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A family of numerical methods for the numerical solution of Blasius' equation.

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1. INTRODUCTION

It is well known that the general initial value problem of order N given by

$$y^{(N)}(x) = f(x, y, y', \ldots, y^{(N-1)}), \quad x > x_0, \tag{1}$$

with initial conditions

$$y(x_0) = y_0 \;, y^{(r)}(x_0) = z_r (r = 1,2,\ldots, N-1), \tag{2}$$

may be expressed as the system of N first order equations

$$\begin{bmatrix} u_1' \\ u_2' \\ . \\ . \\ u_{N-1}' \\ \\ u_N \end{bmatrix} = \begin{bmatrix} 0 & 1 & & & & \\ & 0 & 1 & & & \\ & & . & . & & \\ & & & . & . & \\ & & & & . & . \\ & & & & 0 & 1 \\ & & & & & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ . \\ . \\ . \\ u_{N-1} \\ u_N \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ . \\ . \\ 0 \\ f(x, u_1, u_2, \ldots, u_N) \end{bmatrix} \tag{3}$$

where $u_1(x) = y(x)$ and $u_s(x) = y^{(s-1)}(x)$ for $s = 2, \ldots, N$. The initial conditions associated with (3) are given by

$$\underline{u}(0) = [y_0, z_1, z_2, \ldots, z_{N-1}]^T, \tag{4}$$

T denoting transpose. Clearly, the solution vector is $\underline{u} = \underline{u}(x) = [u_1, u_2, \ldots, u_N]^T$.

A particular example of (1) is Blasius' equation

$$y'''(x) = -\frac{1}{2} y(x)\, y''(x), \quad x > x_0 \tag{5}$$

with Initial conditions

$$y(x_0) = y_0, \;\; y'(x_0) = z_1 \;, \; y''(x_0) = z_2. \tag{6}$$

Blasius' equation was originally a boundary value problem: numerical results were reported by Howarth [1] and the problem was formulated as an initial value problem in a recent paper by Radok and Chan [3]. Allied to (3), the Blasius initial value problem can be expressed as the first order system

$$\begin{bmatrix} u_1' \\ u_2' \\ u_3' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -\frac{1}{2}u_1 u_2 \end{bmatrix} \tag{7}$$

with initial conditions

$$\underline{u}(0) \equiv \underline{u}_0 = [y_0, z_1, z_2]^T \tag{8}$$

or as

$$D\underline{u}(x) \equiv \underline{u}'(x) = M\underline{u}(x) + \underline{F}(\underline{u}) \; ; \quad \underline{u}(x) = \underline{u}_0 \tag{9}$$

where D = diag{d/dx} is of order three,

$$M = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

$\underline{u} = \underline{u}(x) = [u_1, u_2, u_3]$ and $\underline{F}(\underline{u}) = [0, 0, -\frac{1}{2}u_1 u_3]^T$.

2. THE FAMILY OF NUMERICAL METHODS

Numerical methods for the solution of the Blasius problem (5), (6) will be obtained by approximating the exponential term in

$$\underline{u}(x+h) = \exp(hD)\underline{u}(x). \tag{10}$$

In equation (10), h is an increment in x so that the independent variable x is discretised in such a way that $x_n = x_0 + nh$ (n=0,1,2,...).

Using the approximation

$$\exp(hD) \simeq (I - \theta hD)^{-1}[I + (1-\theta)hD], \tag{11}$$

where $0 \leq \theta \leq 1$ is a parameter and I is the identity matrix of order three, leads to

$$(I - \theta hD)\underline{U}(x+h) = [I + (1-\theta)hD]\underline{U}(x) \tag{12}$$

which, using (9), gives the numerical method

$$(I - \theta hM)\underline{U}^{n+1} - \theta h\underline{F}^{n+1} = [I + (1-\theta)hM]\underline{U}^n + (1-\theta)h\underline{F}^n. \tag{13}$$

In (13) $\underline{U}^n = \underline{U}(x_n)$ denotes the solution of the numerical method and $\underline{F}^n = \underline{F}(\underline{U}^n)$ with n = 0,1,2,...

Equation (13) describes a family of numerical methods for the solution of (9) and thus of the Blasius initial value problem (5), (6). Choosing $\theta = 0$ is equivalent to the use of the (0,1) Padé approximant to replace exp(hD) in (10) and leads to the well known explicit Euler method for solving (9); choosing $\theta=1$ is equivalent to the use of the (1,0) Padé approximant to exp(hD) and leads to the fully implicit or backward Euler method for solving (9); and choosing $\theta = \frac{1}{2}$ is equivalent to the use of the (1,1) Padé approximant to exp(hD) and gives the modified Euler or trapezoidal rule for solving (9).

