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Abstract   A polygonal patch is defined to fill an n-sided hole within a C^2 parametric continuous rectangular patch complex.

1. <u>Introduction</u>

In this note a polygonal surface patch interpolant is presented which will fill an n-sided hole within a C^2 parametric continuous rectangular patch complex. The polygonal patch interpolant is based on a convex combination construction, with appropriate choice of the boundary data to ensure geometric continuity of order 2 with the adjoining rectangular patches (GC^2).

A method for solving the general GC^k problem is presented in [Gregory and Hahn '87]. This method involves the C^k reparameterization of the surface around the hole so that it is defined on the exterior of a regular polygon. The method for the particular C^2 case proposed here is different, however, in that it involves the direct use of geometric GC^2 continuity conditions in the development of the patch.

We will make use of geometric continuity conditions between patches in the total derivative form given in [Gregory and Hahn '86]. The present paper can be considered as a sequel to this earlier paper in that it provides a solution to the polygonal patch problem posed there. We begin by briefly reviewing the conditions for geometric continuity between patches. Further details can be found in the earlier paper or in [Hahn '87] of these proceedings, where a full discussion of geometric continuity is given together with appropriate references.

2. <u>Conditions for Geometric Continuity between Patches</u>

Let $p : \Omega_p \rightarrow \mathbb{R}^3$ and $q : \Omega_q \rightarrow \mathbb{R}^3$ be two parametric surface patches defined on closed polygonal domains $\Omega_p \subset \mathbb{R}^2$ and $\Omega_q \subset \mathbb{R}^2$. (We are specifically concerned with the case of Ω_q a rectangle and Ω_p an n-sided polygon.) Assume that p and q are regular C^2 maps and let $e_p : [0,1] \rightarrow \mathbb{R}^2$ and $e_q : [0,1] \rightarrow \mathbb{R}^2$ be regular parametric representations of boundary segments of Ω_p and Ω_q respectively. Then the following proposition is based on Lemma 2.1 of [Gregory and Hahn '86], where, for notational convenience, we have reversed the roles of p and q from the earlier paper.

<u>Proposition 1</u>. The patches p and q have a GC^2 join across their respective boundary segments e_p and e_q, if there exists a C^2 diffeomorphism $\phi : \mathbb{R}^2 \to \mathbb{R}^2$, defined in a neighbourhood of e_p, which is such that:

(i)   (<u>Domain continuation</u>) $e_q = \phi \circ e_p$ and interior points of Ω_q are mapped from exterior points of Ω_p.

(ii) (<u>Patch continuity</u>) Given any (non-zero) transversal vector field U(s) defined on $e_p(s)$, then

$$\partial^i p\big|_{e_p(s)}\left(U^i(s)\right) = \partial^i (q \circ \phi)\big|_{e_p(s)}(U^i(s)), \ \ i = 0,1,2, \text{ for all } 0 \le s \le 1. \tag{2.1}$$

Here, $\partial p\big|_x$ is the first derivative linear map, $\partial^2 p\big|_x$ is the second derivative summetric bilinear map at $x \in \mathbb{R}^2$ and $U^i(s)$ denotes the i-tuple $(U(s),...,U(s))$.

Expanding the right hand sides of (2.1) using the chain and product rules, and writing

$$V(s) = \partial\phi\big|_{e_p(s)}(U(s)), W(s) = \partial^2\phi\big|_{e_p(s)}(U(s), U(s)), \tag{2.2}$$

leads to the following proposition, the sufficiency of which is proved in [Gregory and Hahn '86], (see also [Hahn '87] for a weaker form of the proposition).

<u>Proposition 2.</u> Let $U:[0,1] \to \mathbb{R}^2$ be a C^2 (non-zero) vector field which is transversal to e_p in an inward direction. Then the patches p and q have a GC^2 join iff. there exist (i) a (non-zero) C^1 vector field $V:[0,1] \to \mathbb{R}^2$, transversal to e_q in an outward direction, and (ii) a C^0 vector field $W:[0,1] \to \mathbb{R}^2$, which are such that

$$p(e_p(s)) = q(e_q(s)), \tag{2.3}$$

$$\partial p\big|_{e_q(s)}(U(s)) = \partial q\big|_{e_q(s)}(V(s)), \tag{2.4}$$

$$\partial^2 p\big|_{e_p(s)}(U(s), U(s)) = \partial^2 q\big|_{e_q(s)}(V(s), V(s)) + \partial q\big|_{e_q(s)}(W(s)) \tag{2.5}$$

Our final proposition is concerned with geometric continuity between patches p and q, when p is defined as a convex combination of two patches p_1 and p_2. It was observed in [Gregory and Hahn '86] that GC^2 continuity between p_1 and q, and between p_2 and q, is not sufficient to ensure GC^2 continuity between the patches p and q. The following proposition, which gives a sufficient

condition for a GC^2 join, will be used in the development of the polygonal patch.

