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Bézier Polynomials over Triangles and the Construction of Piecewise C^r Polynomials

By

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Introduction

Bézier polynomials and their generalization to tensor-product surfaces provide a useful tool in surface design (Bézier 1970, 1977; Forrest 1972). They were developed as early as 1959 by de Casteljau at Citroën but owe their name to P. Bézier from Renault who was first to employ them in car body design in the late sixties.

De Casteljau 1959 also describes triangular patches, but these scarcely received any attention until Sabin 1977. Farin 1979 generalizes and extends results obtained by de Casteljau and Sabin, sharing their restrictions to domains that consist of congruent triangles only.

The present paper restates some of the results of Farin 1979, including a short outline of the univariate case, and then generalizes them to Bezier polynomials defined over arbitrary triangles; formulas describing C^r continuity of adjacent triangular patches are provided.

The last two sections give applications of the theory: the C^1 Clough-Tocher scheme is generalized to the C^2 case and a formula for the dimension of the linear space of piecewise C^r polynomials (of degree n) is derived.

I Univariate Bézier Polynomials

1. Definition

A Bézier polynomial $B_n\phi$ is defined by

$$(1)\quad [B_n\phi](t) = \sum_{i=0}^{n} b_i B_i^n(t)$$

where the B_i^n are Bernstein polynomials

$$(2)\quad B_i^n(t) = \binom{n}{i} t_i (1-t)n-i \;;\; o \le i \le n$$

and ϕ is the piecewise linear function joining the points $(\frac{i}{n}, b_i); 0 \le i \le n$. ϕ is called the Bézier polygon associated with $B_n\phi_i^{*)}$; the b_i are called Bézier ordinates of $B_n\phi$.

Since the B_i^n satisfy

$$(3)\quad B_i^n(t) \le 0; 0 \le i \le n ; 0 \le t \le 1 ,$$

$$(4)\quad \sum_{i=0}^{n} B_i^n(t) = 1 ,$$

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*) In classical approximation theory, $B_n\phi$ is called the "Bernstein approximant" to ϕ (Davis 1975) the graph of $B_n\phi$, $0 \le t \le 1$, lies in the convex hull of the graph of ϕ. (Bézier 1970, Bézier 1976, Forrest 1970).

2. Degree Elevation

Every polynomial of degree n can be written as a polynomial of degree n+1 ; let $E\phi$ be a polygon joining points $\left(\frac{i}{n+1}, b_i^*\right)$, $0 \le i \le n+1$. If

(5) $$b_i^* = \phi\left(\frac{i}{n+1}\right) = \frac{i}{n+1} b_{i-1} + \left(1 - \frac{i}{n+1}\right) b_i \ , \ 0 \le i \le n+1 \ ,$$

it is easy to show that

(6) $$B_n\phi = B_{n+1} E\phi$$

3. Derivatives

For the r-th derivative of $B_n\phi$ we find

(7) $$\frac{d^r}{dt^r}[B_n\phi]\ (t) = \frac{n!}{(n-r)!} \sum_{i=0}^{n-r} \Delta^r b_i B_i^{n-r}(t)$$

This yields immediately

(8a) $$\frac{d^r}{dt^r}[B_n\phi]\ (0) = \frac{n!}{(n-r)!} \Delta^r b_o$$

(8b) $$\frac{d^r}{dt^r}[B_n\phi]\ (1) = \frac{n!}{(n-r)!} \Delta^r b_{n-r} \ ,$$

i.e. the r-th derivative at an endpoint depends only on the (r+1) adjacent Bézier ordinates.

We also note that for $\zeta \varepsilon [a,b]$ instead of $t \varepsilon [0,1]$, (7) becomes

$$(9) \quad \frac{d^r}{d\zeta^r} [B_n \phi](\zeta) = \frac{n!}{(n-r)!} \cdot \frac{1}{(b-a)^r} \sum_{i=0}^{n-r} \Delta^r b_i B_i^{n-r}(\zeta)$$

4. Recursive Algorithm (de Casteljau 1959)

The B_i^n satisfy a recurrence relation

$$(10) \quad B_i^n(t) = (1-t)\, B_i^{n-1}(t) + t\, B_{i-1}^{n-1}(t)$$

(with $B_i^n(t) = 0$ for $i < 0$ or $i > n$).

(10) allows to expand $B_n\phi$ in terms of Bernstein polynomials of lower degree :

$$(11) \quad [B_n\phi](t) = \sum_{i=0}^{n-r} b_i^r(t) B_i^{n-r}(t) \quad , \quad 0 \le r \le n,$$

where the $b_i^r(t)$ are defined by

$$(12) \quad b_i^r(t) = (1-t)\, b_i^{r-1} + t\, b_{i+1}^{r-1}$$

$$b_i^o(t) = b_i .$$

Since

$$(13) \quad [B_n\phi](t) = b_0^n(t) ,$$

(12) provides an easy and stable algorithm for the numerical evaluation of $[B_n\phi](t)$. This is illustarted in fig. 1

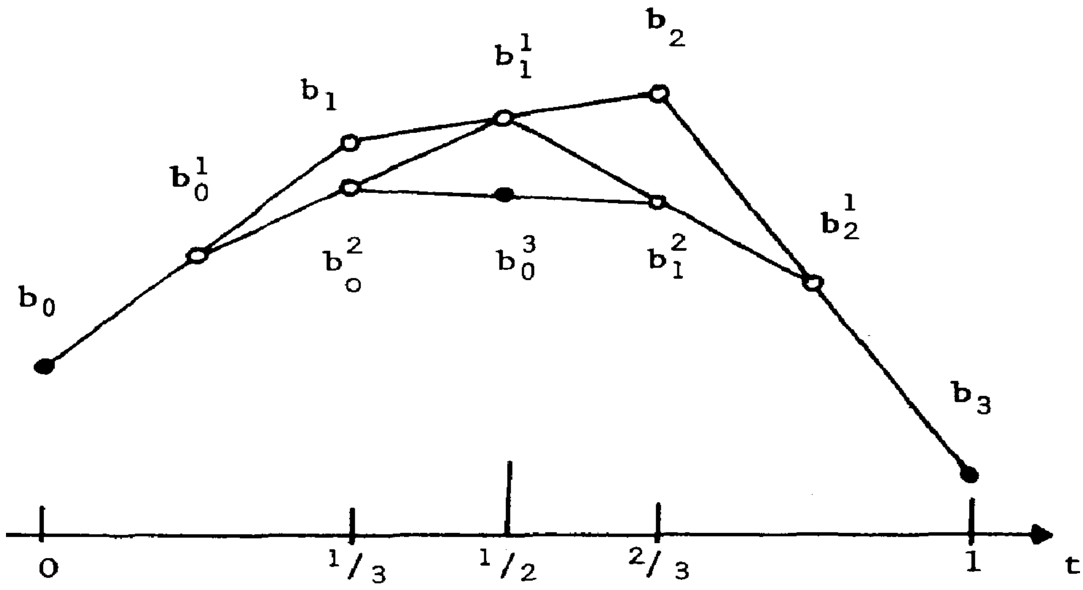

Fig. 1 construction of $[B_3\phi](\frac{1}{2})$

One can show that

$$(14)\quad b_i^r(t) = \sum_{j=0}^{r} b_{i+j} B_j^r(t) \; ; \quad \begin{matrix} 0\le r\le n \\ 0\le i\le n-r \end{matrix}$$

The $b_i^r(t)$ can also be used to determine the r-th derivative of $B_n\phi$;

$$(15)\quad \frac{d^r}{dt^r} [B_n\phi](t) = \frac{n!}{(n-r)!} \sum_{i=0}^{n-r} b_i^{n-r}(t)\, B_i^r(t)$$

For r=1, (15) states that $b_0^{n-1}(t)$ and $b_1^{n-1}(t)$ determine the tangent to $[B_n\phi](t)$ and, for r=2, that $b_0^{n-2}(t)$, $b_1^{n-2}(t)$, $b_2^{n-2}(t)$ determine the osculating parabola.