Implementing the family of methods is easy; with $\theta=0$ the solution at x_{n+1} is given by

$$\begin{aligned} U_1^{n+1} &= U_1^n + hU_2^n , \\ U_2^{n+1} &= U_1^n + hU_2^n , \\ U_3^{n+1} &= U_3^n - \tfrac{1}{2}hU_1^n U_3^n \end{aligned} \tag{14}$$

for n = 0,1,2,... . With $0 < \theta \le 1$ the solution is obtained by solving a nonlinear algebraic system of the form

$$\underline{\phi}^{n+1} \equiv \underline{\phi}(\underline{U}^{n+1}) = \underline{0} , \tag{15}$$

where $\underline{0}$ is the zero vector of order three and the elements of $\underline{\phi}^{n+1}$ are

$$\begin{aligned} \phi_1^{n+1} &= \phi_1(\underline{U}^{n+1}) = U_1^{n+1} - \theta hU_2^{n+1} - U_1^n - (1-\theta)hU_2^n , \\ \phi_2^{n+1} &= \phi_2(\underline{U}^{n+1}) = U_2^{n+1} - \theta hU_3^{n+1} - U_2^n - (1-\theta)hU_3^n , \\ \phi_3^{n+1} &= \phi_3(\underline{U}^{n+1}) = U_3^{n+1} + \tfrac{1}{2}\theta hU_1^{n+1} U_3^{n+1} - U_3^n + \tfrac{1}{2}(1-\theta)hU_1^n U_3^n . \end{aligned} \tag{16}$$

The elements J_{rs}^{n+1} (r,s = 1,2,3) of the Jacobian of $\underline{\phi}^{n+1}$ are

$$\begin{aligned} &J_{11}^{n+1} = 1 , \quad J_{12}^{n+1} = -\theta h , \quad J_{13}^{n+1} = 0 \\ &J_{21}^{n+1} = 0 , \quad J_{22}^{n+2} = 1 , \quad J_{23}^{n+1} = -\theta h \\ &J_{31}^{n+1} = \tfrac{1}{2}\theta hU_3^{n+1} , \quad J_{32}^{n+1} = 0 , \quad J_{33}^{n+1} = 1 + \tfrac{1}{2}\theta hU_1^{n+1} \end{aligned} \tag{17}$$

and the solution vector $\underline{U}^{n+1}$ is found from (15) by applying the Newton-Raphson method

$$J^{n+1}_{(k)}\underline{z}(k) = -\underline{\phi}^{n+1}_{(k)} , \quad (18)$$

$$\underline{U}^{n+1}_{(k+1)} = \underline{U}^{n+1}_{(k)} + \underline{z}(k) , \quad (19)$$

in which the subscript k= 0,1,2,... is the iterate number. In (18) and (19) $\underline{z}(k)$ is the correction vector which is determined by solving the linear algebraic system (18). It is an easy matter to write a computer program to implement (13) from (14) or (16), (17), (18), (19).

The local truncation error vector $\underline{L}[\underline{u}(x);h]$ of (12)/(13) is given by

$$\underline{L}[\underline{u}(x);h] = \{(\tfrac{1}{2}-\theta)h^2D^2 + (\tfrac{1}{6}-\tfrac{1}{2}\theta)h^3D^3 + (\tfrac{1}{24}-\tfrac{1}{6}\theta)h^4D^4 + \ldots\}\underline{u}(x) \quad (20)$$

from which it follows that the method is first order accurate with error constant $C_2 = \frac{1}{2}-\theta$ (see Lambert [2]) for $\theta \neq \frac{1}{2}$. but is second order accurate with error constant $C_3 = -\frac{1}{12}$ for $\theta = \frac{1}{2}$ The global truncation vector $\underline{G}[\underline{u}(x);h]$is given by

$$\underline{G}[\underline{u}(x);h] = \{(\tfrac{1}{2}-\theta)hD^2 + (\tfrac{1}{6}-\tfrac{1}{2}\theta)h^2D^3 + (\tfrac{1}{24}-\tfrac{1}{6}\theta)h^3D^4 + \ldots\}\underline{u}(x). \quad (21)$$

Stability of the family of methods (13) is investigated in relation to the usual single test equation

$$w'(x) = \lambda w(x) \;\; ; u(x_0) = u_0 \quad (22)$$

in which $\lambda < 0$ is real. Equation (10) thus becomes

$$w(x+h) = \exp(h\lambda)w(x) \quad (23)$$

and it follows from (11), using W to represent the solution of a numerical method, that

$$W^{n+1} - \left(\frac{1+(1-\theta)h\lambda}{1-\theta h\lambda}\right)W^n \quad (24)$$

