computing

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Key words, rational, shape preserving, monotonic, convex, interpolation

Abstract

An explicit representation of a C^1 piecewise rational cubic function is developed which can be used to solve the problem of shape preserving interpolation. It is shown that the interpolation method can be applied to convex and/or monotonic sets of data and an error analysis of the interpolant is given. The scheme includes, as a special case, the monotonic rational quadratic interpolant considered by the authors in [1] and [5]. However, the requirement of convexity necessitates the generalization to the rational cubic form employed here.

1. Introduction

The problem of shape preserving interpolation has been considered by a number of authors. Fritsch and Carlson [4] and Fritsch and Butland [3] have discussed the piecewise cubic interpolation of monotonic data. Also, McAllister, Passow and Roulier [6] and Passow and Roulier [9] consider the piecewise polynomial interpolation of monotonic and convex data. In particular, an algorithm for quadratic spline interpolation is given in McAllister, and Roulier [7]. An alternative to the use of polynomials, for the interpolation of monotonic data, is the application of piecewise rational quadratic functions, as described by the authors in references [1] and [5].

In this paper we describe a piecewise rational cubic function which can be used to solve the problem of shape preserving interpolation. The rational cubic includes the rational quadratic function as a special case. However, the rational quadratic is not necessarily applicable to the interpolation of convex data and this necessitates the generalization to the rational cubic form employed here.

The paper begins with a definition and error analysis of the rational cubic interpolant. The application of the interpolant to monotonic and/or convex sets of data is then discussed in Section 3. It is shown that $O(h^4)$ error bounds can be expected when exact derivative information is given at the data points. For the case where the derivatives are not known, these have to be estimated and various schemes for this are considered. Finally, in Section 4, examples of the rational interpolants applied with various derivative schemes are given.

2. The Rational Cubic Interpolant

Let (x_i, f_i) $i = 1,\ldots,n$ be a given set of data points, where $x_1 < x_2 < \ldots < x_n$. Let

$$h_i = x_{i+1} - x_1 ,$$

$$\Delta_i = (f_{i+1} \; f_i)/h_i . \qquad (2.1)$$

A piecewise rational cubic function $s \in C^1[x_1, x_n]$ is defined as follows. For $x \in [x_i, x_{i+1}]$ let

$$\theta = (x - x_i)/h_i . \qquad (2.2)$$

Then

$$s(x) = P_i(\theta)/Q_i(\theta), \qquad (2.3)$$

where

$$P_i(\theta) = f_{i+1}\,\theta^3 + (r_i f_{i+1} - h_i d_{i+1})\,\theta^2(1-\theta) + (r_i f_i + h_i d_i)\theta(1-\theta)^2 + f_i(1-\theta)^3 , \qquad (2.4)$$

$$Q_i(\theta) = \theta^3 + r_i\,[\theta^2(1-\theta) + \theta(1-\theta)^2] + (1-\theta)^3$$

$$= 1 + (r_i - 3)\,\theta\,(1-\theta) . \qquad (2.5)$$

The rational cubic has the following interpolatory properties

$$s(x_i) = f_i , \quad s(x_{i+1}) = f_{i+1} ,$$

$$s^{(1)}(x_i) = d_i , \; s^{(1)}(x_{i+1}) = d_{i+1} , \qquad (2.6)$$

where $s^{(1)}$ denotes differentiation with respect to x and the d_i denote derivative values given at the knots x_i.

The parameter r_i is to be chosen such that

$$r_i > -1 \qquad (2.7)$$

which ensures a strictly positive denominator in the rational cubic. When r_i=3 the rational cubic clearly reduces to the standard cubic

Hermite polynomial. For our purposes r_i will be chosen to ensure that the interpolate preserves the monotonic or convex shape of the data. This choice requires a knowledge of $s^{(1)}(x)$ and $s^{(2)}(x)$ which are given in the relevant sections below.

Remark It should be noted that the interpolant will define a non-linear operator, since the r_i. will be dependent on the data. However the interpolant to the zero function is zero. Also, the interpolant to the data $K + f_i$, $i = 1,\ldots,n$, where K is a constant, is K. + s(x), provided the r_i, are independent of such translations. This will be the case for the choices of r_i in this paper.

An error bound for the rational cubic is given by the following theorem.