5. C^r –Continuity

Suppose we are given a Bézier polygon ϕ with Bézier ordinates b_i over $t\varepsilon[0,1]$. We seek a Bézier polygon ψ with Bézier ordinates C_i over $\zeta\varepsilon[1,2]$ such that the two polynomials defined by ϕ and ψ form a function in $C^r[0,2]$. From (8a) and (8b) we get the conditions

(16) $\Delta^{\rho} b^{n-\rho} = \Delta^{\rho} c_o \ ; \ 0 \leq \rho \leq r.$

(16) implies that, for fixed ρ, the Bézier polynomials defined by $b^{n-\rho}, b^{n-\rho+1}, \ldots, b_n$ and $c_o, c_1, \ldots, c_\rho$ coincide since all their derivatives coincide at t=1 resp. ς=0.

Hence

$$\sum_{i=o}^{\rho} b_{n-\rho+i} B_i^{\rho}(t) = \sum_{i=o}^{\rho} c_i B_i^{\rho} \ (1) .$$

since $\varsigma = t - 1$. This is true for all t, i.e. also for t=2:

$$\sum_{i=o}^{\rho} b_{n-\rho+i} B_i^{\rho}(2) = \sum_{i=o}^{\rho} c_i B_i^{\rho} \ (1) .$$

The right-hand side equals c_ρ, and we get

(17) $c_\rho = \sum_{i=o}^{\rho} b_{n-\rho+i} B_i^{\rho} \ (2) \quad ; \quad 0 \leq \rho \leq r.$

Note that this is equivalent to:

(18) $c_\rho = b_{n-\rho}^{\rho} \ (2) \quad ; \quad 0 \leq \rho \leq r.$

We can define the second Bézier polynomial over $[1,\beta]$ instead of $[1,2]$; in this case, (18) becomes

(19) $c_\rho = b^{\rho}_{n-\rho}(\beta)$.

Thus we have a condition for C^r–continuity given by (16) and a construction given by (17), Note that fig. 1 can be interpreted as the construction of the b^n_o , b^{n-1}_1 , ..., b^o_n from the b^o_o , b^1_o , ..., b^n_o to obtain C^n– continuity. We also note that the two corresponding Bézier polynomials coincide with the original polynomial given by b^o_o , b^o_1 , ..., $b^o{}_n$.

II Bezier Polynomials over a Triangle

1. Definition

We consider a triangle T in the plane with vertices P_1, P_2, P_3 and edges e_1, e_2, e_3 in which we assume barycentric coordinates defined such that for each point P in the plane

$$P = uP_1 + vP_2 + wP_3$$

where

$$0 \leq u, v, w \leq 1 \text{ for all } P \varepsilon T,$$

$$u + v + w = 1,$$

and

$$u = \frac{[P_3\, P\, P_2]}{[P_1\, P_2\, P_3]}, \quad v = \frac{[P_1\, P\, P_3]}{[P_1\, P_2\, P_3]}, \quad w = \frac{[P_1\, P\, P_2]}{[P_1\, P_2\, P_3]}.$$

Here, $[P_3\, P\, P_2]$ denotes the area of the triangle P_1, P, P_2 etc.

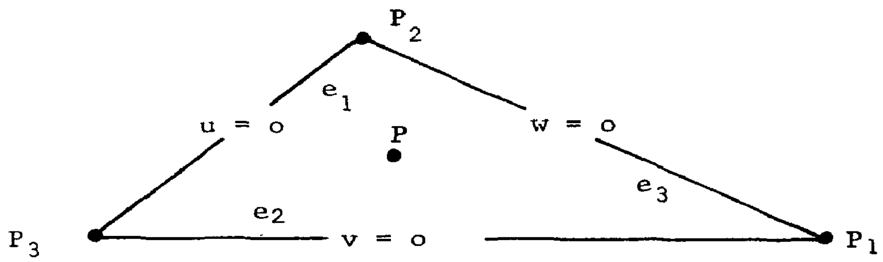

Fig. 2: Triangle T with barycentric coordinates.

We define Bernstein polynomials $B^n_{\underset{\sim}{i}}(\underset{\sim}{u})$ over T:

$$(20) \qquad B^n_{\underset{\sim}{i}}(\underset{\sim}{u}) = \frac{n!}{i!\,j!\,k!} u^i v^j w^k \; ; \; \begin{matrix} u + v + w = 1 \\ i + j + k = n \end{matrix} \qquad \begin{matrix} \underset{\sim}{u} = (u, v, w) \\ \underset{\sim}{i} = (i, j, k) \end{matrix}$$

Since the $B^n_{\underset{\sim}{i}}(\underset{\sim}{u})$ are terms of

$$(21) \qquad (u + v + w)^n = \sum_{\substack{i+j+k=n \\ i,j,k>0}} \frac{n!}{i!j!k!} u^i v^j w^k ,$$

we have immediately

$$(22) \qquad B^n_{\underset{\sim}{i}}(\underset{\sim}{u}) \leq 0 \quad \text{for } 0 \leq u, v, w \leq 1,$$

$$(23) \qquad \sum_{\underset{\sim}{i}}^{n} B_{\underset{\sim}{i}}^{n}(\underset{\sim}{u}) \equiv 1.$$

The summation $\sum_{\underset{\sim}{i}}^{n}$ in (23) is short for the one used in (21).

The $\frac{1}{2}(n+1)(n+2)$ polynomials $B_{\underset{\sim}{i}}^{n}$ form a basis for the linear space of all bivariate polynomials of degree n.

A <u>Bézier polynomial over T</u> is defined by

$$(24) \quad [B_n\phi](\underset{\sim}{u}) = \sum_{\underset{\sim}{i}}^{n} b_{\underset{\sim}{i}} \; B_{\underset{\sim}{i}}^{n}(\underset{\sim}{u}) \;,$$

where ϕ is the piecewise linear function determined by the points $(\frac{i}{n}, \frac{j}{n}, \frac{k}{n}, b_{\underset{\sim}{i}})$. The $b_{\underset{\sim}{i}}$ are called <u>Bézier ordinates</u> of $B_n\phi$; ϕ is called the <u>Bézier net</u> of $B_n\phi$.*) (22) and (23) imply that the graph of $B_n\phi$; lies in the convex hull of the graph of ϕ. The structure of ϕ is illustrated in fig. 3.

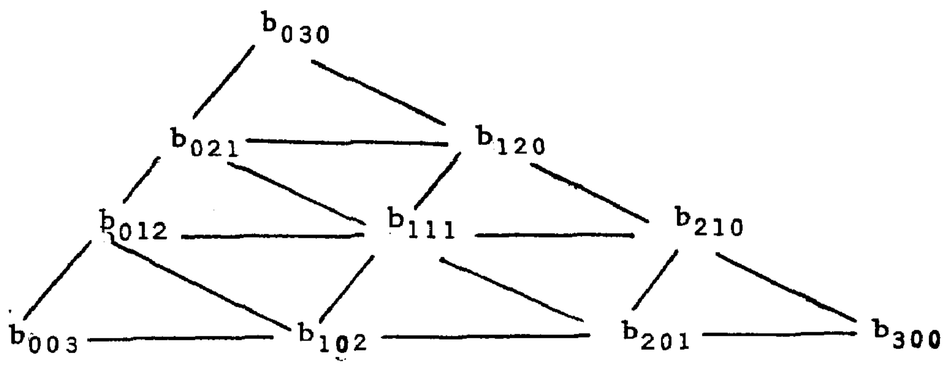

Fig. 3: Structure of ϕ for n = 3

We also note that the boundary curves of $B_n\phi$ are the (univariate) Bézier polynomials determined by the boundary points of ϕ.

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*) This notation is chosen to be like the one in the univariate case to point out the similarity of both methods. No confusion should arise, however, since the meaning of $B_n\phi$, ϕ, etc. will be clear from the context.

2. Degree elevation

Every bivariate polynomial of degree n can be written as a polynomial of degree n + 1; let $E\phi$ be a net determined by points $(\frac{i}{n+1}, \frac{j}{n+1}, \frac{k}{n+1}, b^*_{\underset{\sim}{i}})$, $i+j+k = n+1$. If

$$(25)\quad b^*_{\underset{\sim}{i}} = (\frac{1}{n-1}\cdot i) = \frac{i}{n+1} b_{i+1,j,k} + \frac{j}{n+1} b_{i,j+1,k} + \frac{k}{n+1} b_{i,j,k+1} \; ; \; i+j+k = n+1,$$

it is easy to show that

$$(26)\quad B_n\phi = B_{n+1}E\phi.$$

This is illustrated in fig. 4 and the following example.

