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Reformulations of Mathematical Programming Problems as linear Complementarity Problems

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0. Abstract

A family of complementarity problems are defined as extensions of the well known Linear Complementarity Problem (LCP). These are

(i.) Second Linear Complementarity Problem (SLCP) which is an LCP extended by introducing further equality restrictions and unrestricted variables,

(ii.) Minimum Linear Complementarity Problem (MLCP) which is an LCP with additional variables not required to be complementary and with a linear objective function which is to be minimized,

(iii.) Second Minimum Linear Complementarity Problem (SMLCP) which is an MLCP but the nonnegative restriction on one of each pair of complementary variables is relaxed so that it is allowed to be unrestricted in value.

A number of well known mathematical programming problems, namely quadratic programming (convex, nonconvex, pseudoconvex nonconvex), bilinear programming, game theory, zero-one integer programming, the fixed charge problem, absolute value programming, variable separable programming are reformulated as members of this family of four complementarity problems.

1. Introduction

Linear Complementarity Problems may be defined as that of solving systems of linear equations in the variables z and w traditionally required to be nonnegative and satisfying a complementarity relationship. Some of these problems have in addition a linear objective function to be minimized. In this paper four linear complementarity problems are distinguished. These are stated below:

(i) Linear Complementarity Problem (LCP)

$$w = q + Mz,\ z \geq 0,\ w \geq 0,\ z^T w = 0,\ z,w \varepsilon R^n . \tag{1}$$

(ii) Second Linear Complementarity Problem (SLCP)

$$w = q + Mz + Nu,$$

$$0 = p + Rz + Su,$$

$$z \geq 0,\ w \geq 0,\ z^T w = 0,\ -\infty < u < +\infty,\ z,w\varepsilon R^n ,\ p,u\varepsilon R^m. \tag{2}$$

(iii) Minimum Linear Complementarity Problem (MLCP)

$$\text{Minimize } p^T z + q^T w + r^T u,$$
$$Pz + Qw + Ru = b,$$
$$z \geq 0,\ w \geq 0,\ z^T w - 0,\ z,w\varepsilon R^n,\ b\varepsilon R^m,\ u\varepsilon R^\ell, \tag{3}$$

and the u-variables may be unrestricted or nonnegative.

(iv) Second Minimum Linear Complementarity Problem (SMLCP)

$$\text{Minimize } p^T z + q^T w + r^T u ,$$

$$Pz + Qw + Ru = b, \tag{4}$$
$$z \geq 0,\ -\infty < w < +\infty,\ z_i w_i = 0,\ i - 1, \ldots, n,\ z,wR^n ,\ u\varepsilon R^\ell ,\ b\varepsilon R^m$$

and the u-variables may be unrestricted or nonnegative.

To date only the LCP has been considered in any detail as a mathematical programming problem. This problem was first proposed by Lemke as an unified approach to solve the bimatrix game and the convex quadratic programming problem ([1,2]). Since then it has been studied extensively both in theoretical and computational aspects ([3,4,5]). Some problems in engineering applications [6,7] and the bimatrix and polymatrix games [8,9] have been studied as LCPs. However the most

Important application of the LCP is to solve convex quadratic programming problems. Portfolio selection problems [10,11,7] and problems involving variational inequalities [12,13,14] reduce to convex quadratic programming problems and have been sited as examples of LCP applications. If some equality constraints and unrestricted variables are introduced in the convex quadratic program then this problem is no longer equivalent to an LCP but can be reduced to an SLCP. The SLCP was proposed precisely for this purpose.

The MLCP was first introduced by Ibaraki [l5] as an alternative method of solving the 0-1 integer programming problem. In this paper it is shown that a number of other discrete optimization problems may also be reduced to the MLCP. This underlies the importance of the MLCP and the need to find efficient methods for solving this problem. Concentrating for the moment on each component of the complementarity condition

$$z_i w_i. = 0,\ z_i \geq 0,\ w_i. \geq 0$$

there is an implied either/or relationship which is comparable to a boolean 0-1 variable. In certain reformulations of optimization problems as linear complementarity problems this relationship is still required but because of the nature of the problem some of the variables are not restricted in sign. This is the basis of the SMLCP where the above relationship appears as

$$z_i.w_i. = 0,\ z_i. \geq 0,\ -\infty < w_i < +\infty.$$

In section 2 of this paper it is shown that a linear programming problem in the general form and its dual are compactly restated as an SLCP. In section 3 it is shown that convex, nonconvex, and pseudoconvex nonconvex quadratic programming problems may be reformulated as an LCP, an SLCP or an MLCP- Reformulation of the bilinear programming problem section 4, matrix games section 5, absolute value programming section 6 and zero—one integer programming section 7 are already reported in the literature. These are presented here for the purpose of completeness. In section 8 a new compact reformulation of the fixed—change problem which uses the linear complementarity relation is presented. In section 9 the latter approach is extended to show that the separable programming problem can be reformulated as an SMLCP. In section 10 solution methods for processing the Linear Complementarity Problems are discussed. Experience of solving a number of test problems taken from different sources is reported elsewhere in [16].

2. Linear Programming

Consider the general linear programming problem

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$$\text{Minimize } \{f(x,y) = c^T x + d^T y, (x,y) \varepsilon K_{\ell p}\}, \tag{5}$$

where

$$K_{\ell p} = \{(x,y) : Ax + By = b,\ Ex + Fy \geq g,\ x \geq 0,\ -\infty < y < +\infty\} \tag{6}$$

Its dual may be stated as ([17, page 154])

$$\text{Maximize } \{h(\pi,u) = b^T \Pi + g^T u, (\pi,u)\, \varepsilon\, {}^{K}{}_{d\ell p}\}, \tag{7}$$

where

$$K_{d\ell p} = \{(\pi, \mu : A^T \Pi \quad + \quad E^T \Pi\ ,\ B^T \Pi + F^T \mu = d,\ \mu \geq 0,\ -\infty < \pi < +\infty\} \tag{8}$$

Suppose that $(\overline{x}, \overline{y})$ is an optimal solution of the linear program (5).