Theorem 2.1 Let $f \in C^4[x_1, x_n]$ and let s be the piecewise rational cubic interpolant such that $s(x_i) = f(x_i)$ and $s^{(1)}(x_i) = d_i$, $i = 1,\ldots,n$. Then for $x \in [x_i, x_{i+1}]$

$$|f(x) - s(x)| \leq \frac{h_i}{4c_i} \max\left\{\left|f_i^{(1)} - d_i\right|, \left|f_{i+1}^{(1)} - d_{i+1}\right|\right\}$$
$$+ \frac{1}{384c_i}\left\{h_i^4 \left\|f^{(4)}\right\| (1 + |r_i - 3|/4) + 4|r_i - 3|(h_i^3 \left\|f^{(3)}\right\| + 3h_i^2 \left\|f^{(2)}\right\|)\right\} \tag{2.8}$$

Where

$$c_i = \begin{cases} (1+r_i)/4 & \text{if } -1 < r_i < 3 \\ 1 & \text{if } r_i \geq 3. \end{cases} \tag{2.9}$$

and $\|\bullet\|$ denotes the uniform norm on $[xi, x_{i+1}]$.

Proof On $[xi, x_{i+1}]$ let $x(\theta) = x_i + \theta h_i$ and $F_i(\theta) = f(x(\theta))$.

Then

$$f(x) - s(x) = F_i(\theta) - P_i(\theta) / Q_i(\theta)$$

where $0 \le \theta \le 1$. Consider

$$\left|F_i(\theta) - P_i(\theta)/Q_i(\theta)\right| \le [\left|F_i(\theta)\ Q_i(\theta) - P_i^*(\theta)\right| + \left|P_i^*(\theta) - P_i(\theta)\right|] / \left|Q_i(\theta)\right|, \quad (2.10)$$

where (cf. (2.4))

$$P_i^*(\theta) = f_{i+1}\theta^3 + (r_i f_{i+1} - h_i f_{i+1}^{(1)})\ \theta(1-\theta) + (r_i f_i + h_i f_i^{(1)})\theta(1-\theta)^2 + f_i(1-\theta.)^3 . \quad (2.11)$$

Then $P_i^*(\theta)$ is the cubic Hermits interpolant to $F_i(\theta)\ Q_i(\theta)$ on $0 \le \theta \le 1$ with the error bound

$$\left|F_i(\theta)\ Q_i(\theta) - P_i^*(\theta)\right| \le \frac{1}{384} \max_{0 \le \theta \le 1} \left|\frac{d^4}{d\theta^4} F_i(\theta)\ Q_i(\theta)\right| ,$$

$$= \frac{1}{384} \max_{0 \le \theta \le 1} \left|F_i^{(4)}(\theta)\ Q_i(\theta) + 4\ F_i^{(3)}(\theta)\ Q_i^{(1)}(\theta) + 6F_i^{(2)}(\theta)\ Q_i^{(2)}(\theta)\right|$$

since $Q_i(\theta)$ is quadratic. Now

$$\left|Q_i(\theta)\right| \le 1 + \left|r_i - 3\right|/4, \left|Q_i^{(1)}(\theta)\right| \le \left|r_i - 3\right|, \left|Q_i^{(2)}(\theta)\right| = 2\left|r_i - 3\right|$$

and $F_i^{(j)}(\theta) \le h_i^{\,j}\ \|f^{(j)}\|$.

Hence

$$\left|F_i(\theta)\ Q_i(\theta) - P_i^*(\theta)\right| \le \frac{1}{384}\Big\{h_i^4\ \|f^{(4)}\|\ (1 + |r_i - 3|/4) + 4h_i^3\ \|f^{(3)}\|\,|r_i - 3|$$

$$+ 12h_i^2\ \|f^{(2)}\|\,|r_i - 3|\Big\} \quad (2.12)$$

Also

$$|p_i^*(\theta) - p_i(\theta)| = |\theta(1-\theta)h_i[\theta(d_{i+1} - f_{i+1}^{(1)}) + (1+\theta)(f_i^{(1)} - d_i)]|,$$

$$\le \frac{1}{4}h_i \max\left\{ f_i^{(1)} + d_i|, |f_{i+1}^{(1)} - d_{i+1}|\right\} \quad (2.13)$$

Finally

$$|Q_i(\theta)| = Q_i(\theta) \geq \begin{cases} 1 & \text{if } r_i \geq 3 \;, \\ 1 - (3 - r_i)/4 & \text{if } -1 < r_i < 3 \,. \end{cases} \tag{2.14}$$

Combining (2.12), (2.13) and (2.14) in inequality (2.10) completes the proof of the theorem.

A direct consequence of Theorem 2.1 which is of relevance in the remaining sections is the following corollary.

