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ON BLENDING-FUNCTION INTERPOLATION

by

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1. Introduction.

The purpose of this note is to discuss the relationship between blending-function methods [2,3,4,] and cross-product methods [6]. A general theorem on projections is quoted. This theorem includes blending-function methods as a special case and leads to simpler proofs of some of Gordon's theorems.

2. Orthogonal Projections.

The following is a theorem in Bachman and Narici [1, p. 414]:

Theorem 1. Let X be a Hilbert space with E_1 and E_2 as orthogonal projections that commute and are onto the closed suhspaces M_1, and M_2, respectively. If $E \equiv E_1 + E_2 - E_1 E_2$, then E is an orthogonal projection onto $\overline{M_1 \cup M_2}$. (Moreover, if I is the identity operator, then $I - E = (I - E_1)(I - E_2)$.)

The application of this theorem to blending-functions is as follows: Let $\varphi_1(x), \ldots, \varphi_k(x)$ be an orthonormal set of functions in $L_2(a, b)$ and $M_1 \equiv \{\sum_i a_i(y)\varphi_i(x) : a_i(y)$ piecewise continuous] and let $\psi_1(y), \ldots \psi_k, (y)$ be an orthonormal set of functions in $L^2(c, d)$ and $M_2 \equiv \{\sum_j b_j(x)\psi_j(y) : b_j(x)$ piecewise continuous] . For a function of two variables,, $F(x,y)$, projections of the from $E_1(F) = P_x(F) \equiv \sum_{i=1}^{k} a_i(y)\varphi_i(x)$ and $E_2(F) = P_y(F) \equiv \sum_{i=1}^{k'} b_j(x)\psi_j(y)$ are considered. For the case of least squares approximation, $a_i(y) \equiv \int_a^b F(x,y)\varphi_i(x)$ and $b_j(x)$ dually [2].

We now use the above to simplify the proof of the following theorem due to Gordon [2].

Theorem 2. ("Bivariate orthogonal expansions")

Let F(x,y) be piecewise continuous on [a,b] x [c,d] . Of all functions of the from $\tilde{f}(x,y) = \sum_{i=1}^{k} g_i(y)\varphi_i(x) + \sum_{i=1}^{k'} h_j(x)\psi_j(y)$, the g_i and h_j piecewise continuous, such that

$$(F, \varphi_i)_{(x)} \equiv \int_a^b F(x, y)\varphi_i(x)dx = \int_a^b \tilde{f}(x, y)\varphi_i(x)dx$$

and

$$(F, \psi_j)_{(y)} \equiv \int_a^b F(x, y)\psi_j(y)dy = \int_a^d \tilde{f}(x, y)\psi_j(y)dy ,$$

the function $f = P_x(F) + P_y(F) - P_xP_y(F)$ uniquely minimizes $\|F - \tilde{f}\|$.

Proof: The fact that f is admissible, i.e.,

$(F,\varphi_i)_{(x)} = (f, \varphi_i)_{(x)}$ and $(F, \psi_j)_{(y)} = (f, \psi_j)_{(y)}$, follows from its definition: Let $\tilde{f}_x \equiv \sum_i a_i(y)\varphi_i(x)$ and $\tilde{f}_y \equiv \sum_j b_j(x)\psi_j(y)$.

By the properties of orthogonal projections, $(F - P_x(F), \tilde{f}_x)_{(x)} = 0$ i.e., $F - P_x(F)$ is orthogonal to M_1. Similarly, $(F - P_y(F), \tilde{f}_y)_{(y)} = 0$. Since the inner product on the space is

$(F, G) = \int_a^b \int_a^b F(x, y)G(x, y)dxdy$ it is obvious that

$(F - P_x(F), \tilde{f}_y) = 0 = (F - P_y)$. Expand

$$\| F - \tilde{f}_x - \tilde{f}_y \| = \| F \|^2 - 2[F, \tilde{f}_x) + (F, \tilde{f}_y)] + \| \tilde{f}_y + \tilde{f}_y \|^2$$

$$= \| F \|^2 - 2[(P_x(F), \tilde{f}_x) + (P_y(F), \tilde{f}_y) + (P_y(F), \tilde{f}_x)$$