By the Fundamental Duality theorem of linear programming ([17, page 159]) the dual (7) also has an optimal solution $(\overline{\pi}, \overline{\mu})$. Furthermore the complementarity slackness property ([I7, page 165]) concerning the nonnegative variables $\overline{x}$ and $\overline{\mu}$ and the slack variables $\overline{v}$ and $\overline{\gamma}$ corresponding to the inequalities in the dual (7) and primal (5) holds. Hence $(\overline{v}, \overline{\gamma}, \overline{x}, \overline{\mu}, \overline{y}, \overline{\pi})$ is a solution of the following SLCP

$$\begin{bmatrix} v \\ \gamma \end{bmatrix} = \begin{bmatrix} c \\ -g \end{bmatrix} + \begin{bmatrix} 0 & -E^T \\ E & 0 \end{bmatrix} \begin{bmatrix} x \\ \mu \end{bmatrix} + \begin{bmatrix} 0 & -A^T \\ F & 0 \end{bmatrix} \begin{bmatrix} y \\ \pi \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} d \\ -b \end{bmatrix} + \begin{bmatrix} 0 & -F^T \\ A & 0 \end{bmatrix} \begin{bmatrix} x \\ \mu \end{bmatrix} + \begin{bmatrix} 0 & -B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} y \\ \pi \end{bmatrix} \tag{9}$$

$$v \geq 0,\ Y \geq 0,\ X \geq 0,\ \mu \geq 0,\ -\infty < y < +\infty,\ -\infty < \pi < +\infty,\ v^T x + \gamma^T \mu = 0.$$

Conversely if $((\overline{v}, \overline{\gamma}, \overline{x}, \overline{\mu}, \overline{y}, \overline{\pi})$ is a solution of the SLCP (9) then both the constraint sets $K_{\ell p}$ and $K_{d\,\ell p}$ are nonempty and

$$f(\overline{x}, \overline{y}) = c^T \overline{x} + d^T \overline{y} = \overline{x}^T (\overline{v} + E^T \overline{\mu} + A^T \overline{\pi}) + \overline{y}^T (F^T \overline{\mu} + B^T \overline{\pi})$$

$$= \overline{x}^T \overline{v} + \overline{\mu}^T (E\overline{x} + F\overline{y}) + \overline{\pi}^T (A\overline{x} + B\overline{y})$$

$$= \overline{x}^T \overline{v} + \overline{\mu}^T (g + \overline{\gamma}) + \overline{\pi}^T b = \overline{x}^T \overline{v} + \overline{\mu}^T \overline{\gamma} + \overline{\mu}^T g + \overline{\pi}^T b = h(\overline{\mu}, \overline{\pi})$$

Hence by the Fundamental theorem of duality (x,y) Is an optimal solution of the linear program (5) and (μ,π) is an optimal solution of the dual (7). Therefore the following theorem holds:

Theorem 1 There is a one-to-one correspondence between the solutions of the linear program (5) and the SLCP (.9).

In (5) set A = B = F = 0 and d = b = 0 and thereby the correspondence between the canonical linear program and the LCP given in [18] is obtained. It may be noted that the existence of the complementarity condition is due to inequalities in the constraint set $K_{\ell p}$ and that a linear program in which all constraints are equalities and all the variables are unrestricted in sign is equivalent to a system of equations.

3. Quadratic Programming

Consider the general quadratic programming problem

$$\text{Minimize } f(x,y) = \begin{bmatrix} c \\ d \end{bmatrix}^T \begin{bmatrix} x \\ y \end{bmatrix} + \frac{1}{2}\begin{bmatrix} x \\ y \end{bmatrix}^T \begin{bmatrix} P & R \\ R^T & Q \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} \quad , \tag{10}$$

$(x,y) \in K_{qp} = \{(x,y) : Ax+By=b ,\ Ex+Fy \geq g ,\ \ x \geq 0 \ \ , -\infty < y < +\infty\},$

where P and Q are symmetric matrices. The first theorem of this section is a generalisation for the quadratic program (10) of a result given by Murty ([17, page 491]).

Theorem 2 If $(\bar{x},\bar{y})$ is an optimal solution of the quadratic program (10) then $(\bar{x},\bar{y})$ is also an optimal solution of the linear program

$$\text{Minimize } \{(c + P\bar{x} + R\bar{y})^T x + (d + R^T\bar{x} + Q\bar{y})^T y, \quad (x,y) \in k_{qp}\} \tag{11}$$

Proof By writing

$$z = \begin{bmatrix} x \\ y \end{bmatrix}, \ q = \begin{bmatrix} c \\ d \end{bmatrix}, \ D = \begin{bmatrix} P & R \\ R^T & Q \end{bmatrix}, \ \bar{z}\begin{bmatrix} \bar{x} \\ \bar{y} \end{bmatrix} \tag{12}$$

the programs(10) and (11) can be written respectively in the forms

$$\text{Minimize } \{q^T z + \frac{1}{2} z^T Dz,\ z \in K_{qp}\},$$

$$\text{Minimize } \{(q + D\bar{z})^T z,\ z \in K_{qp}\},$$

The same proof of Murty's result can now be used to establish the theorem.

The dual of the linear program (11) takes the form

$$\text{Maximize } \{b^T \pi + g^T \square : (\pi.\mu) \in K_{d\ell pa}\}, \tag{13}$$

where

$$K_{d\ell pa} = \{(\pi(\mu) : A^T\pi + E^T \le c + P\bar{x} + R\bar{y},$$
$$B^T\pi + F^T\mu = d + R^T\bar{x} + Q\bar{y}, \mu \ge 0, -\infty < +\pi < +\infty\}.$$

Therefore Theorem 3 follows from the theorems 1 and 2 and is stated below.

Theorem 3 If $(\bar{x},\bar{y})$ is an optimal solution of the quadratic program (10) there exist vectors $\bar{\pi}$, $\bar{\mu}$, $\bar{v}$, and $\bar{\gamma}$ such that $(\bar{x}, \bar{y}, \bar{\pi}, \bar{\mu}\ \bar{v}, \bar{\gamma})$ is a solution of the SLCP

$$\begin{bmatrix} v \\ \gamma \end{bmatrix} = \begin{bmatrix} c \\ -g \end{bmatrix} + \begin{bmatrix} P & -E^T \\ E & 0. \end{bmatrix}\begin{bmatrix} x \\ \mu \end{bmatrix} + \begin{bmatrix} R & -A^T \\ F & 0. \end{bmatrix}\begin{bmatrix} y \\ \pi \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} d \\ -b \end{bmatrix} + \begin{bmatrix} R^T & -F^T \\ A & 0. \end{bmatrix}\begin{bmatrix} x \\ \mu \end{bmatrix} + \begin{bmatrix} Q & -B^T \\ B & 0. \end{bmatrix}\begin{bmatrix} y \\ \pi \end{bmatrix} \tag{14}$$

$v \ge 0$, $y \ge 0$, $x \ge 0$, $\square \ge 0$, $-\infty < y < +\infty$, $-\infty < \pi < +\infty$, $v^T x + \gamma^T \mu = 0$.

