Birational geometry and mirror symmetry of Calabi–Yau pairs

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(joint work with Alessio Corti)

A Calabi–Yau (CY) pair (X, D_X) consists of a normal projective variety X and a reduced sum of integral Weil divisors D_X such that $K_X + D_X \sim_{\mathbb{Z}} 0$.

The pair (X, D_X) has (t, dlt) (resp. (t, lc)) singularities if X is terminal and (X, D_X) divisorially log terminal (resp. log canonical).

A birational map $(X, D_X) \overset{\varphi}{\dashrightarrow} (Y, D_Y)$ is called volume preserving if for any geometric valuation E with centre on X and on Y, the equality $a_E(K_X + D_X) = a_E(K_Y + D_Y)$ holds.

One can define an invariant of the volume preserving class of a (t, lc) CY pair as follows. Recall that the dual complex of a dlt pair $(Z, D_Z = \sum D_i)$ is a regular cell complex constructed by attaching an $(|I|-1)$-dimensional cell for every irreducible component of $\bigcap_{i\in I} D_i$ a non-empty intersection of components of D_Z. De Fernex, Kollár and Xu show that the PL homeomorphism class of the dual complex is a volume preserving birational invariant of a dlt pair [**?**]. By [**?**], a (t, lc) CY pair (X, D_X) has a volume preserving (t, dlt) modification $(\widetilde{X}, D_{\widetilde{X}}) \to (X, D_X)$. Abusing notation, I call dual complex of (X, D_X) the PL homeomorphism class of the dual complex of a volume preserving (t, dlt) modification. The dual complex is denoted $\mathcal{D}(X, D_X)$; it is an invariant of the volume preserving birational equivalence class of (X, D_X).

The underlying varieties of CY pairs range from Calabi–Yau to Fano varieties, but X being Fano is not a volume preserving birational invariant of (X, D_X). A CY pair (X, D_X) has *maximal intersection* if $\dim \mathcal{D}(X, D_X) = \dim X - 1$. In other words, (X, D_X) has maximal intersection if a volume preserving (t, dlt) modification has a 0-dimensional log canonical centre. Since the dual complex is a volume preserving birational invariant, so is the property of having maximal intersection. Pairs with maximal intersection have some *Fano-type* properties, in a sense made precise by the following result.

Theorem 1. [**?**] *Let (X, D_X) be a CY pair with maximal intersection. Then, there is a volume preserving birational map $(X, D_X) \dashrightarrow (Z, D_Z)$ to a CY pair whose boundary fully supports a big and semiample divisor.*

Note that having maximal intersection is a "degenerate" condition; a CY pair (X, D_X) whose underlying variety is Fano does not have maximal intersection in general.

Example 2. A toric pair (X_Σ, D_Σ) is a CY pair formed by a toric variety and the reduced sum of its toric invariant divisors. A volume preserving birational map to a toric pair is called a toric model. Any (t, lc) CY pair with a toric model has maximal intersection.

Example 3. In dimension 2, this is an equivalence: CY pairs with maximal intersection are precisely those with a toric model.

The existence of a toric model for a pair is difficult to determine. The results of [?] state criteria that characterise toric pairs, but it is not clear whether or how such criteria could be extended to characterise CY pairs with a toric model.

A motivation to understand better the birational geometry of CY pairs and their relation to toric pairs comes from mirror symmetry. Most known constructions of mirror partners make use of toric features of the varieties or pairs considered, such as the existence of a toric model. In an exciting development, Gross, Hacking and Keel propose a construction of the mirror partner of CY pairs with maximal intersection; they conjecture:

Conjecture 4. [?] *Let (Y, D_Y) be a simple normal crossings CY pair with maximal boundary such that D_Y supports an ample divisor (in particular $U = Y \setminus D_Y$ is affine). Let $R = k[\mathrm{Pic}(Y)^\times]$, Ω the canonical volume form on U and*

$$U^{\mathrm{trop}}(\mathbb{Z}) = \{ \text{ divisorial valuations: } k(U) \setminus \{0\} \to \mathbb{Z} | v(\Omega) < 0\} \cup \{0\}.$$