$$+ (P_x(F), \tilde{f}_y) - (P_y(F), \tilde{f}_x) - (P_x(F), \tilde{f}_y)]$$

$$+ \| \tilde{f}_x + \tilde{f}_y \|^2 .$$

Since $P_x P_y (F) - P_y(F)$ is orthogonal M_1 , i.e.,
$(PxPy(F) - P_y(F), \tilde{f}_x)_{(x)} = 0 = (P_x P_y(F) - P_y(F), \tilde{f}_x)$, we have that

$$\| F - \tilde{f}_x - \tilde{f}_y \|^2 = \| F \|^2 - 2[(P_x(F) + P_y(F) - P_x P_y(F), \tilde{f}_x + \tilde{f}_y)] + \| \tilde{f}_x + \tilde{f}_y \|^2$$

$$= \| F \|^2 - \| P_x(F) + P_y(P) - P_x P_y(F) \|^2$$

$$+ \| P_x(F) + P_y(F) - P_x P_y(F) - \tilde{f}_x - \tilde{f}_y \|^2 ,$$

from which the conclusion follows. Q.E.D.

The set M_1 is $\{\sum_i a_i(y)\varphi_i(x)]\}$; M_2 is $\{\sum_j b_j(x)\psi_j(y) \}$,

with $M_1 \cup M_2$ then being $\{\sum_i a_i(y)\varphi_i(x) + \sum_j b_j(x)\psi_j(y)\}$.

Now $M_1 \cup M_2 = \{\sum_{i,j} B_{ij}\varphi_i(x)\psi_j(y)\}$ and the tensor (cross –) product approximation to F is $E_1E_2(F)$. It is the best approximation to F from $M_1 \cap M_2$ [2] and hence is the orthogonal projection of F onto $M_1 \cap M_2$. Now $M_1 \cap M_2 \subseteq M_1 \cup M_2$ implies that $||F - E_1(F) - E_2(F) + E_1E_2(F)||^2 \leq ||F - E_1E_2(F)||^2$. The next theorem gives a precise statement of the improvement obtained.

<u>Theorem 3.</u> Under the above conditions.

$$||F- E_1E_2(F)||^2 - ||F- E_1(F) - E_2(F) + E_1E_2(F)||^2$$

$$= ||E_1(F) - E_2E_1(F)||^2 + ||E_2(F) - E_2E_1(F)||^2$$

$$= ||E_1(F) - E_2(F)||^2 \gamma \{||E_1(F)|| - ||E_2(F)||\}^2 \geq 0 \quad .$$

<u>Proof</u>; By Theorem 1,

$$||F - E_1(F) - E_2(F) + E_1E_2(F)||^2 = ||F||^2 - ||E_1(F) + E_2(F) - E_1E_2(F)||^2 \quad .$$

Using successively the facts that $(F - E_2(F),E_2(F)) = 0$, $(E_1[F- E_2(F), E_2(F)) = 0$, and that E_1 and E_2 commute,

we find that

$$||F - E_1(F) - E_2(F) +E_1E_2(F)||^2 =||F||^2 - ||E_2(F)||^2 - ||E_1(F)||^2 +|| E_1E_2(F)||^2 .$$

By the above remarks concerning $'E_1E_2(F)$, $|| F- E_1E_2(F)||^2 =$

$||F||^2 - ||E_1E_2(F)||^2$, from which the conclusion follows. Q.E.D.

3. <u>Connection with Stancu's results</u>.

For interpolation along sections, different definitions of the projections P_x and P_y are required than for the above least squares interpolation. If $L_i(F) = g_i(y)$, $i = \overline{1,K}$, and

$M_j(F) = h_j(x)$, $j = \overline{1, K^1}$, are required, then the corresponding $\varphi_i(x)$ and $\psi_j(y)$ are required to be biorthonormal with respect to the linear functionals L_i and M_j, respectively.

Let $P_x(F) \equiv \sum_i L_i(F)\varphi_i(x)$ and $P_y(F) \equiv \sum_j M_j(F)\psi_j(y)$. If we let

(1) $$R_B(F) \equiv F(x,y) - P_x(F) - P_y(F) + P_xP_y(F) ,$$

then $R_B(F)$ can be related to the cross-product remainder. In Stancu's [6] notation, the cross-product remainder can be represented as $R(F) = T(F) - \sum_{i,j} B_j(A_i(F))$, where $T = T_2T_1$, $T_1(F) = \sum_i A_i(F) + R_1(F)$, $T_2(F) = \sum_j B_j(F) + R_2(F)$,

and T_1 operates on the function $F(x,y)$ as a function of its first variable and T_2 dually. For this situation, Stancu shows that

(2) $$R(F) = R_1(T_2(F)) + R_2(T_1(F)) - R_2(R_1(F)) .$$

For interpolation along sections, T, T_1, and T_2 are all point evaluations at (x,y), $A_i(F) \equiv L_i(F)\varphi_i(x)$, and $B_j(F) \equiv M_j(F)\psi_j(y)$.