The conditions(14) are usually called the Kuhn-Tucker conditions (19). The converse of this theorem is not in general true. However, if the matrixes positive semi-definite the converse does hold as set out in the following theorem.

Theorem 4 If the matrix D as in (12) is positive semi-definite there is a one-to-one correspondence between the optimal solution $(\bar{x}, \bar{y})$ of the quadratic program (10) and the solution $(\bar{x}, \bar{y}, \bar{\pi}, \bar{\mu}, \bar{v}, \bar{\gamma})$ of the SLCP (14).

Proof Because of the last theorem it is only necessary to prove that $f(z) \geq f'(\bar{z})$ for any $z \in K_{qp}$. But

$$f(z) - f(\bar{z}) - (q + D\bar{z})^T, (z - \bar{z}) + 1 (z - \bar{z})^T D (z - \bar{z}) \quad (15)$$

Since $(q + D\bar{z})^T (z - \bar{z})$ a $\geq$ by theorem 2 and the second term of the right-hand side of (15) is nonnegative (D is positive semi-definite) and the result follows,

A quadratic program whose matrix is positive semi-definite is called a Convex Quadratic Program ([20, chapter 9]). The equivalence of a convex quadratic program whose constraints are all inequalities to an L*CP* ([l8]) follows from theorem 4 by setting F=A=B=R=Q=0 and d = b - 0. It also follows that a convex quadratic program with only equality constraints and all variables unrestricted in sign is equivalent to a system of equations.

For the case in which all the variables of (10) are required to be nonnegative, that is, R=Q=B=F=0and d = 0, it can be shown ([20, page 132], [21], [22]) that theorem 4 also holds if the positive semi-definiteness condition is replaced by

$$\begin{bmatrix} P & c \\ c^T & 0 \end{bmatrix} \leq 0 \text{ and has a unique negative eigenvalue and c } \neq 0 \quad (16)$$

In this case the quadratic program is called a Pseudoconvex Nonconvex Quadratic Program ([20, chapter 9]).

If neither the matrix D is positive semi-definite nor the condition (16) holds the quadratic program is no longer equivalent to an SLCP. However, any optimal solution $(\bar{x}, \bar{y})$ of the quadratic program (10) has to satisfy the conditions (14). If this happens, then

$$F(\overline{x},\overline{y}) = \frac{1}{2}(c^T\overline{x} + d^T\overline{y}) + \frac{1}{2}[(c + p\overline{x} + R\overline{y})^T\overline{x} + (d + R^T\overline{x} + Q\overline{y})^T\overline{y}]$$
$$= \frac{1}{2}(c^T\overline{x} + d^T\overline{y} + b^T\overline{\pi} + g^T\overline{\mu})$$

by theorem 2. Hence any nonconvex quadratic program with a finite optimal solution is equivalent to the following MLCP

Minimize { ½ ($c^T x + d^T y + b^T \Pi + g^T \mu$) , ($x, y, \pi, \mu, v, y$) Є K_{mlcp} } (17)

where

K_{mlcp} = {(x, y, π, v, γ) : (x, y, π, μ, v, y) satisfies (14)}

If all the constraints are equalities and all the variables are unrestricted then the nonconvex quadratic program (10) is equivalent to a linear program with the same characteristics. Therefore this quadratic program is equivalent to a system of equations (it is assumed that the quadratic program has a finite optimal solution).

So in almost all cases any quadratic programming problem is equivalent to a linear complementarity problem or to a system of equations. All the different cases are summarized below.

(i) Convex Quadratic Program - is equivalent to

- (a) an LCP if all the constraints are inequalities,
- (b) an SLCP if some constraints are equalities,
- (c) a system of equations if there are no inequalities among the constraints.

(ii) Pseudoconvex Nonconvex Qaudratic Program - is equivalent to

- (a) an LCP if all the constraints are inequalities,
- (b) an SLCP if there are some equality constraints.

(iii) Nonconvex Quadratic Program - if the quadratic program has a finite optimal solution it is equivalent to

- (a) an MLCP if there are some inequalities among the constraints ,
- (b) a system of equations if there are no inequalities.

Obviously the nonnegative restrictions on the variables are considered as inequalities in this summary. Note that a quadratic program has a finite optimal solution if and only if its nonempty constraint set K_{qp} ($K_{qp} \neq \Phi$) is bounded or the function is bounded from below over K_{qp}. ([23, 24]).

4. Bilinear Programming

The Bilinear Programming problem [*25 ,* 26] is another well known problem of mathematical programming which consists of minimizing a bilinear form subject to linear constraints. It can then be stated as

$$\text{Minimize} \quad f(x,y) = \begin{bmatrix} p^1 \\ p^2 \end{bmatrix}^T \begin{bmatrix} x^1 \\ x^2 \end{bmatrix} + \begin{bmatrix} q^1 \\ q^2 \end{bmatrix}^T \begin{bmatrix} y^1 \\ y^2 \end{bmatrix} + \begin{bmatrix} x^1 \\ x^2 \end{bmatrix}^T \begin{matrix} R & S \\ U & V \end{matrix} \begin{bmatrix} y^1 \\ y^2 \end{bmatrix} \tag{18}$$

$$(x1,x2) \in k_{b\ell x} \quad , \quad (y^1, y^2) \in k_{b\ell y} \quad ,$$

where

$$K_{b\ell y} = \{(x^1.x^2) : A_1 x^1 + A_2 x^2 \geq a \ , C_1 + C_2 x^2 = c \ , x^1 \geq 0, -\infty < x^2 < +\infty\} \ ,$$

$$k_{b\ell y} = \{(y^1,y^2) = B_1 y^1 = B_2 y^2 \geq b \ , D_1 y^1 = D_2 y^2 = d \ , y^1 \geq 0 \ , -\infty < y^2 < +\infty \}$$

and $R \in R^{m|xn|}, S \in R^{m|xn|2}, U \in R^{m2xn|}, V \in R^{m2xn2}, ,X^1 \in R^{m1}, X^2 \in R^{m2},$
$Y^1 \in R^{n4}, y^2 \in R^{n2}.$