Corollary 2.1 Let $x \in [x_i, x_{i+1}]$.

(i) If $d_i - f_i^{(1)} = 0(h_i^2) = d_{i+1} - f_{i+1}^{(1)}$ and $r_i - 3 = 0(h_i)$

then $|f(x) - s(x)| = 0(h_i^3)$.

(ii) If $d_i - f_i^{(1)} = 0(h_i^3) = d_{i+1} - f_{i+1}^{(1)}$ and $r_i - 3 = 0(h_i^2)$

then $|f(x) - s(x)| = 0(h_i^4)$.

The above theorem and corollary show that r_i should ideally be such that $r_i - 3 = 0(h^2)$. We now consider how r_i can be chosen to preserve the monotonic or convex shape of the data, whilst maintaining this optimal $0(h^2)$ requirement.

3. Shape Preserving Interpolation

3.1 Monotonic Data

For simplicity of presentation, we assume a monotonic increasing set of data so that

$$f_1 \leq f_2 \leq \ldots \leq f_n, \tag{3.1}$$

or equivalently

$$\Delta_i \geq. 0, \; i = 1, \ldots, n-1. \tag{3.2}$$

(The case of a monotonic decreasing set of data can be treated in a similar manner.) For a monotonic interpolant s(x), it is then necessary that the derivative parameters should be such that

$$d_i \geq 0 \ , \ i = 1,\ldots,n \ . \tag{3.3}$$

Now s(x) is monotonic increasing if and only if

$$s^{(1)}(x) \geq 0 \tag{3.4}$$

for all $x \in [x_1, x_n]$. For $x \in [x_i, x_{i+1}]$ it can be shown, after some simplification, that

$$s^{(1)}(x) = \frac{d_{i+1}\theta^4 + \alpha_i\theta^3(1-\theta) + \beta_i\theta^2(1-\theta)^2 + \gamma_i\theta(1-\theta)^3 + d_i(1+\theta)^4}{[1 + r_i - 3)\,\theta\,(1-\theta)]^2} \ , \tag{3.5}$$

where

$$\alpha_i = 2(r_i\Delta_i - d_i) \ ,$$

$$\beta_i = (r_i^2 + 3)\Delta_i - r_i(d_i + d_{i+1}) \ ,$$

$$\gamma_i = 2(r_i\Delta_i - d_{i+1}) . \tag{3.6}$$

Thus sufficient conditions for monotonicity on $[x_i, x_{i+1}]$ are

$$\alpha_i \geq 0, \beta_i \geq 0, \ \gamma_i \geq 0 \tag{3.7}$$

where the necessary conditions $d_i \geq 0$ and $d_{i+1} \geq 0$ are assumed.

If $\Delta_i > 0$ (strict inequality) then a sufficient condition for (3.7) is

$$r_i \geq (d_i + d_{i+1})/\Delta_i \ . \tag{3.8}$$

In particular, if

$$r_i = 1 + (d_i + d_{i+1})/\Delta_i \tag{3.9}$$

then the rational cubic defined by (2.3) -(2.5) reduces to the rational quadratic form

$$s(x)=\frac{f_{i+1}\theta^2+\Delta_i^{-1}(f_{i+1}d_i+f_id_{i+1})\theta(1-\theta)+f_i(1-\theta)^2}{\theta^2+\Delta_i^{-1}(d_i+d_{i+1})\theta(1-\theta)+(1-\theta)^2}, \tag{3.10}$$

for which $d_i \geq 0$ and $d_{i+1} \geq 0$ are necessary and sufficient conditions for a monotonic increasing interpolant. It should be noted that if $\Delta_i = 0$, then $d_i = d_{i+1} = 0$ and

$$s(x) = f_i = f_{i+1} \tag{3.11}$$

is a constant on $[x_i, x_{i+1}]$.

The rational quadratic form (3.10) has been investigated in detail elsewhere by the authors, see references [1] and [5]. It is worth remarking that (3.9) gives $r_i - 3 = (d_i + d_{i+1} - 2\Delta_i)/\ \Delta I$ and it can then be shown that

$$r_i - 3 = d_i - f_i^{(1)}\ d_{i+1} - f_{i+1}^{(1)} + 0(h_i^2)\ .$$

Thus, Theorem 2.1 and its corollary show that (3.9) is a good choice for r_i, since the optimal $0(h^4)$ bound on the interpolation error can be achieved if d_i and d_{i+1} are chosen with $0(h^3)$ accuracy.