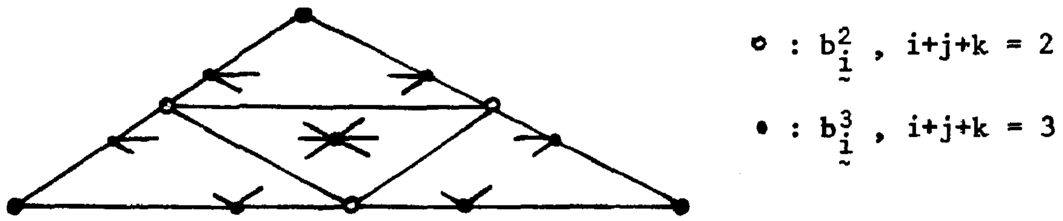

fig. 4: Elevation of degree from 2 to 3.

Example 1: The following two Bézier nets determine the same polynomial:

ϕ:

		0		
	3		0	
0		0		0

$E\phi$:

			0			
		2		0		
	2		1		0	
0		0		0		0

3. Derivatives

Let $\underset{\sim}{u} = \underset{\sim}{u}(s)$ be the equation of a straight line in terms of the barycentric coordinates of T, e.g. $\underset{\sim}{u}(s) = (1-s)\underset{\sim}{u}_0 + s\underset{\sim}{u}_1$ with two points $\underset{\sim}{u}_0, \underset{\sim}{u}_1$.

Hence

(27) $\frac{d^r}{ds^r}\underset{\sim}{u}(s) = \underset{\sim}{0}$ for $r > 1$

(here, $\underset{\sim}{0} = (0,0,0)$).

For $r = 1$, we set $\dot{\underset{\sim}{u}} = \frac{d}{ds}\underset{\sim}{u}(s)$

Since $u + v + w = 1$ we have $\dot{u} + \dot{v} + \dot{w} = 0$.

The term $\dot{\underset{\sim}{u}}$ defines a direction with respect to which we can take directional derivatives. We set $D^r_{\dot{\underset{\sim}{u}}} := \frac{d^r}{ds^r}$.

For the Bernstein polynomials $B^n_{\underset{\sim}{i}}$ we get

<u>Theorem 1</u>: Set $\underset{\sim}{\lambda} = (\lambda,\mu,\nu)$. Then

(28) $D^r_{\dot{\underset{\sim}{u}}}\ B^n_{\underset{\sim}{i}}(\underset{\sim}{u}) = \frac{n!}{(n-r)!}\sum^r_{\underset{\sim}{\lambda}} B^r_{\underset{\sim}{\lambda}}(\dot{\underset{\sim}{u}})\ B^{n-r}_{\underset{\sim}{i}-\underset{\sim}{\lambda}}(\underset{\sim}{u})$

<u>Remarks</u>: (a) The term $B^r_{\underset{\sim}{\lambda}}(\dot{\underset{\sim}{u}})$ is well-defined even if the sum of the arguments does not equal 1. (b) For $\underset{\sim}{i}, \underset{\sim}{\lambda}$ that do not satisfy $\underset{\sim}{i} - \underset{\sim}{\lambda} \geq \underset{\sim}{0}$ (componentwise), we set $B^{n-r}_{\underset{\sim}{i}-\underset{\sim}{\lambda}}(\underset{\sim}{u}) = 0$.

<u>Proof</u>

i) For triple products of functions of one variable s the Leibniz formula

$$\frac{d^r}{ds^r}\ f(s) . g(s) . h(s) = \sum^r_{\underset{\sim}{\lambda}} \frac{r!}{\lambda!\mu!\nu!}\ f^{(\lambda)}(s) . g^{(\mu)}(s) . h^{(\nu)}(s)$$

is true.

ii) Because of the linearity of u(s), v(s), w(s) repeated applications of the chain and product rules yield:

$$\frac{d^\lambda}{ds^\lambda} f[u(s)] = f^{(\lambda)}(u) \cdot \dot{u}^\lambda \qquad \text{etc.}$$

iii) Setting $f(u(s)) = [u(s)]^i$ etc., we obtain

$$D^r_{\dot{\underset{\sim}{u}}} \; B^n_{\underset{\sim}{i}}(\underset{\sim}{u}) = \frac{d^r}{ds^r} \frac{u!}{i!j!k!} [u(s)]^i \, [v(s)]^j \, [w(s)]^k$$

$$= \sum_\lambda^r \frac{r!}{\lambda!\mu!\nu!} \dot{u}^\lambda . \dot{v}^\lambda . \dot{w}^\lambda \; . \; \frac{n!}{(i-\lambda)!(j-\mu)!(k-\nu)!} u^{i-\lambda} v^{j-\mu} w^{k-\nu}$$

This implies

<u>Theorem 2</u>: The r-th directional derivative wrt $\dot{\underset{\sim}{u}}$ of a Bézier polynomial $B_n\phi$ over T is given by

$$(29) \quad D^r_{\dot{u}} \; [B_n\phi](\underset{\sim}{u}) = \frac{n!}{(n-r)!} \sum_i^{n-r} \sum_\lambda^r b_{i+\lambda} B^r_\lambda(\dot{\underset{\sim}{u}}) \; B^{n-r}_i(\underset{\sim}{u})$$

We note that this can be rearranged to

$$(30) \quad D^r_{\dot{u}} \; [B_n\phi](\underset{\sim}{u}) = \frac{n!}{(n-r)!} \sum_\lambda^r B^r_\lambda(\dot{\underset{\sim}{u}}) \sum_i^{n-r} b_{i+\lambda} B^{n-r}_i(\underset{\sim}{u}).$$

4. <u>Recursive Algorithm (de Casteljau 1959)</u>

Let us define $\underset{\sim}{a}_1 = (1,0,0)$, $\underset{\sim}{a}_2 = (0,1,0)$, $\underset{\sim}{a}_3 = (0,0,1)$.

<u>Lemma 3</u>: The B^n_i satisfy a recurrence relation

$$(31) \quad B^n_{\underset{\sim}{i}}(\underset{\sim}{u}) = u \cdot B^{n-1}_{\underset{\sim}{i}-\underset{\sim}{a}_1}(\underset{\sim}{u}) + v. \, B^{n-1}_{\underset{\sim}{i}-\underset{\sim}{a}_2}(\underset{\sim}{u}) + w. B^{n-1}_{\underset{\sim}{i}-\underset{\sim}{a}_3}(\underset{\sim}{u}), \quad i+j+k = n$$

Proof: Use the identity

$$\frac{n!}{i!k!k!} = \binom{n}{i} \binom{n-i}{j}$$

and the recursion formula for binomial coefficients.

This lemma allows to expand $B_n\phi$ in terms of Bernstein polynomials of lower degree, with polynomial coefficients $b^r_{\underset{\sim}{i}}(\underset{\sim}{u})$:

<u>Theorem 4</u>:

(32) $$[B_n\phi](\underset{\sim}{u}) = \sum_{i}^{n-r} b^r_{i}(\underset{\sim}{u})\, B^{n-r}_{i}(\underset{\sim}{u}) \quad , \quad 0 \le r \le n$$

where the $b^r_{\underset{\sim}{i}}(\underset{\sim}{u})$ are defined by

(33) $$\begin{cases} b^r_{\underset{\sim}{i}}(\underset{\sim}{u}) = u \cdot b^{r-1}_{\underset{\sim}{i}+\underset{\sim}{a}_1}(\underset{\sim}{u}) + v . b^{r-1}_{\underset{\sim}{i}+\underset{\sim}{a}_2}(\underset{\sim}{u}) + w \cdot b^{r-1}_{\underset{\sim}{i}+\underset{\sim}{a}_3}(\underset{\sim}{u}) \; ; \; i+j+k = n-r \\ b^0_{\underset{\sim}{i}}(\underset{\sim}{u}) = b_{\underset{\sim}{i}} \end{cases}$$

<u>Proof</u> is by induction on r. (32) is true for r = 0.