The problem (18) is a special quadratic program since it can be written in the form

$$\text{Minimize} \quad f(x, y) = \begin{bmatrix} p \\ q \end{bmatrix}^T \begin{bmatrix} x \\ y \end{bmatrix} + \frac{1}{2} \begin{bmatrix} x \\ y \end{bmatrix}^T \begin{bmatrix} \overline{R} & \overline{S} \\ \overline{S}^T & \overline{U} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \ , \tag{19}$$

$$(x, y) \in k_{qb\ell} = \{(x, y) : \overline{A}x + \overline{B}y \geq \overline{a} \ , \overline{C}x + \overline{D}y = \overline{b} \ , x \geq 0 \ , -\infty < y < +\infty\} \quad ,$$

by writing

$$p=\begin{bmatrix}p^1\\q^1\end{bmatrix},\ q=\begin{bmatrix}p^2\\q^2\end{bmatrix},\ x=\begin{bmatrix}x^1\\y^1\end{bmatrix},\ y=\begin{bmatrix}x^2\\y^2\end{bmatrix},\ \overline{R}=\begin{bmatrix}0 & R\\R^T & 0\end{bmatrix},\ \overline{S}=\begin{bmatrix}0 & S\\U^T & 0\end{bmatrix},$$

$$\overline{U}=\begin{bmatrix}0 & V\\V^T & 0\end{bmatrix},\ \overline{A}=\begin{bmatrix}A_1 & 0\\0 & B_1\end{bmatrix},\overline{C}=\begin{bmatrix}C_1 & 0\\0 & D_1\end{bmatrix},\overline{D}=\begin{bmatrix}C_2 & 0\\0 & D_2\end{bmatrix},\overline{B}=\begin{bmatrix}A_2 & 0\\0 & B_2\end{bmatrix},$$

$$a=\begin{bmatrix}a\\b\end{bmatrix},\ \overline{b}\begin{bmatrix}c\\d\end{bmatrix}\quad .$$

Note that (19) is a nonconvex quadratic program, since its matrix is not positive semi-definite [16]. Therefore the bilinear program (18) is equivalent to an MLCP, if $K_{qb\ell}$ is bounded or the function f(x,y) is bounded from below over $K_{qb\ell}$. However it is possible to reduce it directly to a more compact MLCP than that obtained by using the quadratic program (19).

The bilinear program (18) can be written in the form:

$$\text{Minimize}\quad \{h(x) + g(x),\quad x \in K_{b\ell y}\}, \tag{20}$$

where

$$h(x) = (p^1)^T x^1 + (p^2)^T x^2,\ g(x) = m_y\text{in}\ (i(y),\ y\in K_{b\ell y}\},$$

$$i(x)=\left[\begin{bmatrix}q^1\\q^2\end{bmatrix}+\begin{bmatrix}R^T & U^T\\S^T & V^T\end{bmatrix}\cdot\begin{bmatrix}x^1\\x^2\end{bmatrix}\right]^{T\bullet}\begin{bmatrix}y^1\\y^2\end{bmatrix} \tag{21}$$

But the dual of the linear program min {i (y) , y ∈ $k_{b\ell y}$ } is the: linear program max {j (u) , u∈ $K_{b\ell y}$ }, where

$$j(u)=b^Tu^1+d^Tu^2,\quad K_{b\ell u}=\{(u^1,u^2)\ :\ B_1^Tu^1+D_1^Tu^2\le q^1+R^Tx^1+U^Tx^2\ ,$$
$$B_2^Tu^1+D_2^Tu^2=q^2+S^Tx^1+V^Tx^2\ ,\ u\ge 0,\quad -\infty<u^2<+\infty\}$$

Now if the linear program min $\{i(y), y \in K_{b\ell y}, \}$ has a finite optimal solution then

$$g(x) = \min\{i(y), y \in K_{b\ell y}\} = -\min\{-j(u), u \in K_{b\ell y}, y \in K_{b\ell y}, 2^T u^1 = w^T y^1 = 0\}$$

where z and w are the slack, variables corresponding to the inequalities of the primal and the dual problems respectively. Note that the linear program min $\{i(y),\ y \in K_{b\ell y}\}$ has a finite optimal solution if $K_{b\ell y} \neq \theta$ is bounded ([31 , page 17]). Therefore if $K_{b\ell y}$ is bounded the bilinear program (18) is equivalent to the following MLCP

$$\text{Minimize } (p^1)^T x^1 + (p^2) x^T x^2 + b^T u^1 + d^T u^2 ,$$

$$\begin{bmatrix} w \\ 0 \\ z \\ 0 \\ v \\ o \end{bmatrix} = \begin{bmatrix} q^1 \\ q^2 \\ -b \\ -d \\ -a \\ -c \end{bmatrix} + \begin{bmatrix} 0 & 0 & -B_1^T & -D_1^T & R^T & U^T \\ 0 & 0 & -B_2^T & -D_2^T & S^T & V^T \\ B_1 & B_2 & 0 & 0 & 0 & 0 \\ D_1 & D_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & A_1 & A_2 \\ 0 & 0 & 0 & 0 & C_1 & C_2 \end{bmatrix} \begin{bmatrix} y^1 \\ y^2 \\ u^1 \\ u^2 \\ x^1 \\ x^2 \end{bmatrix} \qquad (22)$$

$$x^1 \geq 0,\ -\infty < x^2 < +\infty,\ y^1 \geq 0, -\infty < y^2 < +\infty,\ u^1 \geq 0,\ -\infty < u^2 < \infty,\ w \geq 0,\ z \geq 0,$$

$$v \geq 0,\ w^T y^1 + z^T u^1 = 0 .$$

As before if all the constraints are equalities and all the variables are unrestricted the bilinear program (18) is equivalent to a system of equations. Finally let (m,n) and $(\overline{m}, \overline{n})$ denote the numbers of rows and columns of the MLCPs obtained by using the quadratic program (19), and the direct approach (22) respectively. Then $\overline{m} = m-(m_1.+m_2)$ and $n = n - (r_1 + r_2)$ where r_1. and r_2 are the number of rows of A_1.and C_1. and illustrate the advantage of this formulation..