3. 2 Convex Data

We assume a strictly convex set of data so that

$$\Delta_1 < \Delta_2 < \ldots < \Delta_{n-1}\ . \tag{3.12}$$

(The case of concave data, where the inequalities are reversed can be treated in a similar way.) To have a convex interpolant s(x), and to avoid the possibility of s(x) having straight line segments, it is necessary that the derivative parameters should satisfy

$$d_1 < \Delta_1, < d_2 < \ \dots \ < \Delta_{i-1} \ < d_i < \Delta_i \ < \ \dots \ < d_n. \tag{3.13}$$

Now s(x) is convex if and only if

$$s^{(2)}(x) \geq 0 \tag{3.14}$$

for all $x \in [x_1 ,x_n]$. After some simplification, it can be shown that for $x \in [x_i,x_{i+1}]$

$$s^{(2)}(x) = \frac{(2/h_i)[\alpha_i\theta^5 + \beta_i\,\theta^4(1-\theta) + \gamma_i\theta^3(1-\theta)^2\delta_i\theta^2(1-\theta)^3 + \varepsilon_i\theta(1-\theta)^4 + \zeta_i(1-\theta)^5]}{[1+(r_i-3)\theta(1-\theta)]^3} \tag{3.15}$$

where

$$\begin{aligned}
\alpha_i &= r_i(d_{i+1} - \Delta_i) - d_{i+1} + d_i \ , \\
\beta_i &= 2[r_i(d_{i+1} - \Delta_i) - \Delta_i + d_i] \ d_{i+1} - \Delta_i \ , \\
\gamma_i &= (r_i + 3)(d_{i+1} - \Delta_i) + 2(d_{i+1} - d_i) \ , \\
\delta_i &= (r_i + 3)(\Delta_i - d_i) + 2(d_{i+1} - d_i) , \\
\varepsilon_i &= 2[r_i(\Delta_i - d_i) - d_{i+1} + \Delta_i] + \Delta_i - d_i \ , \\
\zeta_i &= r_i(\Delta_i - d_i) - d_{i+1} + d_i) \ .
\end{aligned} \tag{3.16}$$

Hence, from (3.15), necessary conditions for convexity are

$$\alpha_i \geq 0 \ \text{ and } \ \zeta_i \geq 0 \tag{3.17}$$

These conditions, together with inequalities (3.13), are also sufficient since we have

$$\beta_i > \alpha_i, \ \varepsilon_i > \zeta_i, \ \gamma_i > 0, \ \delta_i > 0$$

in (3.15). Thus, from (3.17), we have the condition that the interpolant is convex if and only if

$$r_i \geq \max\left\{\frac{d_{i+1}-d_i}{d_{i+1}-\Delta_i} \quad , \quad \frac{d_{i+1}-d_i}{\Delta_i - d_i}\right\}$$

$$= 1 + M_i / m_i , \tag{3.18}$$

where

$$M_i = \max \{d_{i+1} - \Delta_i , \ \Delta_i - d_i\},$$

$$m_i = \max \{d_{i+1} - \Delta_i , \ \Delta_i - d_i\}, \tag{3.19}$$

and the necessary conditions (3.13) are assumed.

We have found two choices of r_i which satisfy (3.18) and produce pleasing graphical results. These are

$$r_i = 2 + M_i/m_i , \tag{3.20}$$

$$\begin{aligned} r_i &= 3 + (M_i/m_i - 1)^2/(M_i/m_i) \\ &= 1 + M_i/m_i + m_i/M_i, \\ &= 1 + (d_{i+1} - \Delta_i)/(\Delta_i - d_i) + (\Delta_i - d_i)/(d_{i+1} - \Delta_i) , \end{aligned} \tag{3.21}$$

the latter being the smaller value. Their use is justified by Theorem 2.1and its corollary as follows. Suppose $d_i - f_i^{(1)} = 0(h_i^2)$ and $d_{i+1} - f_{i+1}^{(1)} = 0(h_i^2)$. Then it can be shown that $M_i/m_i = 1 + 0(h_i)$. Thus r_i. $-3 = 0(h_i)$ for (3.20) and $r_i - 3 = 0(h_i^2)$ for (3.21). In practice, therefore, we prefer the use of (3,21), since the optimal $0(h^4)$ bound on the interpolation error can be achieved if $0(h^3)$ derivative values are given.