Induction:

$$[B_n\phi](\underset{\sim}{u}) = \sum_{i}^{n-r} b^r_{i}(\underset{\sim}{u})\, B^{n-r}_{i}(\underset{\sim}{u})$$

$$\overset{(31)}{=} \sum_{\underset{\sim}{i}}^{n-r} b^r_{\underset{\sim}{i}}(\underset{\sim}{u})\, [u . B^{n-r-1}_{\underset{\sim}{i}-\underset{\sim}{a}_1}(\underset{\sim}{u}) + v . B^{n-r-1}_{\underset{\sim}{i}-\underset{\sim}{a}_2}(\underset{\sim}{u}) + w\, B^{n-r-1}_{\underset{\sim}{i}-\underset{\sim}{a}_3}(\underset{\sim}{u})]$$

$$= \sum_{i}^{n-r-1} [u \cdot b^r_{i-a^1}(\underset{\sim}{u}) + v \cdot b^r_{i-a^2}(\underset{\sim}{u}) + w \cdot b^r_{i-a^3}(\underset{\sim}{u})]\, B^{n-r-1}_{i}(\underset{\sim}{u})$$

$$\overset{(33)}{=} \sum_{i}^{n-r-1} b^{r+1}_{i}(\underset{\sim}{u})\; B^{n-r-1}_{i}(\underset{\sim}{u})$$

Since

$$[B_n\phi](\underset{\sim}{u}) = b^n_{\underset{\sim}{0}}(\underset{\sim}{u}),$$

(33) provides an algorithm for the evaluation of $[B_n\phi](\underset{\sim}{u})$. This is illustrated in figure 5.

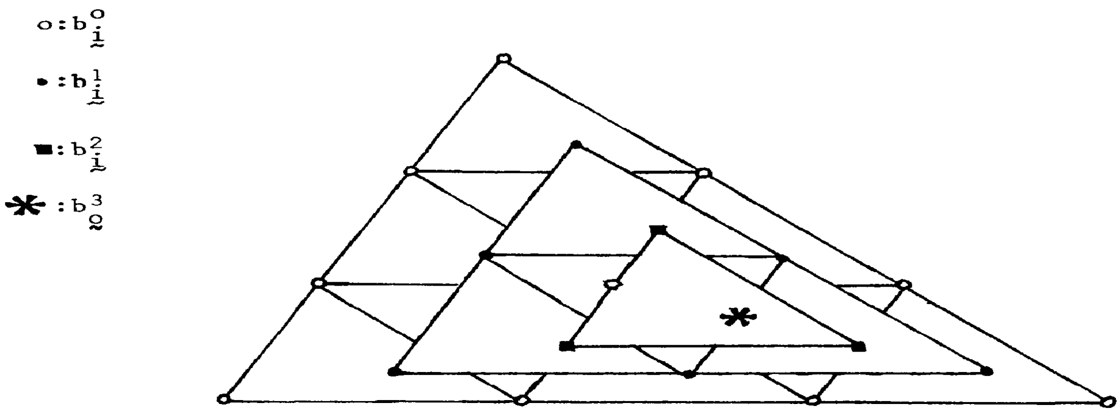

Figure 5: Construction for $[B_3\phi](\frac{1}{2},\frac{1}{4},\frac{1}{4})$

The $b^r_{\underset{\sim}{i}}(\underset{\sim}{u})$ have an explicit form similar to (14):

$$(34) \qquad b^r_{\underset{\sim}{i}}(\underset{\sim}{u}) = \sum_{\underset{\sim}{\lambda}}^{r} b_{\underset{\sim}{i}-\underset{\sim}{\lambda}} B^r_{\underset{\sim}{\lambda}}(\underset{\sim}{u}) \; ; \; i+j+k = n-r.$$

To prove this one checks that (34) is consistent with the recursive definition (33) of the $b^r_{\underset{\sim}{i}}(\underset{\sim}{u})$.

With (34), we can simplify (30) to

$$(35) \qquad D^r_{\underset{\sim}{u}}[B_n\phi](\underset{\sim}{u}) = \frac{n!}{(n-r)!}\sum_{\underset{\sim}{\lambda}}^{r} b^{n-r}_{\underset{\sim}{\lambda}}(\underset{\sim}{u})\, B^r_{\underset{\sim}{\lambda}}(\dot{\underset{\sim}{u}}) \; .$$

Hence to take the rth (directional) derivative of B_n, we first perform n-r steps of the evaluation algorithm (33) to obtain the $b^{n-r}_{\underset{\sim}{\lambda}}(\underset{\sim}{u})$ and then evaluate the Bezier polynomial (35) using the same algorithm, but now with weights $\dot{u}$, $\dot{v}$, $\dot{w}$ instead of u, v, w. For r = 1,· (35) means that the $b^{n-1}_{\underset{\sim}{i}}(\underset{\sim}{u})$ determine the tangent plane to $[B_n\phi](\underset{\sim}{u})$- for r = 2 we see that the osculating paraboloid is determined by the $b^{n-2}_{\underset{\sim}{i}}(\underset{\sim}{u})$.

Another possibility to compute $D^r_{\underset{\sim}{u}}B_n$ is given by

$$(36) \qquad D^r_{\underset{\sim}{u}}[B_n\phi](\underset{\sim}{u}) = \frac{n!}{(n-r)!}\sum_{\underset{\sim}{i}}^{n-r} b^r_{\underset{\sim}{i}}(\dot{\underset{\sim}{u}})\, B^{n-r}_{\underset{\sim}{i}}(\underset{\sim}{u}) \; ,$$

which is proved from (29) (Farin 1979). We have thus a second method to compute the rth derivative of $B_n\phi$: first, perform r steps of

algorithm (33) with weights $\dot{u}, \dot{v}, \dot{w}$ to obtain the $b_i^r(u)$, then evaluate the Bézier polynomial (36) using (33) with weights u, v, w. Actually, one can switch from one method to the other at each step. *

Let us now evaluate derivatives across a boundary of T, say e_3, this implies w = 0. From (36) we see that the rth derivative of $B_n\phi$ is a Bézier polynomial of degree n-r with Bézier ordinates $b_{\underset{\sim}{i}}^r(\underset{\sim}{u})$. On the boundary e_3, this Bézier polynomial will only depend on those $b_{\underset{\sim}{i}}^r(\underset{\sim}{u})$ for which k = 0.

Therefore $D_{\underset{\sim}{\dot{u}}}^r[B_n\phi]|_{e_3}$ depends only on the r + 1 parallels (of Bézier ordinates) to e_3.

Note also that $D_{\underset{\sim}{\dot{u}}}^r[B_n\phi]|e_3$ is an (n-r)th degree <u>univariate</u> polynomial in v:

$$(37)\quad D_{\underset{\sim}{\dot{u}}}^r[B_n\phi]|e_3 = \frac{n!}{(n-r)!}\sum_{j=0}^{n-r} b_{\underset{\sim}{i}_3}^r(\underset{\sim}{\dot{u}})\, B_j^{n-r}(v)$$

where $\underset{\sim}{i}_3$ is short for (n-r-j, j,0).

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* The relationship of this statement with the univariate case becomes clear if we view terms in $\underset{\sim}{\dot{u}}$ as generalizations of the difference operator Δ.

III Composite Surfaces

1. C^r-Continuity between adjacent triangles

Let a Bézier polynomial $B_n\phi_1$ be defined over a triangle $T_1 = P_1 P_2 P_3$. Let a second triangle $T_2 = P_1 P_4 P_2$ with

$$P_4 = u_0 P_1 + v_0 P_2 + w_0 P_3$$

be given. We seek a Bézier polynomial $B_n\phi_2$ defined over T_2 that has C^r-continuity with $B_n\phi_1$ along the common edge $\overline{P_1 P_2}$.