Defining $\overline{h} = h$, so that $\overline{h} < 0$, the term

$$S_\theta = \frac{1+(1-\theta)\overline{h}}{1-\theta\overline{h}} \quad (25)$$

is variously called the *amplification factor* or *amplification symbol* or *symbol* of the numerical method defined by (10), (11). In the case of (9), λ is usually taken to be the real part (assumed negative) of an eigenvalue of the Jacobian of $M\underline{u}(x)+\underline{F}(\underline{u})$. For $\lambda< 0$ real, the stability interval of the numerical method is the range of values of $\bar{h}$ for which $|S_\theta| < 1$. It is easy to verify that (12)/(13) has a finite stability interval for $0\le\theta<\frac{1}{2}$ and that (12)/(13) is absolutely stable or A-stable for $\frac{1}{2}\le\theta\le 1$. Furthermore, it is easy to verify that $\theta = 1$ is the only L-stable member of the family described by (13). Clearly the methods are applicable also to general initial value problems of higher order.

3. GLOBAL EXTRAPOLATION OF THE NUMERICAL SOLUTION

Introducing a slight change of notation, suppose that the discretization of x used so far is called Grid 1 and consists of the points $x_n^{(1)}$ (n=0,1,2,...) and suppose further that the numerical solution is sought at some fixed point $X=x_Q^{(1)}=x_0+Qh$. Here, the superscript refers to Grid 1 and the chosen numerical method is used Q times to integrate from $x=x_0$ to $x = x_0+ Qh$. The global truncation error at X is given by (21).

Suppose now that the interval of integration is divided into 2Q subintervals each of width $\frac{1}{2}h$ giving a second discretization to be called Grid 2 consisting of the points $x_i^{(2)}=x_0+\frac{1}{2}ih\,(0,1,...,2Q)$. Clearly the points $x_r^{(2)}$ (r =0,2,4,...,2Q). of Grid 2 are coincident with the points of Grid 1 . The global error for Grid 2 has the form

$$\underline{G}^{(2)}[\underline{u}(x);\tfrac{1}{2}h]=\{\tfrac{1}{2}(\tfrac{1}{2}-\theta)hD^2+\tfrac{1}{4}(\tfrac{1}{6}-\tfrac{1}{2}\theta)h^2D^3+\tfrac{1}{8}(\tfrac{1}{24}-\tfrac{1}{6}\theta)h^3D^4+...\}\underline{u}(x). \qquad (26)$$

Suppose, with another slight change of notation, that $\underline{U}_{(1)}^{Q}$ and $\underline{U}_{(2)}^{2Q}$ are the solutions obtained at X on Grids 1 and 2 respectively. Then it may be shown that, when $\theta\neq\frac{1}{2}$, the globally extrapolated solution

$$\underline{U}^{(E)}=\underline{U}_{(2)}^{2Q}+(1-\alpha)\underline{U}_{(1)}^{Q}, \qquad (27)$$

where α is some parameter, is second order accurate provided α = 2, for then the term in h in the associated global extrapolated error function

$$\underline{G}^{(E)} = \alpha\underline{G}^{(2)} + (1-\alpha)\underline{G}^{(1)} \tag{28}$$

vanishes (here $\underline{G}^{(1)}$ is given by (21)).

It is easy to show that the global extrapolation procedure just described, when applied to the method with $\theta = \frac{1}{2}$, gives a third order method provided $\alpha = \frac{4}{3}$

Global extrapolation thus improves the numerical methods developed in §2 for the numerical solution of Blasius' equation by one order of accuracy.

4. NUMERICAL RESULTS

Numerical results were obtained for the Blasius initial value problem

$$y''' = -\tfrac{1}{2}yy'' \;\; ; \;\; y(0) = y'(0) = 0 \;\; , y''(0) = 0.33206 \tag{29}$$

which was used by Radok and Chan [3], Four numerical methods were used; these were

Method A: the first order L-stable method obtained by writing θ = 1 in (11),

Method B: the second order extrapolation of Method A,

Method C: the second order A-stable method obtained by writing $\theta = \frac{1}{2}$ in (11),

Method D: the third order extrapolation of Method C.

The solution vector $\underline{U} = [y, y', y'']^T$ was computed for x = 0(h)9 using step sizes h = 1, 0.5, 0.25, 0.2, 0.1, 0.05, 0.025, 0.02, 0.01.