<u>Proposition 3</u>. Let $p_i:\Omega_p \to \mathbb{R}^3$, $i=1,2$, have GC^2 joins with $q:\Omega_q \to \mathbb{R}^3$ in the sense of Proposition 2. Thus

$$p_i(e_p(s) = q(e_q(s)) \tag{2.6}$$

$$\partial p_i \big|_{e_p(s)}(U(s)) = \partial p \big|_{e_q(s)}(V_i(s)) \tag{2.7}$$

$$\partial^2 p_i \big|_{e_p(s)}(U(s)), U(s)) = \partial^2 q \big|_{e_q(s)}(V_i(s), V_i(s)) + \partial q \big|_{e_q(s)}(W_i(s)) \tag{2.8}$$

for appropriately defined vector fields $U(s)$, $V_i(s)$, $W_i(s)$, $i=1,2$. Let $p:\Omega_p \to \mathbb{R}^3$ be defined by

$$P(x) = W_1(x)P_1(x) + W_2(x)P_2(x) \tag{2.9}$$

where $w_i:\Omega_p \to \mathbb{R}$, $i=1,2$ are C^2 functions such that

$$(w_1 + w_2)\big|_{e_p(s)} = 1 \text{ and } \partial^i(w_1 + w_2)\big|_{e_p(s)} = 0, \ \ j = 1,2. \tag{2.10}$$

Then a sufficient condition that p and q have a GC^2 join is that

$$V_1(s) \equiv V_2(s).$$

The proof of this proposition is straight forward since it is readily shown that

$$p(e_p(s)) = q(e_q(s)),$$

$$\partial p \big|_{e_p(s)}(U(s)) = \partial q \big|_{e_q(s)}(V(s)),$$

$$\partial^2 p \big|_{e_p(s)}(U(s), U(s)) = \partial^2 q \big|_{e_q(s)}(V(s), V(s))$$

$$+ \partial q \big|_{e_q(s)}(w_1(s)W_1(s) + w_2(s)W_2(s)),$$

where $w_i(s) = w_i(e_q(s))$ and $V(s) \equiv V_1(s) \equiv V_2(s)$. The essential point to note in Proposition 3 is that we have assumed the particular case where $V_1(s) \equiv V_2(s)$ and hence that p_i and q, i=1,2, have <u>identical</u> GC^1 joins.

3. <u>The Polygonal Patch Problem</u>

Let q_i, i=0,...,n-1, form a C^2 parametric continuous rectangular patch complex around an n-sided hole. More specifically, let $q_i:[0,2] \times [0,1] \to \mathbb{R}^3$ be such that

$$\partial_{0,j}q_i(s,1) := \frac{\partial^j}{\partial t^j} q_i(s,t)\Big|_{t=1}, \quad j=0,1,2, \tag{3.1}$$