5, Matrix Games

A Bimatrix Game [8] is a game in which two players with loss matrices A and B ($A \in R^{mxn}$, $B \in R^{mxn}$) try to reduce their losses as much as possible. The Row Player has m pure strategies which are identified with the rows of A and the Column Player has n pure strategies which correspond to the columns of B. If the Row Player uses its i^{th} pure strategy and the Column Player uses its j^{th} pure strategy then their respective losses are given by a_{ij}., and b_{ij}. Using mixed strategies

$$x=(x_1,\ldots,x_m)^T,\ y=(y_1,\ldots,y_n)^T,\ \sum_{i=1}^{m} x_i = 1,\ \sum_{i=1}^{m} y_i = 1,\ x \geq 0,\ \ y \geq 0. \quad (23)$$

their respectively losses are $x^T Ay$ and $x^T By$. Therefore a bimatrix game $\Gamma(A,B)$ seeks a pair of strategies $(\bar{x},\bar{y})$ such that

$$\begin{aligned} &\bar{x}^T A\bar{y} \leq x^T A\bar{y} \quad \text{for all straregies } x \text{ satisfying } (23), \\ &\bar{x}^T B\bar{y} \leq \bar{x}^T By \quad \text{for all straregies } y \text{ satisfying } (23), \end{aligned} \qquad (24)$$

A pair $(\bar{x},\bar{y})$ satisfying (24) is called an Equilibrium Point of the Bimatrix Game $T(A,B)$. The problem of finding an equilibrium point is equivalent to an LCP as stated below.

Theorem 5 $(\bar{x},\bar{y})$ is an equilibrium point of T(A,B) if and only if $(\bar{x},\bar{y},\bar{u},\bar{v})$ is a solution of the LCP

$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} -e^m \\ e^n \end{bmatrix} + \begin{bmatrix} 0 & \overline{A}^T \\ \overline{B} & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}, \begin{bmatrix} u \\ v \end{bmatrix} \geq 0, \begin{bmatrix} x \\ y \end{bmatrix} \geq 0, \begin{bmatrix} u \\ v \end{bmatrix}^T \begin{bmatrix} x \\ y \end{bmatrix} = 0 \qquad (25)$$

where $\overline{A} = A + \theta_0 E,\ \ \overline{B} = B - \theta_0 E,\ \ \theta_0 > \max\ \{|\, a_{ij}\,|, |\, b_{ij}\,|, i = 1,\ldots, m,\ j = 1,\ldots, n\}$, $E = (e_{ij})$ with $e_{ij} = 1$ for all $i = 1,\ldots,m$ and $j = 1,\ldots,n$ and e^m and e^n are vectors of order m and n whose components are all ones.

This theorem is due to Eaves [27] and note that $\overline{A} > 0$ $\overline{B} < 0$ Lemke and Howson [8] previously established a similar result but with $\overline{A} > 0$ and $\overline{B} > 0$ The advantage of Eaves' formulation is discussed later.

In [26] Konno defines a Constrained Bimatrix Game $\Gamma'(A,B)$ which is a bimatrix game in which the choice of the mixed strategies x and y are subject to more general constraints, that is, the sets of mixed strategies are defined as

$$\begin{aligned} k_{bgx} &= \{x \in R^m : Px \geq p,\ Qx = q,\ (e^m)^T x = 1, x \geq 0\} \\ K_{bgy} &= \{\ y \in R^n\!: Ry \geq r,\ Sy = s,\ (e^n)^T\ y = 1,\ y \geq 0\} \end{aligned} \qquad (26)$$

Hence a pair of strategies $(x,y) \in K_{bgx} \times K_{bgy}$ is called an Equilibrium Point of the constrained bimatrix gamer T(A,B) if and only if

$$\bar{x}^T A\bar{y} = \min\ \{x^T A\bar{y},\ \ x \in K_{bgx}\},\ \bar{x}^T B\bar{y} = \min\{\bar{x}^T B\bar{y}\ y \in K_{bgy}\} \qquad (27)$$

Two linear programs are set out in (27). Hence considering their duals and applying theorem 1 the following result is obtained.

Theorem 6 $(\bar{x},\bar{y})$ is an equilibrium point of $T'(A,B)$ if and only if $(\bar{x},\bar{y})$ is a solution of the SLCP

$$\begin{bmatrix} u \\ v \\ z \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ -P \\ 0 \\ -r \end{bmatrix} + \begin{bmatrix} 0 & -P & A & 0 \\ P & 0 & 0 & 0 \\ B^T & 0 & 0 & -R^T \\ 0 & 0 & R & 0 \end{bmatrix} \begin{bmatrix} x \\ \alpha \\ y \\ \beta \end{bmatrix} + \begin{bmatrix} -e^m & 0 & -Q^T & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -e^n & 0 & -S^T \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \alpha_o \\ \beta_o \\ \delta \\ \gamma \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ -q \\ -s \end{bmatrix} + \begin{bmatrix} (e^m)^T & 0^T & 0^T & 0^T \\ 0^T & 0^T & (e^n)^T & 0^T \\ Q & 0 & 0 & 0 \\ 0 & 0 & S & 0 \end{bmatrix} \begin{bmatrix} x \\ \alpha \\ y \\ \beta \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \alpha_o \\ \beta_o \\ \delta \\ \gamma \end{bmatrix} \quad (28)$$

$x \geq 0$, $y \geq 0$, $a \geq 0$, $B \geq 0$, $u \geq 0$, $v \geq 0$, $z \geq 0$, $w \geq 0$, $-\infty < 6 < +\infty$, $-\infty < y < +\infty$,
$x^T u + a^T v + y^T z + ß^T w == 0$, $-{}^{\circ\circ} < a_0 < +\infty 0$, $-\infty < ß_0 < -\infty$.