Then, denoting by V the free R-module with basis $U^{\mathrm{trop}}(\mathbb{Z})$, V has a natural finitely generated R-algebra structure whose structure constants are non-negative integers determined by counts of rational curves on U. The associated fibration $p\colon \mathrm{Spec(V)} \to \mathrm{Spec(R)} = \mathrm{T_{Pic(Y)}}$ is a flat family of affine log CY varieties with maximal boundary. Letting $K = \mathrm{Ker}\{\mathrm{PicY} \to \mathrm{Pic(U)}\}$, the map p is T_K-equivariant. The quotient family $\mathrm{Spec(V)/T_K} \to \mathrm{T_{Pic(U)}}$ depends only on U and is the mirror family to U.

Versions of this conjecture are proved for cluster varieties in [?], but relatively few examples are known.

In this talk, I present some examples of (t, lc) CY pairs with maximal intersection which do not have a toric model because their underlying varieties are birationally rigid.

One can construct volume preserving (t, dlt) modifications of these pairs that have relatively mild singularities and for which one expects to be able to compute the punctured Gromov-Witten invariants appearing in the conjecture. There is no known construction of mirror partners for these pairs, and they would be natural examples on which to test and study the conjecture.

Example 5. Consider the pair (X, D_X) where:

$$X = \{x_0^4 + x_1^4 + x_2^4 + x_0x_1x_2x_3 + x_4(x_3^3 + x_4^3) = 0\} \text{ and } D_X = X \cap \{x_4 = 0\}.$$

The quartic X is rigid because it is nonsingular. The unique singular point $p = (0{:}0{:}0{:}1{:}0)\}$ of D_X is locally analytically equivalent to a $T_{4,4,4}$ singular point $0 \in \{x^4 + y^4 + z^4 + xyz = 0\}$. A volume preserving (t, dlt) modification of (X, D_X) is obtained by taking a resolution of the cusp singularity of D_X.

Example 6. (Example due to R. Svaldi) Consider the smooth cubic 3-fold

$$X = \{x_0x_1x_2 + x_1^3 + x_2^3 + x_3q + x_4q' = 0\},$$

where q, q' are general conics in $x_0, \cdots, x_4$ with $(q(1,0,0,0,0), q'(1,0,0,0,0)) \neq (0,0)$. Let $\Pi = \{x_3 = x_4 = 0\}$ and $D = \{x_3 = 0\} + \{x_4 = 0\}$. The section $\Pi \cap X$

is a cubic with a node at $p = (1{:}0{:}0{:}0{:}0)$; $p \in D_X$ is locally analytically equivalent to $0 \in \{x^2y^2 - z^2 = 0\}$. A volume preserving (t, dlt) modification of (X, D_X) can be constructed, showing that (X, D_X) has maximal intersection.

Example 7. Consider the pair (X, D_X) where:

$$X = \{x_1^2x_2^2 + x_1x_2x_3l + x_3^2q + x_4f_3 = 0\}, D_X = X \cap \{x_4 = 0\},$$

where l is a general linear form and q a general conic in $x_0, \cdots, x_3$ and f_3 is a general cubic in $x_0, \cdots, x_4$. The surface D_X is non normal as it has multiplicity 2 along $L_1 = \{x_1 = x_3 = x = 4 = 0\}$ and $L_2 = \{x_2 = x_3 = x = 4 = 0\}$; the point $p = L_1 \cap L_2$ is locally analytically equivalent to $0 \in \{x^2y^2 - z^2 = 0\}$. The quartic X has three ordinary double points lying on $L_1 \cap \{f_3 = 0\}$ and three ordinary double points lying on $L_2 \cap \{f_3 = 0\}$. As X has less than 9 ordinary double points, X is birationally rigid. A volume preserving (t, dlt) modification of (X, D_X) can be constructed, showing that (X, D_X) has maximal intersection.

References

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