<u>Remark</u> In the above we have assumed strictly convex data. Otherwise if Δ_i. $= \Delta_{i+1}$then on $[x_i, x_{i+1}]$ we must have $d_i = d_{i+1} = \Delta_i$. As would be expected, the rational cubic then reduces to the stright line segment

$$s(x) = (1-\theta)\, f_i + \theta f_{i+1} ,$$

with an equivalent result on $[x_{i+1}, x_{i+2}]$.

3.3 Convex and Monotonic Data

We now consider the possibility that the data satisfy both the monotonic increasing condition (3.1) and the strictly convex condition (3.12). The derivative parameters must then satisfy the inequalities

$$0 \le d_1 < \Delta_1 < d_2 < \ldots < \Delta_{i+1} < d_i < \Delta_I < \ldots < d_n . \qquad (3.22)$$

Any convex interpolant must then also be monotonic. This result follows since

$$s^{(1)}(x) = \int_{x_1}^{x} s^{(2)}(x)dx + s^{(1)}(x_1) ,$$

$$= \int_{x_1}^{x} s^{(2)}(x)dx + d_1 .$$

Hence $d_1 \ge 0$ and the convexity condition $s^{(2)}(x) \ge 0$ imply that $s^{(1)}(x) \ge 0$ for $x \in [x_1, x_n]$. Thus the convex interpolaton method of the previous subsection is also suitable for the interpolation of convex and monotonic data. This result is confirmed by the fact that

$$1 + M_i/m_i \ge (d_i + d_{i+1})/\Delta_i$$

for data satisfying (3.22). Thus the convexity condition (3.18) is sufficient to ensure that the monotonicity condition (3.8) is satisfied.

It should be noted that if the data is convex but not strictly convex, then the interpolant can produce straight line (and hence monotonic) segments, as observed in the previous subsection.

3.4 Approximations for the Derivative Parameters

In most applications, the derivative parameters d. will not be given and hence must be determined from the data (x_i,f_i), i =1,...,n. An obvious choice is the $0(h^2)$ three point difference approximation

$$d_i - (h_i \Delta_{i-1} + h_{i-1} \Delta_i) / (h_{i-1}+h_1) , \quad i = 2,\ldots,n-1 \qquad (3.23)$$

with end conditions

$$d_1 = (1+h_1/h_2)\ \Delta_1 - (h_1/h_2)\ \Delta_{3,1},\ \Delta_{3,1} = (f_3-f_1) / (x_3-x_1)$$
$$d_n = (1+h_{n-1}/h_{n-2})\ \Delta_{n-1} - (h_{n-1}/h_{n-2})\ \Delta_{n,n-2}\ , \Delta_{n,n-2} = (f_n-f_{n-2}) / (x_n-x_{n-2}) \qquad (3.24)$$

These arithmetic mean approximations are suitable for the convex interpolation problem, since they satisfy inequalities (3.13). However, for the interpolation of monotonic increasing data, (3.24) may give negative results, thus violating the necessary conditon (3.3). Also (3.23) does not define a continuous functional on the space of monotonic C^1 functions, since we can have lim $d_i \neq 0$ as either lim $\Delta_{i-2}=0$or lim $\Delta_i = 0$.

Alternative $0(h^2)$ approximations which avoid the above problems are the geometric means

$$d_i = \Delta_{i-1}^{h_i/(h_{i-1}+h_i)} \Delta_i^{h_{i-1}/(h_{i-1}+h_i)} , \quad i = 2 ,\ldots, n-1 \qquad (3.25)$$

with end conditions

$$d_1 = \Delta_1^{(1+h_1/h_2)} \Delta_{3,1}^{-h_1/h_2},$$

$$d_n = \Delta_{n-1}^{(1+h_{n-1}/h_{n-2})} \Delta_{n,n-2}^{-h_{n-1}/h_{n-2}} . \qquad (3.26)$$

These approximations, which are discussed in detail in Delbourgo and

Gregory [2], are suitable for the interpolation of monotonic data. Furthermore, if the data is monotonic and convex, then the geometric mean approximations are also appropriate, since they satisfy inequalities (3.22). Reference [2] also considers the use of harmonic mean approximations. However, we do not discuss these here.

The above $0(h^2)$ derivative approximations give $0(h^3)$ bounds on the interpolation error, see Corollary 2.1. The use of $0(h^3)$ derivative approximations for monotonic interpolation is discussed in detail in reference [2]. Unfortunately these approximations do not necessarily satisfy the convexity constraints and the existence of $0(h^3)$ approximations which a priori satisfy such constraints is an open question. Finally it should be noted that the rational quadratic (3.10) can be used to construct a C^2 rational spline which interpolates strictly monotonic data. This is discussed in detail in Delbourgo and Gregory [1] where it is shown that the spline produces $0(h^3)$ derivative approximations.