Let the barycentric coordinates in T_i be $\underset{\sim}{u}_i$, $i = 1,2$. Then there xists a linear transformation

(38) $\underset{\sim}{u}_2 = \underset{\sim}{u}_1.A \quad , \quad \dot{\underset{\sim}{u}}_2 = \dot{\underset{\sim}{u}}_1.A$

with a nonsingular matrix A such that

$$\underset{\sim}{a}_1 = \underset{\sim}{u}_0.A \quad , \quad \underset{\sim}{a}_2 = \underset{\sim}{a}_2.A \quad , \quad \underset{\sim}{a}_3 = \underset{\sim}{a}_1.A$$

(see also Fig. 6). We find for A:

$$A = \frac{1}{w_0}\begin{bmatrix} 0 & 0 & w_0 \\ 0 & w_0 & 0 \\ 1 & -v_0 & -u_0 \end{bmatrix} \quad ; \quad w_0 \neq 0$$

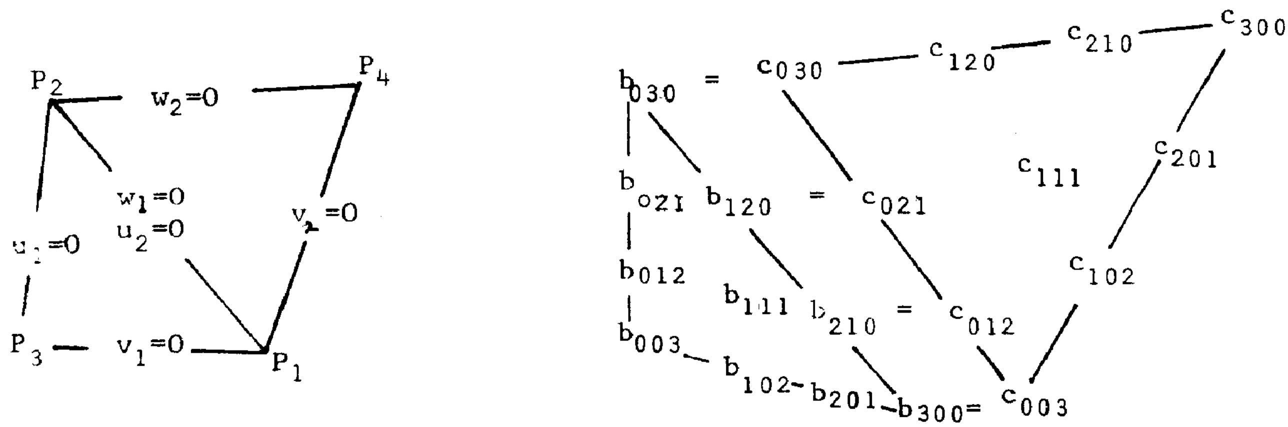


Figure 6. Two adjacent (cubic) triangles

Let the Bézier ordinates of $B_n\phi$, be $b_{\underset{\sim}{i}}$, those of $B_n\phi$ be $c_{\underset{\sim}{i}}$.

The rth cross-boundary derivative with respect to some direction $\dot{\underset{\sim}{u}}_1$(resp. $\dot{\underset{\sim}{u}}_2$) of $B_n\phi_i$ is determined by the (r+1) rows of Bezier ordinates in T_i parallel to the edge e_3. The next theorem gives a simple method to compute the relevant $c_{\underset{\sim}{i}}$ from the relevant $b_{\underset{\sim}{i}}$.

<u>Theorem 5</u>: With the above notations $B_n\phi_1$ and $B_n\phi_2$ have C^r-continuity along e_3 if and only if

$$(39) \quad c_{\not\in\rho, j, n-\rho-j} = b^{\rho}_{n-\rho-j, j, 0}(\underset{\sim}{u}_0) \; ; \quad \begin{matrix} 0 \le \rho \le r \\ 0 \le j \le n-\rho \end{matrix}$$

<u>Example 2</u>: For r = 1, (39) becomes for $\rho = 0$:

$$c_{0,j,n-j} = b_{n-j,j,0} \; ; , \; 0 \le j \le n$$

and for $\rho = 1$:

$$c_{1,j,n-1-j} = b^1_{n-1-j,j,0}(\underset{\sim}{u}_0) \qquad 0 \le j \le n-1$$

$$= u_0\, b_{n-j,j,0} + v_0\, b_{n-1-j,j+1,0} + w_0\, b_{n-1-j,j,1}.$$

The first equation ensures that $B_n\phi_1$ and $B_n\phi_2$ have a common boundary curve. The second equation states that every shaded quadrilaterial in fig, 7 is plane. (Figure 7 shows the projection into the plane only).

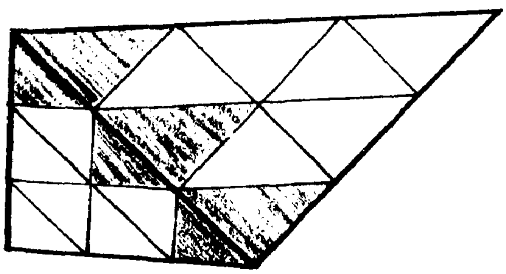


Fig. 7 C^1-continuity between adjacent cubic triangles.

<u>Proof</u> of theorem 5:

Let $\underset{\sim}{i}_1$ be of the form (0, j, k) and $\underset{\sim}{i}_3$ of the form (i, j, 0). (37) gives the C^r-condition

$$\sum_{j=0}^{n-\rho} b^{\rho}{}_{\underset{\sim}{i}3}\ (\dot{\underset{\sim}{u}}_1)\ B^{n-\rho}{}_j(v) = \sum_{j=0}^{n-\rho} c^{\rho}_{\underset{\sim}{i}1}(\dot{\underset{\sim}{u}}_2)\ B^{n-\rho}_j(v);\ 0 \le \rho \le r.$$

Comparison of coefficients yields

$$(40)\quad b^{\rho}{}_{\underset{\sim}{i}3}\ (\dot{\underset{\sim}{u}}_1) = c^{\rho}{}_{\underset{\sim}{i}1}\ (A\dot{u}_1)\ ; \qquad \begin{matrix} 0 \le \rho \le r \\ 0 \le j \le n-\rho \end{matrix}$$

The term $b^{\rho}{}_{\underset{\sim}{i}3}\ (\dot{\underset{\sim}{u}}_1)$ can be viewed as the ρ-th directional derivative of the Bézier polynomial defined by $b^{\rho}{}_{\underset{\sim}{i}3}\ (\underset{\sim}{u}_1)$; the same is true for $c^{\rho}{}_{\underset{\sim}{i}1}\ (A\dot{u}_1)$. These two polynomials coincide in their derivatives up to order ρ: hence they are equal:

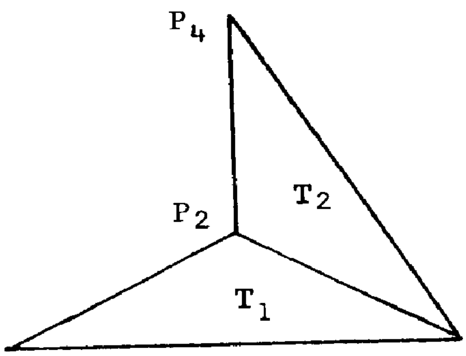
$$b^{\rho}{}_{\underset{\sim}{i}3}\ (\underset{\sim}{u}_1) = c^{\rho}{}_{\underset{\sim}{i}1}\ (A\dot{u}_1)\ ; \qquad \begin{matrix} 0 \le \rho \le r \\ 0 \le j \le n-\rho \end{matrix}\ .$$

This is true for all $\underset{\sim}{u}_1$, e.g. also for $\underset{\sim}{u}_1 = \underset{\sim}{u}_0$:

$$\begin{aligned} b^{\rho}{}_{\underset{\sim}{i}3}\ (\underset{\sim}{u}_0) &= c^{\rho}{}_{\underset{\sim}{i}1}\ (A\underset{\sim}{u}_0) \\ &= c^{\rho}{}_{\underset{\sim}{i}1}\ (1,0,0) \qquad \begin{matrix} 0 \le \rho \le r \\ 0 \le j \le n-\rho \end{matrix} \\ &= c_{\rho,j,n-\rho-j} \end{aligned}$$

<u>Example 3</u>: Consider the two triangles below. Let P_2 be the centroid of P_1, P_4, P_3, such that

$P_4, = 3P_2 - P_1 - P_3$.



P_4 has barycentric coordinates $\underset{\sim}{u}_0 = (-1, 3, -1)$ with respect to T_1.

Let ϕ_1 be defined over T_1 by the Bézier ordinates

$$\begin{array}{ccccccc} & & & 4 & & & \\ & & 3 & & 2 & & \\ & 0 & & 0 & & 1 & \\ 0 & & 0 & & 0 & & 1 \end{array}$$

We seek ϕ_2, defined over T_2, such that $B_3\phi_1$ and $B_3\phi_2$ have common first and second cross-boundary derivatives along $\overline{P_1,P_2}$.