<u>Theorem 3.</u> Under the above conditions,

(3) $$R_B(F) = R_2R_1(F).$$

<u>Proof</u> : $R(F) = T(F) - \sum_{i,j} B_j(A_i(F))$. Subtract $R_1(T_2(F)) + R_2(T_1(F)) - 2R_2(R_1(F))$ from both sides of this equation.

Thus

$$(4) \qquad R_2(R_1(F)) = R(F) - \sum_{i,j} B_j(A_i(F)) - R_1(T_2(F))$$

$$- R_2(T_1(F)) + 2R_2(R_1(F)).$$

Now

$$R_1(T_2(F)) + R_2(T_1(F)) - 2R_2(R_1(F))$$

$$= R_1(T_2(F) - R_2(F)) + R_2(T_1(F) - R_1(F))$$

$$= R_1(\sum_j B_j(F)) + R_2(\sum_i A_i(F))$$

$$= T_1(\sum_j B_j(F)) - \sum_i A_i(\sum_i A_i(\sum_i B_j(F))$$

$$+ T_2(\sum_i A_i(F)) - \sum_i B_j(\sum_j B_j(\sum_i A_i(F))$$

$$= \sum_j B_j(F) - 2\sum_{i,j} A_i(B_j(F)) + \sum_i A_i(F) \quad .$$

Substitution of this in equation (4) yields

$$R_2(R_1(F)) = F(x,y) - \sum_i A_i(F) - \sum_i A_i(F) - \sum_j B_j(F) + \sum_{i,j} A_i(B_j(F) \equiv R_B(F) \quad .$$

Q.E.D,

Gordon has derived remainder terms for specific examples that are of the form $R_B(F) = R_2R_1(F)$. The following corollary shows that this is a general result.

<u>Corollary</u>. Let $T = T_2T_1$ be a bounded linear functional . that commutes with R_1 and R_2 .

Then $$R_B(T(F)) = T(F) - \sum_i L_i(T_2(F))\ T_1(\varphi_i(x)) - \sum_j M_j(T_1(F))\ T_2(\psi_j(y)) + \sum_{i,j} L_i\ M_j\ (F)\ T_1(\varphi_i(x))\ T_2(\psi_j(y)) \ .$$

Proof: Apply $T = T_2 T_1$ to equation (1).

The importance of this corollary is that, when the problem functional T, which operates on functions of two variables, can be written as a composition of linear functionals T_1, and T_2 which operate on functions of one variable, then $R_B(T(F)) = R_2R_1(T(F))$ and the appropriate blending-function approximation is obtained by operating with T on the interpolatory blending-function. (The latter is the procedure used in practice.) The point is that R_1 and R_2 are the one-dimensional interpolation remainders throughout, instead of being e.g., quadrature remainders if $T(F) = \int_c^b \int_a^b F(x, y)dxdy$. If the varibles in T cannot be separated into the product of a T_1 and T_2, then the above does not hold. However, blending-function methods are inherently of a (generalized) cross-product type in that $R_B(\varphi_1(x)g(y)) = 0 = R_B(f(x)\psi_j(y))$, i.e., the precision is of a rectangular type and spaces analogous to Sard's [5] $B_{p,q}$ are appropriate.

We remark in conclusion that the use of projections can simplify other proofs, e.g., the minimum norm property for interpolating blending-functions [3]. In addition, it leads to $(P_x + P_y - P_x P_y)(F)$ as the approximation to use, since $F - P_x(F)$ and $P_y(F) - P_x P_y(F)$ are both orthogonal to M_1, $F - P_y(F)$ and $P_x(F) - P_y P_x(F)$ are both orthogonal to M_2, and hence $(P_x + P_y - P_x P_y)(F)$ is orthogonal to $M_1 \cup M_2$. (Equivalently, the factorization $I - E = (I - E_1)(I - E_2)$ of Theorem 1 could be considered,)

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