4. Numerical Results

We consider the application of the rational schemes to two sets of data. The first is the monotonic and convex set defined by $f(x) = 1/x^2$ on [-2,-0.2], with the interpolation points at x =-2, -1, -0.3 and -0.2. This is the example used by McAllister et al [6], Since few data points are given, this is a fairly severe test of any scheme, particularly one where the derivatives are estimated from the data. Also, we cannot expect the rational interpolants to reproduce $1/x^2$, because of the non-linear nature of the interpolation method.

Figure 1 shows the application of the rational cubic scheme of subsection 3.2 to the above data, where r. is defined by (3.21). The graphs (i) and (ii) are respectively the interpolants with the arithmetic and geometric $0(h^2)$ derivative approximations of subsection 3.4, and graph (iii) is that with the known exact derivatives. As expected from the theory, all graphs are convex but the graph with the arithmetic derivative approximations is not monotonic. It can be seen that the graph with the exact derivative settings gives the best result.

Since the data is monotonic, the rational quadratic scheme of subsection 3.1 is also applicable. Figure 2 shows the application of this scheme with various choices of the derivative parameters. These are (i) the $0(h^2)$ geometric approximations of subsection 3.4, (ii) the C^2 spline approximations of Delbourgo and Gregory [1], and (iii) the known exact derivatives. All curves are monotonic but (i) exhibits an inflexion. Curves (ii) and (iii) give good results. It should be noted that exact end conditions have been used for the C^2 spline scheme. The alternative use of geometric approximations to the end derivatives give comparable results for this set of data. The curves illustrate that although the monotonic rational quadratic schemes are not a priori convex, in practice they might be so. An a posteriori test for convexity is the necessary and sufficient condition (3.18).

Our second set of data consists of points uniformly spaced at 15° intervals over a half or quarter circle. The half circle of points is a convex but not monotonic set and the set of points on the quarter circle is convex and monotonic.

The results of applying the convex rational cubic schemes and the monotonic rational quadratic schemes to the circle data are given in Figures 3 and 4. Figure 3 shows that convexity is assured for all choices of the derivative parameters, the arithmetic settings being suitable for the convex half circle data and the geometric settings being appropriate for the convex and monotonic quarter circle data. The choice of exact derivatives has once more produced a good result, although here the end derivative values ±50, which replace the infinite gradients of the circle data, have been set by trial and error.

It can be seen from Figure 4 that the monotonic rational quadratic scheme with geometric derivative approximations has a slight inflexion in the curve. The choice of exact derivatives or the C^2 spline approximations have again produced good curves, where at the end, where the quarter circle has infinite gradient, we have set the derivative d_n by trial and error. Too large a value of d_n creates an inflexion in the last interval and it is of interest to compare the behaviour of the rational quadratic and rational cubic schemes as the end condition d_n is made large. Figure 5 illustrates this for the case $d_n = 1000$ with exact derivative settings elsewhere. Since the rational quadratic has only to maintain monotonicity, the graph begins to behave in a step function manner. However, the additional convex constraint on the rational cubic eliminates this behaviour and instead produces a straight line almost vertical section at the end.

5. Conclusion

A shape preserving piecewise rational cubic scheme has beendescribed which can be used to interpolate convex and/or monotonic data. The method seems to produce visually pleasing C^1 curves and good error bounds can be expected, particularly when exact derivative information is given at the interpolation points.