defines the boundary data along the i'th segment of the hole for $0 \le s \le 1$, see Figure 1. (In practice, each q_i may consist of a sub-complex of rectangular patches but for descriptive purposes it is convenient to represent this as one parametric surface.)
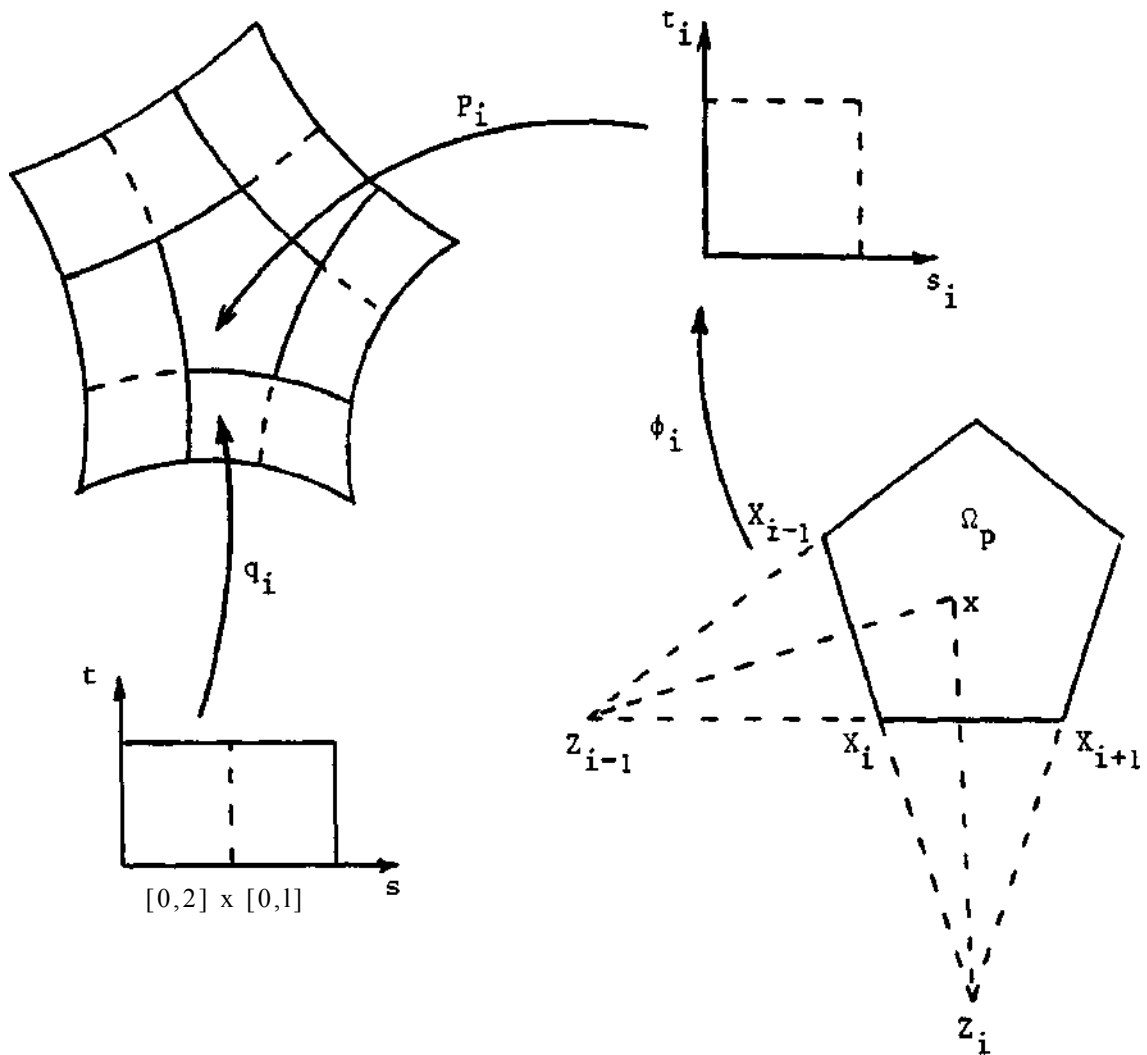

Figure 1

We assume that the patches q_i, $i = 0,\ldots, n-1$, form a C^2 parametrically continuous patch complex in the sense that the composed map

$$(s,t) \rightarrow \begin{cases} q_{i-1}(2-t,1+s), & (s,t) \in [-1,0] \times [0,2] \\ q_i(s,t), & (s,t) \in [0,2] \times [0,1] \end{cases} \tag{3.2}$$

is C^2 continuous (i.e. parametrically C^2) for all $i=0,\ldots,n-1$. In many practical applications the composed map will also be $C^{2,2}$ continuous and, for simplicity, we assume this case here. In particular, $C^{2,2}$ continuity of the composed map at the corner $(0,1)$ gives the "compatibility conditions"

$$\partial_{j_1,j_2} q_i(0,1) = (-1)^{j_2} \partial_{j_2,j_1} q_{i-1}(1,1), \quad 0 < j_1, j_2 < 2. \tag{3.3}$$

Let $\Omega_p \subset \mathbb{R}^2$ be an n-sided polygon with sides of length unity, centre the origin 0, vertices X_i, $i=0,\ldots,n-1$ and edges E_i, $i=0,\ldots,n-1$, parameterized by

$$E_i(s) = (1-s)X_i + sX_{i+1}. \tag{3.4}$$

Then, following the approach of [Charrot and Gregory '84], our polygonal patch $p:\Omega_p \rightarrow \mathbb{R}^3$ will take the form

$$p(x) = \sum_{i=0}^{n-1} w_i(x)p_i(x), \quad x \in \Omega_p, \tag{3.5}$$

where $p_i:\Omega_p \rightarrow \mathbb{R}^3$ is a parametric surface patch interpolant which will be constructed such that it has GC^2 joins with q_{i-1} along the edge E_{i-1} and q_i along the edge E_i.