Following Radok and Chan [3], the results of Haworth [1] were taken to be the theoretical solution of the problem. Haworth showed that the solution y(x) increases linearly (approximately) for x > 4, that y'(x) $\rightarrow$1 (from below) as x increases, and that y"(x) $\rightarrow$ 0 (from above) as x increases indicating that Methods A and B should model the behaviour of y"(x) fairly accurately.

It was found that Methods A,B,C,D each gave accurate representations of all three components y, y', y" of the solution vector $\underline{U}$, the accuracy increasing as h was decreased.

Comparing the results obtained with those of Haworth [1] and the best results of Radok and Chan [3] which are given in the Appendix, it was found that for x $x \geq 5$ and

(i) for h=0.01, Method A gave results for y(x) which were almost as accurate as those of Radok and Chan, while the results for $y^1(x)$ and y"(x) were more accurate. Results for h-0.01 are given in Table 1,

(ii) for $h \leq 0.25$, Method B gave results for $y(x)_{>}$ $y'(x)$, y"(x) which were closer to those of Haworth than those of Radok and Chan. Results for h=0.25 are given in Table 2;

(iii) for $h \leq 0.25$, Method C, which is of the same order as Method B, also gave more accurate results than the method reported by Radok and Chan in [3];

(iv) for all values of h tested, Method D gave results for all of y, y', y" which were closer to those of Haworth than those of Radok and Chan. Results for h =1 are given in Table 3.

Overall, the methods proposed in the present paper were found to give more accurate results for larger values of x than the method of Radok and Chan [3], when tested on the model Blasius problem (29). Following Radok and Chan, all results are given to five decimal places .

5. SUMMARY

A family of numerical methods has been developed for the numerical solution of the Blasius initial value problem $y'''(x) = -\frac{1}{2}y(x)\,y''(x)$, $x > x_0$ with $y(x_0)$, $y'(x_0)$, $y''(x_0)$ given. The methods were tested on a problem from the literature and were seen to give good accuracy for higher values of x.

The family of methods may be used to solve general initial value problems of higher order.

REFERENCES

1. L. Howarth, On the solution of the laminar boundary layer equations, Proc. Roy. Soc. London A 164 (1938) 547-579.
2. J. D. Lambert, Computational Methods in Ordinary Differential Equations, John Wiley and Sons, Chichester, 1973.
3. R. Radok and P.S. Chan, A linearizing algorithm for nonlinear differential equations, Comp. Meth. Appl. Mech. Engng. 54 (1986) 245-253.

Table 1: Values of y, y' and y" for x = 0(1)9 using h = 0.01

	X	A	B	C	D
y	0	0.00000	0.00000	0.00000	0.00000
	1	0.16719	0.16557	0.16557	0.16557
	2	0.65266	0.65004	0.65003	0.65003
	3	1.39890	1.39683	1.39681	1.39682
	4	2.30551	2.30578	2.30576	2.30576
	5	3.27975	3.28331	3.28329	3.28330
	6	4.27258	4.27966	4.27964	4.27965
	7	5.25868	5.27928	5.27927	5.27927
	8	6.26514	6.27925	6.27925	6.27925
	9	7.26162	7.27925	7.27925	7.27926
y'	0	0.00000	0.00000	0.00000	0.00000
	1	0.32965	0.32978	0.32978	0.32978
	2	0.62882	0.62977	0.62977	0.62977
	3	0.84373	0.84605	0.84605	0.84605
	4	0.95223	0.95552	0.95552	0.95552
	5	0.98799	0.99154	0.99155	0.99155
	6	0.99543	0.99898	0.99898	0.99898
	7	0.99640	0.99992	0.99993	0.99993
	8	0.99648	1.00000	1.00000	1.00000
	9	0.99649	1.00000	1.00000	1.00000
y"	0	0.33206	0.33206	0.33206	0.33206
	1	0.32274	0.32301	0.32301	0.32301
	2	0.26597	0.26675	0.26675	0.26675
	3	0.16060	0.16136	0.16136	0.16136
	4	0.06403	0.06423	0.06423	0.06423
	5	0.01599	0.01591	0.01591	0.01591
	6	0.00246	0.00240	0.00240	0.00240
	7	0.00023	0.00022	0.00022	0.00022
	8	0.00001	0.00001	0.00001	0.00001
	9	0.00000	0.00000	0.00000	0.00000