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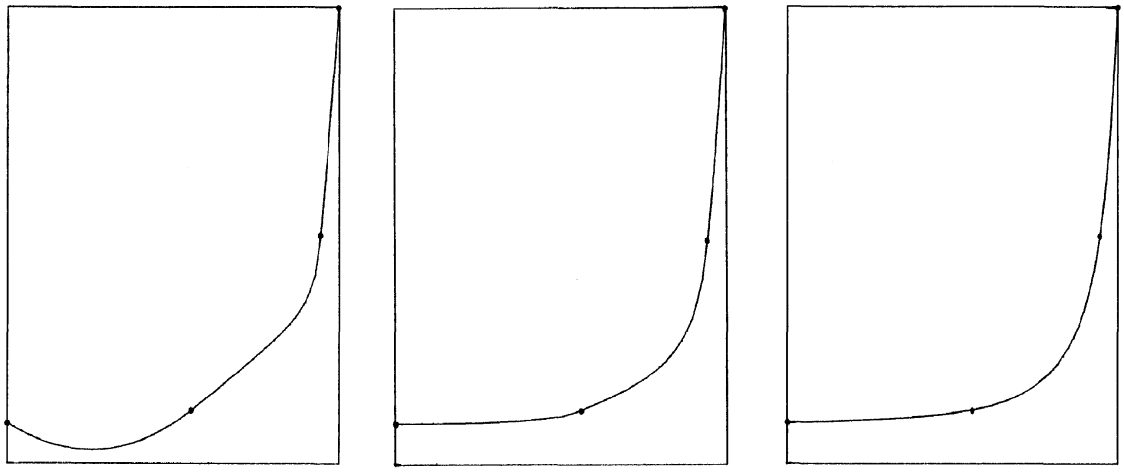
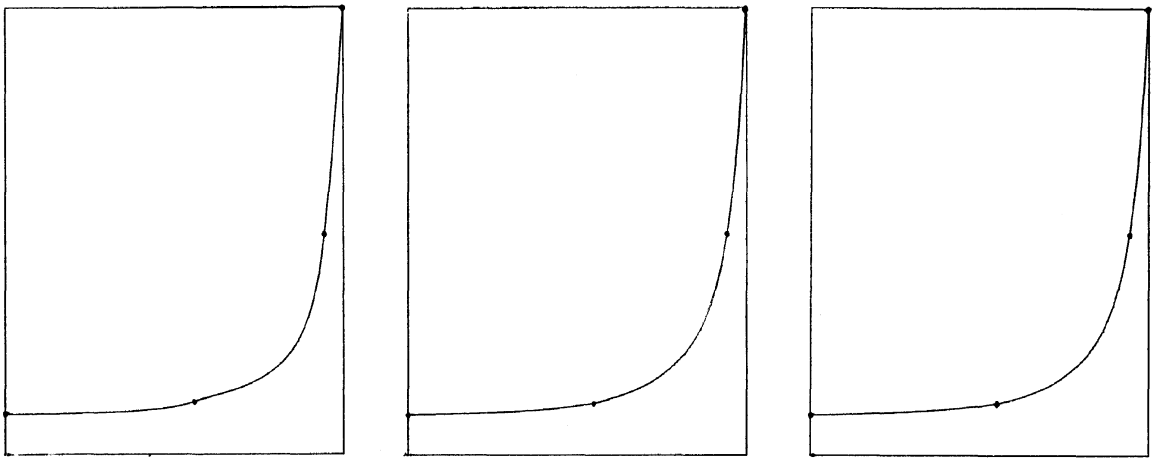


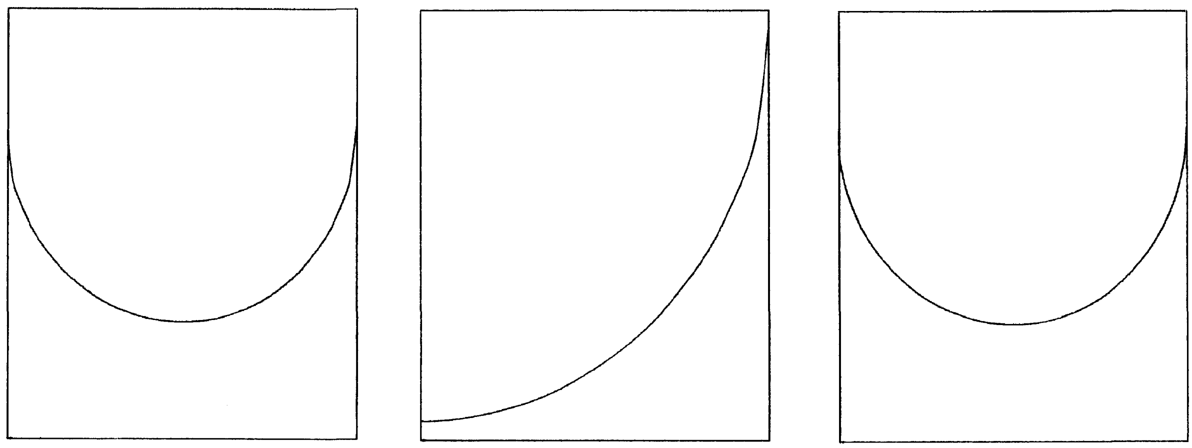
(i) arithmetic derivative values (ii) geometric derivative values (iii) exact derivative values

Fig. 1. Convex rational cubics for $f(x) = 1/x$ on $[-2,-0.2]$.