Theorem 5 suggests the following construction of the Bézier ordinates $c_{\underset{\sim}{i}}$ of ϕ_2 :

<u>Step 1</u>:

$$c_{1,\ j,\ 2-j} \ = \ b^{1}_{2-j,j,0}\,(-1,3,-1).$$

The scheme of the $b^1_{\underset{\sim}{i}}$ is given by

$$\begin{array}{ccccc} & & \underline{7} & & \\ & 9 & & \underline{5} & \\ 0 & & 0 & & \underline{2} \end{array}$$

and hence the underlined numbers are the desired $c_{1,\ j,\ 2-j}$.

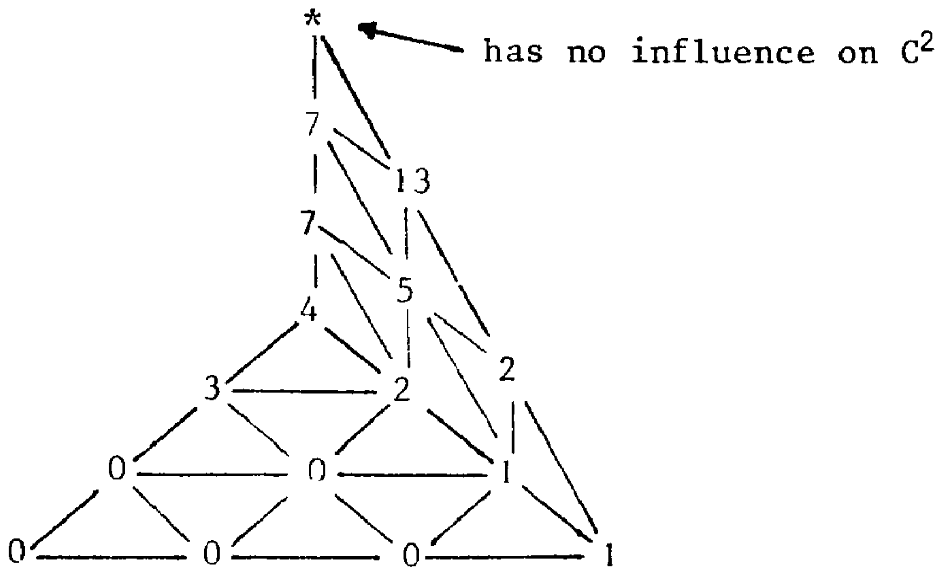
<u>Step 2:</u>

$$c_{2,\ j,\ 1-j} \ = \ b^{2}_{1-j,j,o}\,(-1,3,-1).$$

The scheme of the $b^2_{\underset{\sim}{i}}$ is given by

$$\begin{array}{ccc} & \underline{7} & \\ 27 & & \underline{13} \end{array}$$

The Bézier ordinates of ϕ_1 and ϕ_2 are therefore



Example 4: Suppose we are given ϕ_1 and ϕ_2 from the previous example and want to verify that the two corresponding Bézier polynomials join in C^2. This is easily accomplished by means of (40): Choosing $\dot{\underset{\sim}{u}}_1$ to be a direction perpendicular to $\overline{P_1P_2}$, we get

$$\dot{\underset{\sim}{u}}_1 = [-1, 3, -2]; \ \dot{\underset{\sim}{u}}_2 = \dot{\underset{\sim}{u}}_1 \cdot A = [2, -3, -].$$

The C^1 condition becomes (for j = O, 1, 2)

$$(41) \quad 3b_{n-1-j,\, j+1,0} = b_{n-j,j,0} + c_{o,j,n-j} + b_{n-1-j,j,1}$$

and for the C^2 condition we get (for j =0,1)

$$(42) \quad \begin{aligned} & b_{n-1-j,j,1} - 3\, b_{n-2-j,\, j+1,1} + b_{n-2-j,\, j,\, 2} \\ & = c_{1,\, j,\, n-1-j} - 3\, c_{1,\, j+1,\, 1-j} + b_{2,j,n-2-j} \, . \end{aligned}$$

2. Degrees of Freedom

Theorem 5 enables us to construct a $B_n\phi_2$ that joins a given $B_n\phi_1$ in C^r. There are $\overline{n-r}$ *) Bezier ordinates in ϕ_2 that we can specify arbitrarily, the remaining ones being fixed by C^r-continuity. Since we could specify all $\overline{n+1}$ Bézier ordinates of ϕ_1 arbitrarily, the piecewise surface given by $B_n\phi_1$ and $B_n\phi_2$ has $\overline{n+1}$ + $\overline{n-r}$ degrees of freedom (d.o.f's).

This construction may be repeated, thus adding $\overline{n-r}$ d.o.f's for each new $B_n\phi$.

We may eventually encounter a situation where a new $B_n\phi$ cannot be added in this fashion because two of its edges are shared by previously constructed Bézier polynomials. One can in fact easily give examples where such a construction must fail. It is always possible, however, to add two Bézier polynomials simultaneously as is shown in figure 8. Our problem is now to determine how many Bézier ordinates in these two triangles can be arbitrarily specified; we call this number of degrees of freedom $\rho(n,r)$.

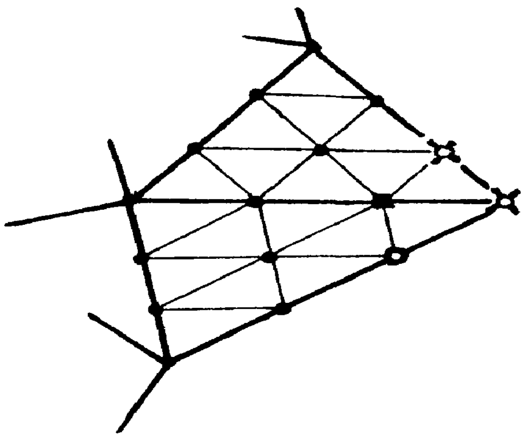


Fig. 8: determination of $\rho(3,1)$

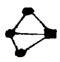
*) We define $\overline{k} = \frac{1}{2}k(k+1) = 1+2+...+k$.

Theorem 6

$$(43)\qquad \rho(n,r)=\begin{cases} n^2 & r=0\\ \overline{r-1}+(n-2r)(n-r) & 1\le r\le \frac{1}{2}(n-1)\\ \overline{n-r-1} & \frac{1}{2}n\le r\le n\\ 0 & r=n\end{cases}$$

The method used for the proof is illustrated in example 5: search every quadrilateral of "side length" r + 1 responsible for C^r for the number of d.o.f.'s it offers, proceeding from left (close to predetermined points) to right. A more detailed description of this procedure is given in Farin 1979.

Remark: The two vertices shared with previously determined polynomials must not form a straight line.

Example 5. (see Fig. 8 and the C^1 conditions in Fig. 7). Let the "•" be determined by C^1, then "■" is fixed because the quadrilateral  must be plane (this justifies the above remark). We can specify the "  " arbitrarily; together with "■", they will determine "**O**". Hence, $\rho(3,1) = 2$.

Consider a triangle T that is subdivided into three subtriangles T_i by its centroid (see Fig. 9). Define τ^r_n to be the linear space of n-th degree polynomials defined over each T_i and joining in C^r. Its dimension is given by

$$(44)\qquad \dim\ \tau^r_n = \overline{n+1}+\rho(n,r)$$

This leads to a somewhat surprising result:

Theorem 7

(45) $\tau_n^{n-1} = \tau_n^n$

Proof: Combining (44) with (43), we get

$$\dim \tau_n^{n-1} = \dim \tau_n^n .$$

(45) follows since is a subspace of τ_n^{n-1}.

A simple consequence is

Corollary 8: Every element of τ_n^r has continuous derivatives of order O, 1 , . . . , r+1 at the centroid of T.

Proof: An element of τ_n^r contains a subtriangle that can be considered an element of τ_{r+1}^r. This subtriangle is responsible for the derivatives at the cetitroid, and an application of Theore 7 completes the proof.

IV Interpolation in τ_n^r `.