The weights $w_i:\mathbb{R}^2 \rightarrow \mathbb{R}^2$ are C^2 functions chosen such that

$$\sum_{i=0}^{n-1} w_i(x) = 1, \quad w_i(x) \geq 0, \text{ for } x \in \Omega_p, \tag{3.6}$$

and

$$\partial^j w_k|_{E_i} = 0, j = 0,1,2, \text{ for } k \neq i, i+1, \ k = 0,\ldots,n-1. \tag{3.7}$$

Thus, to investigate the GC^2 join of p with q_i along the edge E_i, we may write p in the form

$$p(x) = w_i(x)p_i(x) + w_{i+1}(x)p_{i+1}(x) + r_i(x) \tag{3.8}$$

where

$$\partial^j r_i|_{E_i} = 0, \ j = 0,1,2. \tag{3.9}$$

Now p_i and p_{i+1} will each have GC^2 joins with the patch q_i along the edge E_i and if the GC^1 joins are chosen to be identical, then, by Proposition 3, the convex combination patch p will have a GC^2 join with the rectangular patch complex. We are thus concerned with constructing the component patches

$p_i : \Omega_p \to \mathbb{R}^3$ to satisfy such special GC^2 conditions. This construction involves the use of coordinate systems ("coordinate charts") (s_i, t_i), $i = 0, \ldots, n-1$, defined in the following section.

4. Coordinate Charts for Interpolation

Let Z_i be the point of intersection of the boundary segments E_{i-1} and E_{i+1}. Then

$$d_i(x) := < X_i - x, Z_i > / \|Z_i\| \tag{4.1}$$

is the perpendicular distance of $x \in \Omega_p$ from the side E_i, where

$$<x,y> \; := \; x_1 y_1 + x_2 y_2 \tag{4.2}$$

is the Euclidean scalar product of $x = (x_1, x_2) \in \mathbb{R}^2$ and $y = (y_1, y_2) \in \mathbb{R}^2$. Coordinate charts are then defined by

$$\phi_i(x) := (s_i(x), t_i(x)) := \left(\frac{d_{i-1}(x)}{d_{i+1}(x) + d_{i-1}(x)}, \frac{d_i(x)}{d_{i-2}(x) + d_i(x)} \right) \tag{4.3}$$

The coordinate chart ϕ_i corresponds to central projections from Z_i and Z_{i-1}. Thus $E_i(s_i)$ is the point of intersection of the line from Z_i to x with the edge E_i and $E_{i-1}(1-t_i)$ is the intersection of the line from Z_{i-1} to x with E_{i-1} see Figure 1. The interpolants $p_i : \Omega_p \to \mathbb{R}^3$ will be constructed as

$$p_i(x) = p_i \circ \phi_i(x) = p_i(s_i(x), t_i(x)), \; x \in \Omega_p, \tag{4.4}$$

where $p_i : [0,1]^2 \to \mathbb{R}^3$ is chosen to have GC^2 joins with q_i on $t_i = 0$ and q_{i-1} on $s_i = 0$. GC^2 continuity of p_i with q_i and q_{i-1} is then guaranteed, since p_i is related to p_i through the coordinate chart diffeomorphism ϕ_i (see Proposition 1). The explicit construction of p_i (and hence p_i) is given in the next section. However, in order to make use of Proposition 3, we first relate the derivative of $p_i = p_i \circ \phi_i$ along the direction $E_i(s) - Z_i$ with the appropriate derivative of p_i in the coordinate chart.

The chain rule gives

$$\partial p_i \big|_{E_i(s)} (E_i(s) - Z_i) = \partial p_i \big|_{(s,0)} \partial \phi \big|_{E_i(s)} (E_i(s) - Z_i) \qquad (4.5)$$

where

$$\partial \phi_i \big|_{E_i(s)} (E_i(s) - Z_i) = \frac{\partial}{\partial t} \phi_i (E_i(s) + t(E_i(s) - Z_i)) \big|_{t=0}. \qquad (4.6)$$