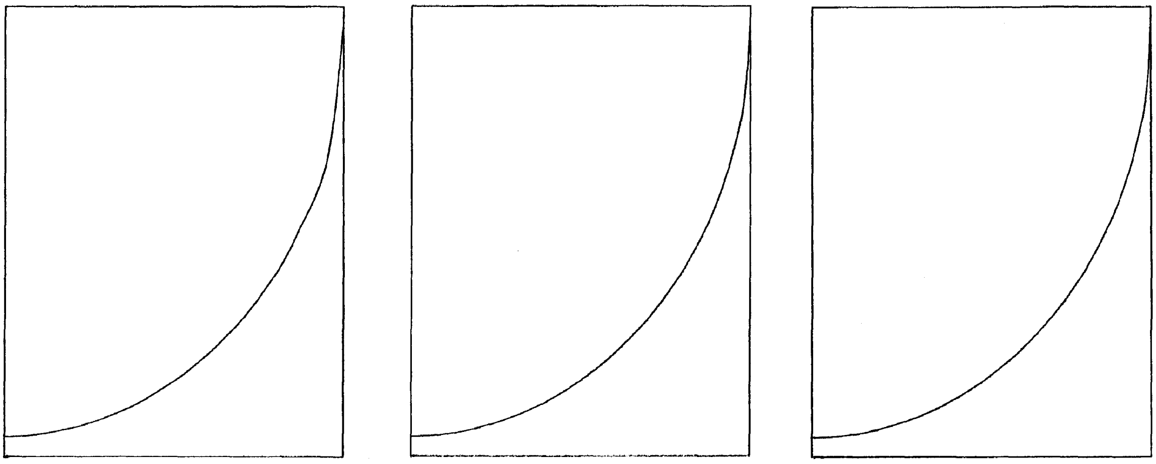
(i) geometric derivative values (ii) c^2 spline derivative values (iii) exact derivative values

Fig. 2. Monotonic rational quadratics for $f(x) = 1/x^2$ on [-2,-0.2].



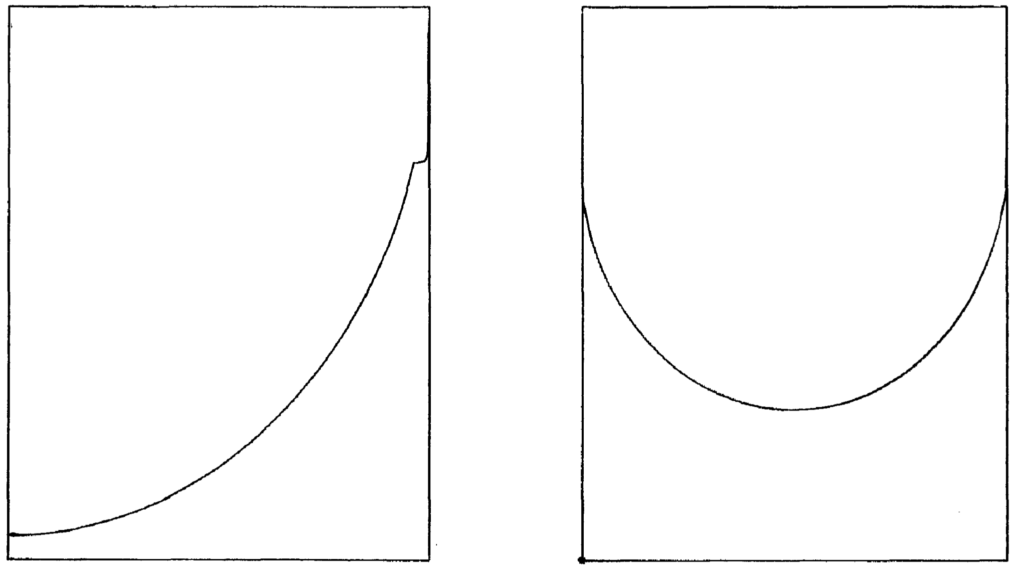
(i) arithmetic derivative values (ii) geometric derivative values (except $d_1 = 0$) (iii) exact derivative values (except $d = -50$, $d_n = 50$)

Fig. 3. Convex rational cubics for circle data.



2(i) geometric derivative values (except $d_1 = 0$)   (ii) C spline derivative values (with $d_1 = 0$, $d_n = 20$)   (iii) exact derivative values (except $d_n = 25$)

Fig. 4. Monotonic rational quadratics for circle data.



(i) monotonic rational quadratic

(ii) convex rational cubic

Fig. 5. The effect of large end conditions for circle data.