1. The case τ_3^1

We define

$$Q_{i,j}^k = \frac{j}{k+1} P_i + \left(1-\frac{j}{k+1}\right) P_{i+1}, \text{*)}$$

$$m = n - 2r - 1 .$$

Let $D^r f(P_i)$; i = 1,2,3; denote all r+1 partials of order 0,1,... of f at P_i; we shall always require that the three $D^r f(P_i)$ be consistent with each other (this is trivially the case if 2r < n)
Let $\dot{\underset{\sim}{u}}_i$ denote a direction not parallel to the edge $P_i\ P_{i+1}$.

*) vertices are counted mod (3) .

Our interpolation problem will be:
Find an element f in τ^r_n that assumes the followi

$$(V_r)\quad D^r f(P_i)\ ;\quad i = 1,2,3$$

$$(E_r)\quad D^{\rho}_{u_i} f(Q^{m+\rho}_{i,j})\ ;\qquad \begin{array}{l} i = 1,2,3 \\ \rho = 0,1,\dots,r \\ j = 0,1,\dots, m+\rho<1. \end{array}$$

Note that (E_r) is void for $m+\rho<1$.

The solution to the interpolation problem given by (V_1) and (E_1) for τ^1_3 is known as the C[1] clough-Tocher scheme (Strang/Fix 1973).

This solution can easily be constructed using Bézier polynomials; it consists of three steps, see also fig. 9.

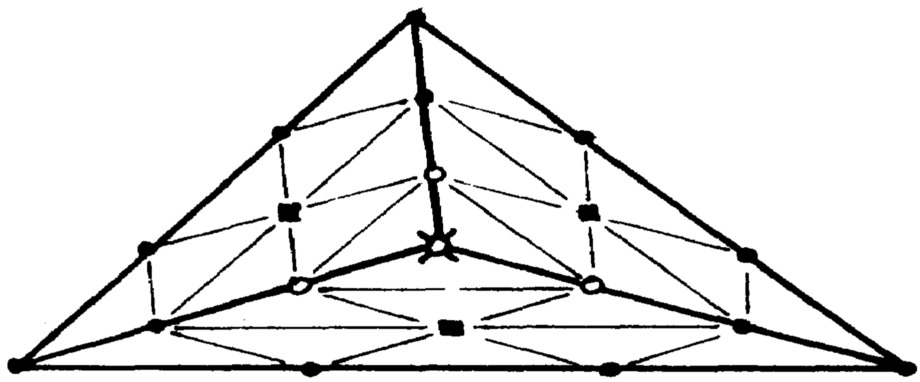


Fig. 9; Constructing the C^1 solution.

1. The Bézier ordinates "•" are given by (V_1); the "■" stem from (E_1).
2. C^1 across the interior vertices determines the "**o**", cf. (41)
3. C^1 at the centro id determines "✗": it has to be the centroid of the "**o**".

Note that this construction also implies the uniqueness of the

interpolant. Moreover, Corollary 8 implies that it has (though constructed in C^1 context) continuous second derivatives at the centroid. This interpolation scheme has cubic precision.

The above case r = 1 cannot be generalized:

<u>Theorem 9:</u> The interpolation problem for τ_n^r given by (V_r) and (E_r) is overdetermined for $r \geq 2$, $n \geq r+2$.

<u>Proof</u>: In fig. 10, let "●" denote Bezier ordinates determined by (V_r) and (E_{r-1}) (for $(n, r) = (5,2)$) .

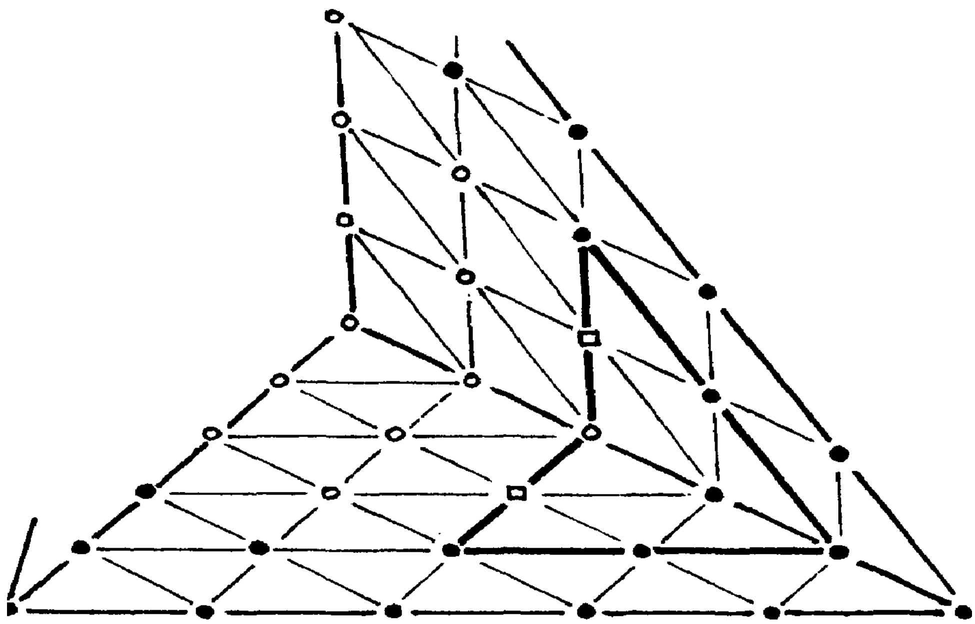


Figure 10: Incompatibility in interpolation problem.

The "fat butterfly" - which is responsible for C^2 continuity, see (42) - is determined by six independent pieces of information, five of which are already fixed; i.e. <u>one</u> of the two Bézier ordinates "□" determines the butterfly completely (by equations (42) and (41). Since (E_r) prescribes both of them arbitrarily, the problem is overdetermined.

2. C^2 Interpolation in τ_6^2

Theorem 9 suggests to consider the following C^2-interpolation problem:

Find an element in τ_6^r that satisfies (E_2) and (V_3') .

V_3' demands that third derivatives parallel to edges are organized as to determine seven (instead of eight) Bézier ordinates per boundary curve. This does not maintain C^3 continuity at the vertices any more, by (a) retains C^2 continuity there and (b) eliminates the incompatibility that caused the failure of schemes using (E_2) and (V_2).

The choice of τ_6^2 is suggested by the following reasoning:
We want to be able to solve the C^2 problem in adjacent triangles. If we were working in, say, τ_5^2 (defined for each triangle), C^2 information along the common edge and (consistent !) C^3 information at the corresponding vertices would not guarantee C^2 continuity between the two interpolants; but it is guaranteed using τ_6^2

We shall now turn to the solution of the C^2 problem.

Since dim τ_6^2 = 37 and (E_2) and (V_3') provide 33 constraints, we may specify four additional Bézier ordinates. Again, the construction of the solution consists of three steps, see Figure 11:

<u>Step 1:</u> The Bezier ordinates "$*$" and the a_i , i=l,..., 15, are given by (V_3) and (E_2).