The Two Person Zero Sum Game is the most celebrated problem in game theory. In this game there are two players but unlike the bimatrix game only one matrix $A \in R^{mxn}$ is considered. The Row Player (RP) has m pure strategies which are the rows of A and the Column Player (CP) n pure strategies identified with the n columns of A. In the matrix, called the Payoff matrix, the element a.. represents the payment made by (CP) to (RP) when (RP) chooses its ith pure strategy and (CP) uses its jth pure strategy. Using the mixed strategies given by (23) (or by (26) in case of a constrained zero—sum game) the expected gain of (RP) is equal to the loss of (CP). Therefore the zero-sum game seeks a pair of strategies $(\bar{x},\bar{y})$ which maximizes the gain of (RP) and minimizes the loss of (CP). Such a pair is called a Saddle-Point and satisfies

$$x^T A\bar{y} \leq x^T A\bar{y} \leq \bar{x}^T Ay \text{ for all mixed strategies}$$
$$(\bar{x},\ \bar{y}),\ (x,y) \text{ satisfying (23) or (26).} \quad (29)$$

The value $\overline{x}^T A \overline{y}$ is called the Value of the Game $\Gamma(A)$. Any zero—sum game can be transformed into a bimatrix game Γ (—A,A) . Since (29) can be rewritten as

$$\overline{x}^T(-A)\overline{y} \leq x^T(-A)\overline{y} \quad , \quad x^T A\overline{y} \leq \overline{x}^T Ay$$

it follows that any saddle—point of $\Gamma(A)$ is an equilibrium point of $\Gamma(-A,A)$. Hence the problem of finding a saddle—point of a zero—sum game (constrained zero—sum game) is equivalent to the LCP (25) (SLCP (28)) where B is replaced by *A,*

Two less known games can also be reduced to linear complementarity problems . These are the Polymatrix Game and the Two Move Game with Perfect Information. The first was introduced by Howson [9] who also proved its equivalence to an LCP. The second was introduced by Dantzig [28] and Konno [26] showed its equivalence toa bilinear program of the form (18) without the unrestricted variables and the equality constraints. Therefore this game is equivalent to an MLCP (if $K_{b\ell y}$ is bounded).

6 Absolute Value Programming

This problem is stated as ([17], page 15)

$$\text{Minimize}\{\sum_{j=1}^{n} c_j\left|x_j\right|, \quad x \in K_{abs}\} \quad ,$$

$$\text{and } K_{abs} = \{x: Ax \geq b, x. \in R^n\} \ . \tag{30}$$

$|x_j|$ is the absolute value of x_j , that is

$$\left|x_j\right| = \begin{cases} x_j & \text{if} \quad x_j \geq 0 \\ -x_j & \text{if} \quad x_j \geq 0 \end{cases} \tag{31}$$

Since x. is an unrestricted variable then it can be written as $x_j = u_j - v_j$ where $u_j \geq 0$ and $v_j \geq 0$. If the complementarity condition $u_j v_j = 0$ is added then $|x_j| = u_j + v_j$ satisfies the relationship of (31). The problem may therefore be stated as an equivalent MLCP

$$\text{Minimize} \quad c^T u + c^T v \tag{32}$$

$$Au - Av \geq b \; u \geq 0, v \geq 0 \text{ and } u^T v = 0$$

7 Zero-one Integer Programming

Zero—one (mixed) integer programming problems are now well established as perhaps the most important mathematical optimization problem. This is mainly because both nonlinear optimization problems and combinatorial optimization problems may be restated as a zero-one (mixed) integer programming problem [29] . A zero-one (mixed) integer programming problem may be stated as

$$\text{Minimize } \{c^T x + d^T u, (x,u) \in K_{01}\} \tag{33}$$

$$\text{where } K_{01} = \{(x,u): Ax + Bu \geq a, u \geq 0, x_j = 0 \text{ or } 1 \text{ for all } j\}$$

If there are no u—variables the problem is known as a zero—one pure integer program. Following Ibaraki [15] since any variable x_j can only take one of the values 0 or 1 it has to satisfy $x_j + y_j = 1$, $x_j \geq 0$, $y_j \geq 0$, $x_j y_j = 0$ and conversely. Therefore any zero-one (mixed) integer program (33) is equivalent to the MLCP

$$\text{Minimize } c^T x + d^T u,$$

$$\text{subject to } \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} e \\ -a \end{bmatrix} + \begin{bmatrix} -I & 0 \\ A & B \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix},$$

$$x \geq 0, y \geq 0, z \geq 0, u \geq 0, x^T y = 0 \tag{34}$$

Finally note that the presence of equality constraints or unrestricted variables does not introduce any relevant modification to the equivalent MLCP.

8 The Fixed Charge Problem

It is perhaps one of the most frequently occurring discrete programming problem and may be defined as

$$\text{Minimize} \left\{ \sum_{j=1}^{n} \varphi_j(x_j), \quad x \in K_{fc} \right\} \tag{35}$$

where $K_{fc} = \{x \in R^n : Ax \geq a, x \geq 0\}$ and

$$\phi_j(x_j) = \begin{cases} 0 & \text{if} \quad x_j = 0 \\ p_j x_j + r_j & \text{if} \quad x_j > 0, \quad r_j \ge 0, \quad p_j \ge 0. \end{cases} \tag{36}$$

The function $\phi_j(x_j)$ is illustrated in Figure 1.

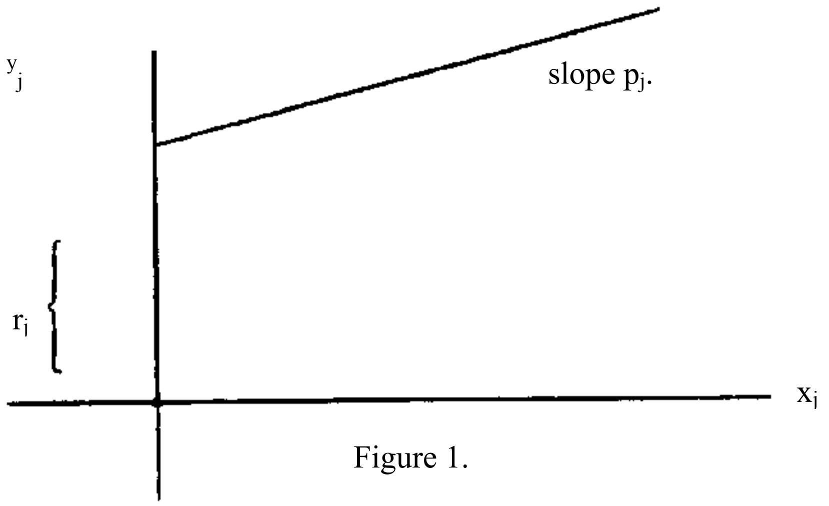

Figure 1.

This problem is usually reformulated as a 0—1 Integer Program [29], and hence from the equivalence shown in the last section it can be reduced to an MLCP. However, it is possible to reformulate it as an MLCP without this intermediate step of a zero-one integer program. The advantage of this direct reduction is that the MLCP obtained is of lower dimension than by the other method.