Table 2: Values of y, y' and y" for x=0(1)9 using h=0.25

	X	A	B	C	D
y	0	0.00000	0.00000	0.00000	0.00000
	1	0.20461	0.16633	0.16543	0.16557
	2	0.70876	0.65385	0.64903	0.65003
	3	1.43459	1.40410	1.39434	1.39682
	4	2.28331	2.31380	2.30214	2.30576
	5	3.18142	3.28978	3.27937	3.28329
	6	4.09391	4.28396	4.27585	4.27965
	7	5.00949	5.28140	5.27569	5.27927
	8	5.92556	6.27919	6.27590	6.27925
	9	6.84170	7.27701	7.27613	7.27925
y'	0	0.00000	0.00000	0.00000	0.00000
	1	0.32478	0.33062	0.32951	0.32978
	2	0.60243	0.63149	0.62893	0.62977
	3	0.78677	0.84652	0.84515	0.84605
	4	0.87590	0.95404	0.95521	0.95552
	5	0.90676	0.98939	0.99168	0.99155
	6	0.91449	0.99681	0.99921	0.99898
	7	0.91592	0.99776	1.00016	0.99993
	8	0.91612	0.99782	1.00023	1.00000
	9	0.91614	0.99782	1.00023	1.00000
y"	0	0.33206	0.33206	0.33206	0.33206
	1	0.31559	0.32339	0.32276	0.32301
	2	0.24794	0.26632	0.26659	0.26675
	3	0.14510	0.15996	0.16167	0.16136
	4	0.06049	0.06357	0.06459	0.06423
	5	0.01795	0.01589	0.01596	0.01591
	6	0.00386	0.00237	0.00236	0.00240
	7	0.00061	0.00017	0.00021	0.00022
	8	0.00007	0.00001	0.00001	0.00001
	9	0.00001	0.00000	0.00000	0.00000

Table 3: Values of y, y' and y" for x = 0(1)9 using h=1.

	X	A	B	C	D
y	0	0.00000	0.00000	0.00000	0.00000
	1	0.29001	0.18896	0.16278	0.16570
	2	0.78805	0.71417	0.63417	0.65000
	3	1.40818	1.48225	1.35891	1.39634
	4	2.08803	2.37882	2.25021	2.30514
	5	2.79280	3.32546	3.22229	3.28281
	6	3.50663	4.28658	4.22080	4.27915
	7	4.22336	5.25073	5.22435	5.27866
	8	4.94093	6.21528	6.22829	6.27855
	9	5.65873	7.17982	7.23222	7.27846
y'	0	0.00000	0.00000	0.00000	0.00000
	1	0.29001	0.34249	0.32557	0.32973
	2	0.49804	0.64097	0.61721	0.62954
	3	0.62013	0.83502	0.83228	0.84588
	4	0.67985	0.92604	0.95031	0.95557
	5	0.70477	0.95669	0.99386	0.99151
	6	0.71383	0.96384	1.00316	0.99886
	7	0.71674	0.96475	1.00394	0.99983
	8	0.71757	0.96466	1.00393	0.99991
	9	0.71779	0.96456	1.00392	0.99991
y"	0	0.33206	0.33206	0.33206	0.33206
	1	0.29001	0.32411	0.31908	0.32298
	2	0.20804	0.25501	0.26420	0.26675
	3	0.12208	0.14668	0.16594	0.16146
	4	0.05973	0.05812	0.07012	0.06417
	5	0.02492	0.01528	0.01699	0.01585
	6	0.00905	0200184	0.00161	0.00246
	7	0.00291	0.00051	0.00004	0.00022
	8	0.00084	0.00040	0.00000	0.00001
	9	0.00022	0.00000	0.00000	0.00000

APPENDIX Values of y, y' and y" for x=0(1)9; H :Howarth's solution [1], RC : Radok and Chan [3].

	X	H	RC
y	0	0.00000	0.00000
	1	0.16557	0.16557
	2	0.65003	0.65003
	3	1.39682	1.39682
	4	2.30576	2.30576
	5	3.28329	3.28290
	6	4.27964	4.27487
	7	5.27926	5.26736
	8	6.27923	6.26023
	9	7.27923	7.25318
y'	0	0.00000	0.00000
	1	0.32979	0.32978
	2	0.62977	0.62977
	3	0.84605	0.84605
	4	0.95552	0.95552
	5	0.99155	0.98955
	6	0.99898	0.99193
	7	0.99992	0.99276
	8	1.00000	0.99295
	9	1.00000	0.99298
y"	0	0.33206	0.33206
	1	0.32301	0.32301
	2	0.26675	0.26675
	3	0.16136	0.16136
	4	0.06424	0.06419
	5	0.01591	0.00646
	6	0.00240	0.00152
	7	0.00022	0.00023
	8	0.00001	0.00003
	9	0.00000	0.00000