After some calculation, which for brevity is omitted, we obtain

$$\phi_i (E_i(s) + t(E_i(s) - Z_i)) = (s, t / [4c^2 s(t+1) + 2c]) \qquad (4.7)$$

where

$$c = \cos(2\pi / n). \qquad (4.8)$$

Hence

$$\partial p_i \big|_{E_i(s)} (E_i(s) - Z_i) = \frac{1}{\gamma(s)} \partial p_i \big|_{(s,0)} (0,1), \qquad (4.9)$$

where

$$\gamma(s) = 4c^2 s + 2c. \qquad (4.10)$$

Similarly

$$\partial p_{i+1} \big|_{E_i(s)} (E_i(s) - Z_i) = \frac{1}{\gamma(1-s)} \partial p_{i+1} \big|_{(0,s)} (1,0). \qquad (4.11)$$

It should be noted that (4.9) and (4.11) describe the fact that differentiation along the direction (0,1) in the coordinate chart ϕ_i corresponds, with appropriate scaling, to differentiation along (1,0) in the coordinate chart ϕ_{i+1}. It is this observation that allows us to make use of Proposition 3.

5. The Polygonal Patch Interpolant

We assume $C^{2,2}$ parametric continuity of the rectangular patch complex q_i, $i = 0,\dots,n-1$, (see Section 3) and define $p_i : [0,1]^2 \to \mathbb{R}^3$ by the Boolean sum Taylor interpolant

$$p_i(s,t) = \sum_{j=0}^{2} \frac{s^j}{j!} \beta^j(t) \partial_{0,j} q_{i-1}(1-t,1)$$

$$+ \sum_{j=0}^{2} \frac{t^j}{j!} \beta^j(s) \partial_{0,j} q_i(s,1)$$

$$- \sum_{j_1=0}^{2} \sum_{j_2=0}^{2} \frac{s^{j_1} t^{j_2}}{j_1! j_2!} \partial_{j_1,j_2} q_i(0,1). \qquad (5.1)$$

Here, β is to be a strictly positive function chosen such that $p_i = p_i \circ \phi_i$ and $p_{i+1} = p_{i+1} \circ \phi_{i+1}$ have identical GC^1 joins with q_i. We first assume that

$$\beta(0) = 1,\ \dot\beta(0) = \ddot\beta(0) = 0, \tag{5.2}$$

and then p_i has the interpolation properties:

$$\partial_{j,0} p_i(0,t) = \beta^j(t)\partial_{0,j} q_{i-1}(1-t,1),\ j = 0,1,2, \tag{5.3}$$

$$\partial_{0,j} p_i(s,0) = \beta^j(s)\partial_{0,j} q_i(s,1),\ \ j = 0,1,2. \tag{5.4}$$

The conditions (5.3) and (5.4) mean that $p_i = p_i \circ \phi_i$ joins q_i and q_{i-1} with GC^2 continuity. We now show that β can be chosen so that the GC^1 joins are identical and hence, by Proposition 3, p will have a GC^2 join with the rectangular patch complex.

From (4.9), (4.11) and the interpolation properties (5.3), (5.4) we

Obtain

$$\partial p_i |E_i(s)^{(E_i(s)-Z_i)} = \frac{\beta(s)}{\gamma(s)}\partial_{0,1} q_i(s,1)$$

$$\partial p_{i+1} |E_i(s)^{(E_i(s)-Z_i)} = \frac{\beta(1-s)}{\gamma(1-s)}\partial_{0,1} q_i(s,1)$$

Thus, for an identical GC^1 join, we require that

$$\beta(s)/\gamma(s) = \beta(1-s)/\gamma(1-s), \tag{5.5}$$

where

$$\gamma(s) = 4c^2 s + 2c. \tag{5.6}$$

Hence

$$\beta(s) = \gamma(s)/\alpha(s) \tag{5.7}$$

say, where from (5.2), (5.5), and (5.6), $\alpha(s)$ must be chosen such that $\alpha(s) = \alpha(1-s)$ and

$$\alpha(0) = 2c,\ \dot\alpha(0) = 4c^2,\ \ddot\alpha(0) = 0.$$

Quintic Hermite interpolation then provides the definition

$$\alpha(s) = 4c^2\left(s^4 - 2s^3 + s\right) + 2c, \tag{5.8}$$

this being a positive concave function on $0 \le s \le 1$. This together with (5.7) and (5.6) defines an appropriate function $\beta(s)$.