Consider for any j = 1,2,...,n the set valued function $\psi_j(x_j)$ defined by

$$\psi_j(x_j) = [0,r_j] \text{ if } x. = 0, \; \psi_j(x_j) = p_j x_j + r_j \text{ if } x_j < 0 \tag{37}$$

A theorem connecting. φ_j and ψ_j is stated and proved below.

Theorem 7

$$\text{Minimum}\left\{\sum_{j=1}^{n} \phi_j(x_j), \;\; x \ge 0\right\} \quad = \text{ Minimum} = \left\{\sum_{j=1}^{n} \psi_j(x_j), \;\; x \ge 0\right\}$$

Proof For $x_i > 0$, $\psi_i(x_i) = \phi_i(x_i)$ and for $x_i = 0$, $\phi_i(x_i) = 0$ and $\psi_i(x_i) = [0,r_i]$. Since $0 \le r_i$ the result follows.

Let $y_j = \psi_j(x_j)$, Then

$$y_j,\ x_j \geq 0,\ y_j \leq P_j x_j + r_j,\ [(P_j X_j + r_j) - y_j]\ x_j = 0,\ j = 1, 2, \ldots, n, \quad (38)$$

Introducing the variables $z_j = (p_j x_j + r_j) - y_j$, $j = 1, 2, \ldots, n$ and the vectors $y = (y_1, \ldots y_n)^T$, $z = (z_1, \ldots, z_n)^T$ the fixed-charge problem (35) is equivalent to the MLCP

$$\text{Minimize} \qquad e^T y,$$

$$\begin{bmatrix} z \\ w \end{bmatrix} = \begin{bmatrix} r \\ -a \end{bmatrix} + \begin{bmatrix} P & -I \\ A & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$x \geq 0, \quad y \geq 0, \quad z \geq 0, \quad w \geq 0, \quad z^T x = 0 \ . \qquad (39)$$

In (39) P is a diagonal matrix whose diagonal elements are P_j, $j = 1, 2 \ldots, n$, $r - (r_1 \ldots, r_n)^T$ and w is the vector of slack variables for the inequalities of K_{fc}. Obviously the existence of equality constraints does not introduce any relevant modification to the MLCP.

9 Variable Separable Programming.

Variable Separable Programming is the most popular nonlinear programming extension of the linear programming methodology. A function $f(x_1, \ldots, x_n)$ is said to be Variable Separable if it can be expressed as a sum of n functions of one argument, that is, if $f(x_1, \ldots x_n) = \sum_{j=1}^{n} \phi_j(x_j)$. A Variable Separable Program may be stated in the following form

$$\text{Minimize} \left\{ \sum_{j=1}^{n} \phi_j(x_j), \ x \in K_{vsp} \right\} \qquad (40)$$

where

$$K_{vsp} = \left\{ x \in R^n : \sum_{j=1}^{n} \psi_{\ell j}(x_j) \geq b_\ell, \ \ell = 1, \ldots, m, \ x_j \in [c_j, d_j] \right\}$$

If the properties of ϕ_j and φ_{lj} . are such that (40) is a convex programming problem then the Variable Separable Program can be reduced to a linear program ([17, pages 11—13])- In the nonconvex case either the zero-one mixed integer formulation [29] or the special order set approach [30] may be exploited to investigate the global solution.

In this section it is shown that this problem (40) can also be transformed to an MLCP. As in other approaches discussed above piecewise linear approximations of the functions $\phi_j(x_j)$ and $\psi_{1j}(x_j)$ in the intervals $[c_j,d_j]$ are used.

Let $\psi(x_j)$ be any one of these functions and $[c_j,d_j]$ be the corresponding interval. Consider the decomposition of this interval defined by k discrete points $c_j. = \mu_1 < \mu_2 <...< \mu_{\kappa-1} < \mu_\kappa = d_j$ and the corresponding values $n_i = \psi(\mu_\iota)$, i=1,2,..,k. The function $\psi(x_j)$ can be represented by a piecewise linear approximation defined by connecting the set of n points $A_i(\mu_\iota.,n_i)$ as shown in Figure 2.

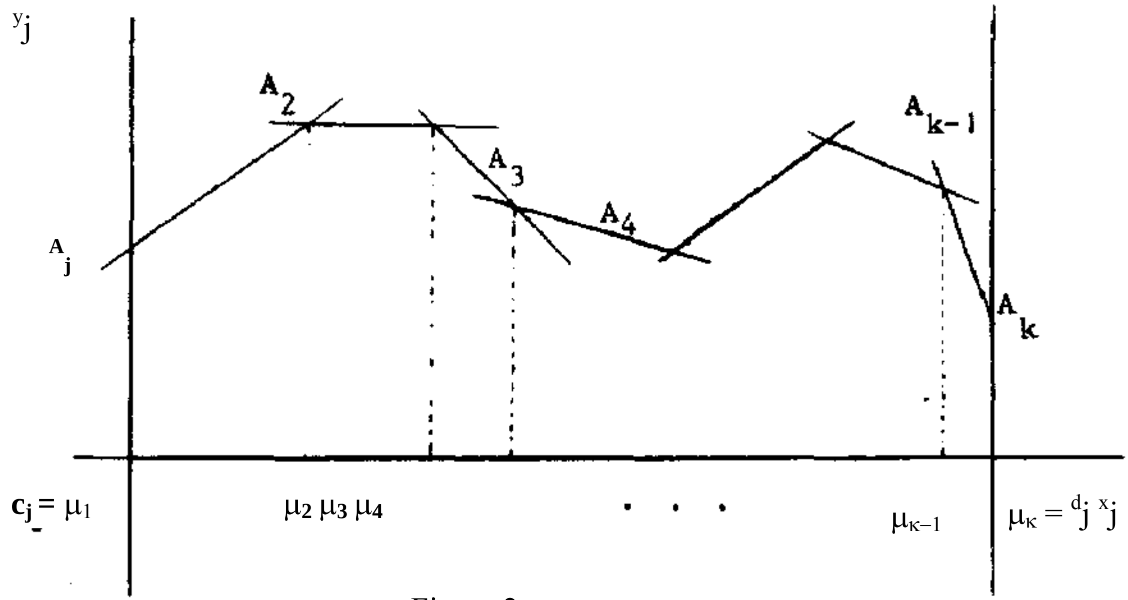

Figure 2

Any $x_j. \in [c_j,d_j]$ can be written as a convex combination of

μ_ι, i = 1 .... ,k as stated below

$$x_j = \sum_{i=1}^{k-1} z_i \ , \ \sum_{i=1}^{k-1} \alpha_i + \sum_{i=2}^{k} \beta_i = 1 \ , \tag{41}$$

$z_i = \alpha_i \mu_i + \beta_{i+1} \mu_{i+1}$, $\alpha_i \geq 0$, $\beta_{i+1} \geq 0$, i=1,....,k-1

Two cases (A) and (B) are considered.