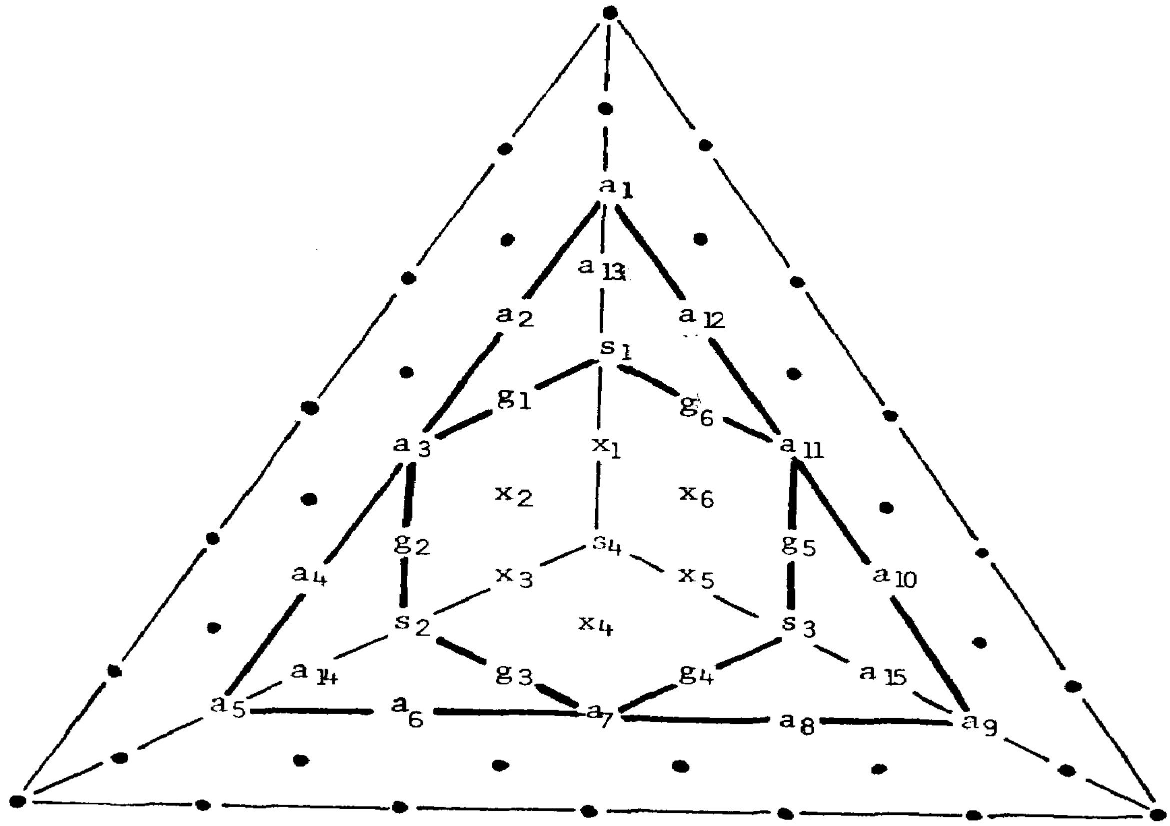


Figure 11: Constructing the C^2 solution.

Step 2: We specify the Bezier ordinates s_1, s_2, s_3 in order to determine the g_i. This is done by solving a 2 x 2 linear system for each of the "fat butterflies", e.g.

(41') $a_{15} + g_5 + g_4 = 3s_3$.

(42') $a_{11} + a_{10} - 3g_5 = a_7 + a_8 - 3g_4$.

These two equations are readily solved for g_4 and g_5.

Step 3: We specify s_4 and determine the x_i by solving a 6 x 6 linear system, the first four equations being applications of (41), the last two corresponding to (42).

The system is

$$(46)\quad \begin{vmatrix} -3 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & -3 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -3 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & -3 & 3 & 0 & 0 \\ 0 & 0 & 0 & -3 & 0 & 3 \end{vmatrix} \cdot \begin{vmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{vmatrix} = \begin{vmatrix} -s_1 \\ -s_2 \\ -s_3 \\ 3s_4 \\ g_3+g_4-g_1-g_2 \\ g_5+g_6-g_3-g_4 \end{vmatrix}$$

and has the solution

$$(47)\quad \begin{vmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{vmatrix} = \frac{1}{72} \begin{vmatrix} -15 & 15 & 9 & 27 & -6 & 2 \\ 18 & 54 & 18 & 54 & -36 & -12 \\ 9 & -9 & 9 & 27 & -6 & -6 \\ 9 & -9 & 9 & 27 & 18 & -6 \\ 6 & -6 & -18 & 18 & 12 & 4 \\ 9 & -9 & 9 & 27 & 18 & 18 \end{vmatrix} \cdot \begin{vmatrix} -s_1 \\ -s_2 \\ -s_3 \\ 3s_4 \\ g_3+g_4-g_1-g_2 \\ g_5+g_6-g_3-g_4 \end{vmatrix}$$

The above choice of the s_i is not the only one possible; but it minimizes the sizes of the linear systems that have to be solved.

Corollary 8 and the derivation of $\rho(n,r)$ yield

Theorem 10 The above scheme has sextic precision. Any interpolant constructed by it has continuous third derivatives at the centroid.

V. The Dimension of τ_n^r (i, b)

Let $\tau(i,b)$ be a simply-connected (but not necessarily convex) triangulation with i interior vertices and b boundary vertices, such that the piecewise linear) boundary curve is a simply closed curve. We exclude triangulations that contain vertices whose star is a convex quadrilateral with the diagonals drawn in.*)
Let τ_n^r (i, b) be the linear space of C^r piecewise polynomials of degree n over τ (i,b).

Theorem 11

$$(48) \quad \dim \tau_n^r(i,b) = \overline{n+1} + i\cdot\rho(n,r) + (b-3)\overline{n-r}$$

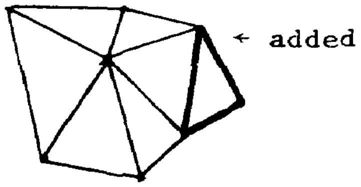
Proof: We use induction on the number of triangles in τ (i ,b) ($\tau(i,b)$ consists of b+2i-2 triangles).

1. If τ (i,b) consists of one triangle only,

$$\dim \tau_n^r(0,3) = \overline{n+1}\ .$$

2. Suppose (48) holds for a simply-connected subtriangulation τ (j, c) of τ (i,b). We add a new triangle to τ (j, c), thus obtaining τ (j',c'). We have to consider two cases.

Case a: j' = j, c' = c+1

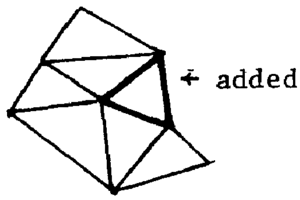


The dimension of $\tau_n^r(j,c)$ is increased by $\overline{n-r}$:

$$\begin{aligned}\dim \tau_n^r(j, c+1) &= \dim \tau_n^r(j,c) + \overline{n-r}\\ &= \overline{n+1} + j\cdot\rho(n,r) + (c-2)\,.\,\overline{n-r}\end{aligned}$$

*) This restriction is a consequence of the remark after theorem 6.

Case b: $\quad j' = j + 1, \quad c' = c-1$



We define

$q(n,r) = \rho(n,r) - \overline{n-r}$.

Since dim $\tau_n^r(i,b)$ was increased by $\rho(n,r)$ when two triangles are added simultaneously (see figure 8), q(n,r) denotes the change if only one triangle is added:

$$\dim \tau_n^r\ (j+1, c-1) = \dim\ \tau_n^r\ (j,c) + q(n,r)^{*)}$$

$$= \overline{n+1} + (j=1)\, p\,(n,r) + (c-4)\ \overline{n-r}$$

Remarks:

1. Every simply-connected triangulation can be constructed by using steps a) and b) from the above proof.

2. If an optimization procedure is applied to a triangulation (Barnhill 1977), the dimension of the corresponding linear spaces does not change.

3. The above proof can be used to construct a basis for τ_n^r (i,b) in terms of Bezier polynomials.

4. For $n \geq 4$, r = 1, theorem 11 coincides with a result obtained by Morgan/Scott, 1975.

   The proof of theorem 11 can easily be adapted to triangulations $\tau'(i,b)$ that have a hole, where the vertices around the hole are considered boundary vertices:

---

*) Thus adding a triangle to τ (i ,b) may decrease dim $\tau_n^r(i,b)$!

Corollary 12

(49) $\dim \tau_n^{\prime r}(i,b) = \overline{n+1} + (i+1)\rho(n,r) + (b-3)\,\overline{n-r}$

Theorem 11 implies that univariate B-splines cannot be generalized :

Theorem 13 No $\tau_n^{n-1}(i,b)$ can contain a non-zero element f such that

i) f is identically zero outside $\tau(i,b)$

ii) $f \in C^{n-1}(\mathbb{R} \times \mathbb{R})$

Proof: Suppose such an f existed. Construct a triangulation $\tau(i',3)$ that contains $\tau(i,b)$. (This is trivially possible since $\tau(i,b)$ is finite.)

i) and ii) imply $f \in \tau_n^{n-1}(i',3)$. Since $\dim \tau_n^{n-1}(i',3) = \overline{n+1}$, we have $\tau_n^{n-1}(i',3) = P_n$ (the linear space of bivariate polynomials of degree $\leq$ n), i.e. f is a (global) polynomial But no non-zero polynomial can satisfy i).

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