We can now summarize the polygonal patch interpolant as

$$p(x) = \sum_{i=0}^{n-1} w_i(x) p_i(s_i(x), t_i(s)), \qquad (5.9)$$

where p_i is defined by (5.1) and the coordinates $(s_i(x), t_i(x))$ are defined by (4.3). An appropriate definition of the weighting functions is

$$w_i(x) = \prod_{j \neq i, i+1} d_j^3 \Bigg/ \left[\sum_{k=0}^{n-1} \prod_{j \neq k, k+1} d_j^3 \right]. \qquad (5.10)$$

6. Shape Control

Equations (5.7) and (5.8) provide one of infinitely many possible definitions for the scaling function $\beta(s) = \gamma(s)/\alpha(s)$. Defining

$$\beta_i(s) = \gamma(s)/\alpha_i(s), \quad \text{where} \quad \alpha_i(s) = v_i s^3 (1-s)^3, \qquad (6.1)$$

provides a method of shape control across the boundaries E_i, for given real parameters v_i. The terms $\beta(t)$ and $\beta(s)$ in (5.1) must then be replaced by $\beta_{i-1}(t)$ and $\beta_i(s)$ respectively.
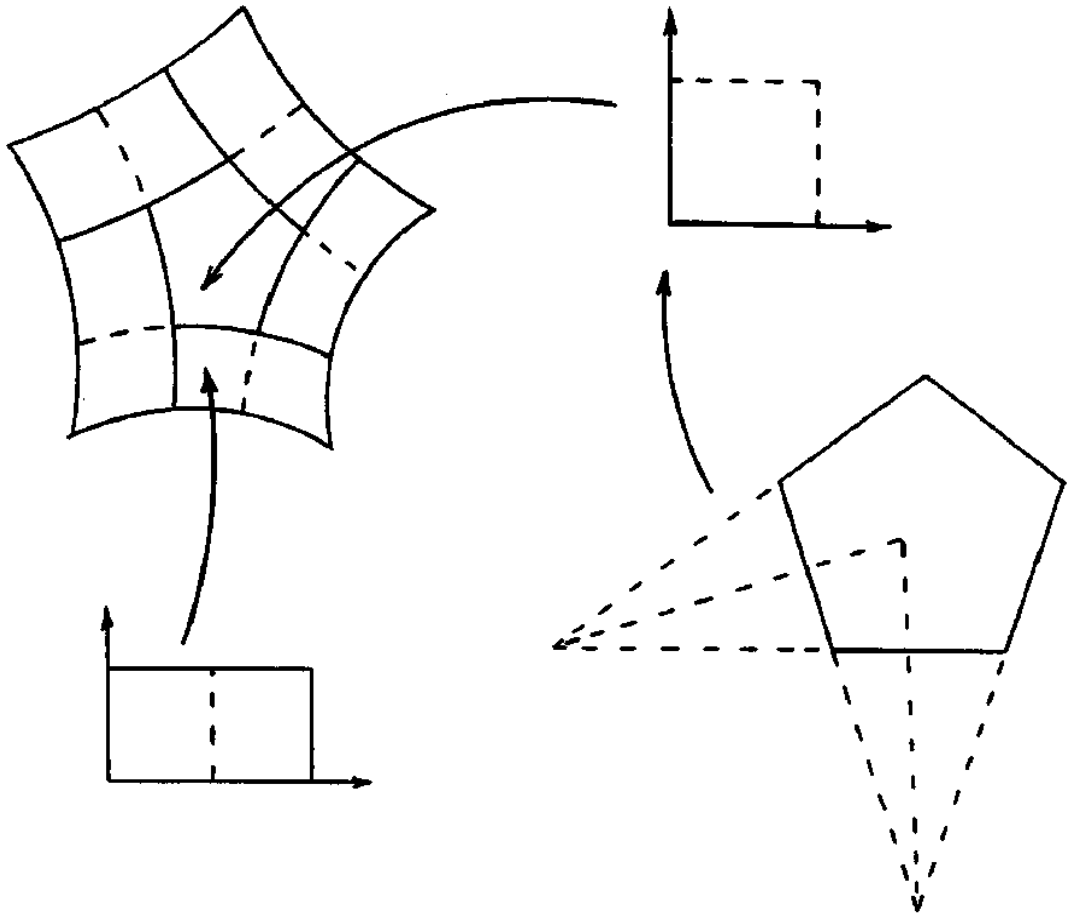
Another possibility for shape control suggested in [Gregory and Hahn '87] is to add a term of the form

$$r(x) \prod_{i=0}^{n-1} d_i^3 \qquad (6.2)$$

to the polygonal patch interpolant p. This term will not affect the GC^2 continuity along the boundary and $r(x)$ may thus be used to control the shape of the polygonal patch interior.

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