(A) $c_j > 0$ In this case $z_i > 0$ for any $i = 1, \ldots, k-1$, since $\mu_\iota > 0$. Let $y_j = a_i x_j + b_i$ be the equation of the straight line $A_i A_{i+1}$. ., $i = 1, \ldots, k-1$. These latter equations together with those set out in (41) constitute a valid representation for the sequence of line segments $\overline{A_i A_{i+1}}$, $i = 1, \ldots, k-1$, if for any (x_j, y_j) there exists an i such that $x_j \in [\mu_., u_{i+1}]$ and $y_j = a_j x_j + b_i$.

Consider the (k-1) variables $w_i. = y_j - (a_i x_j + b_i)$. The following theorem shows that complementarity conditions may be added to the set of equations stated above in order to obtain a valid representation for the line segments $\overline{A_i A_{i+1}}$.

Theorem 8 - Suppose that the following two conditions hold:

Property 1

$$z_s w_s = 0 \text{ for } s = 1, \ldots, k-1,$$

Property 2

$$\text{if } w_i. = w_t = 0 \text{ and } | i-t | > | \text{ then } z_i = 0 \text{ or } z_t = 0.$$

In this case for any (x_j, y_j) there is an i such $x_j \in [u_i ., u_i - 1]$ and $y_j = a_i x_j. + b_i$

Proof: For any (x_j, y_j) there must exist an i such that $w_i. = 0$. In fact if $w_s \neq 0$ for all s 1, . . . ,k-l then by Property 1 $z_s = 0$ for all $s = 1, \ldots, k-1$, which is impossible since $c_j > 0$. Hence there is an i such that $w_i. = 0$ and $y_j = a_i x_j + b_i$. There are two cases as stated below.

(i) If $z_i. = 0$ then Property 1 and Property 2 imply that $z_s = 0$ for any s such that $| s-i | > 1$. If $x_j = u_i$ or $x_j. = \mu_{\iota+1}$, the theorem is proved. Otherwise $w_{i-1} \neq 0$ and $w_{i+1}. \neq 0$, whence $z_{i-1} = z_{i+1} = 0$ by Property 1. Hence $X_j = z_i. = a_i \beta_{i+1} \mu_{\iota+1}$, $a_i + \beta_{i+1} = 1$, $\alpha_i \geq 0$, $B_{i+1} \geq 0$ Therefore $x_j \varepsilon [\mu_\iota, \mu_{\iota+1}]$ and the theorem is proved.

(ii) If $z_i. = 0$ and since $c_j > 0$ there must exist a t such that $z_t > 0$. Then $w_t = 0$ by property 1 and case (i) is obtained if the index i is replaced by the index t.

By this theorem any occurrence of $\varphi(X_J)$ in a variable separable program can be replaced by y. and the conditions:

$$x_j = \sum_{i=1}^{k-1} z_i, \qquad \sum_{i=1}^{k-1} \alpha_i + \sum_{i=2}^{k} ß_i = 1$$

$$z_j = a_i \mu_i + B_{i+1} \mu_{i+1} , \tag{42}$$

$$w_i. = y_j - (a_i x_j + b_i),$$

$$\alpha_j \geq 0, ß_{i+1} \geq 0, z_i \geq 0, z_i w_i = 0, i. = 1,. .. ,k\text{-}1$$

and
$-\infty < w_i < +\infty$, i = 1 , . . . ,k- 1 , and Property 2, that is, if $w_i = w_t = 0$ and $| i\text{-}t | > 1$ then $z_i = 0$ or $z_t = 0$. (43)

If $\psi (x_j)$ is convex over $[c_j , d_j]$ then the condition

$$w_t \geq 0, t =1 \ldots\ldots\ldots\ldots. k\text{-}1 \tag{44}$$

can replace (43) as is shown below. If (44) holds and $w_i. = 0$ then (by the convexity property of φ) $w_s > 0$ for any s such that $|s—i| > 1$. Hence Property 2 is satisfied by default

A similar argument can be used to show that if $\psi (x_j)$ is concave over $[c_j,d_j]$ then the condition (44) can replace conditions (43) if w_t are redefined as $w_t = a_t x_j + b_t - y_j$, t = 1 , . . ,k— 1 .

(B) $c_j \leq 0$ In this case there exists an e>0 such that $c_j + \varepsilon_j > 0$ and by a change of variable x. = x.+ ε_j case (A) is at hand.

The properties referred above show that any variable separable program can be restarted as an MLCP or an SMLCP with special conditions. These two cases are summarized below.

(i) All the functions $\phi_j(x_j)$ and $\psi_{lj}(x.)$ are either convex or concave over their intervals $[c_j,d_j]$ and therefore the variable separable programming problem is equivalent to an MLCP.

(ii) At least one of the functions $\phi_j(x_j)$ or $\psi_{\ell j}(x_j)$ is neither convex nor concave over its interval - the variable separable programming problem is equivalent to an SMLCP with further restrictions stated as Property 2 for a valid representation of any function $\psi(X_j)$ which is neither convex nor concave over its interval.

In [l6] it is shown that a simple modification of the algorithms for the MLCP makes it possible to find the solution of the variable separable programming problem by exploring the underlying SMLCP.

10 Discussions

The range of problems which may be reformulated as one of a family of the LCP's is found to be quite wide. Some problems can be highly nonlinear as in section 9 or difficult from a combinatorial point of view as in sections 3, 7, 8. The authors are of the opinion that only tree search based methods can be computationally successful for such a wide variety of problems.

A number of tree search methods for solving the general LCP (that is making no assumption concerning the nature of the M-matrix (i)) SLCP, MLCP and SMLCP have been developed and these are reported in [16]. A set of quadratic programming, bilinear programming, zero-one integer programming, fixed charge and nonlinear variable separable programming problems taken from known sources have been reformulated as appropriate LCP's. The results of investigating these problems by the tree search methods and the boundedness property of some of these problems are also presented in [16].

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