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NUMERICAL SOLUTION OF A FREE BOUNDARY PROBLEM BY INTERCHANGING DEPENDENT AND INDEPENDENT VARIABLES

by

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ABSTRACT

The classical problem of seepage of fluid through a porous dam is solved to illustrate a new approach to more general free boundary problems. The numerical method is based on the interchange of the dependent variable, representing velocity potential, with one of the independent space variables, which becomes the new variable to be computed. The need to determine the position of the whole of the free boundary in the physical plane is reduced to locating the position of the separation point on a fixed straight—line boundary in the transformed plane.

An iterative algorithm approximates within each single loop both a finite-difference solution of the partial differential equation and the position of the free boundary. The separation point is located by fitting a 'parabolic tail' to the finite-difference solution.

1. Introduction

A free boundary problem involves the solution of an elliptic partial differential equation subject to conditions on a boundary, .part of which is unknown in position and shape. The most familiar model problem refers to the seepage of water through an earth dam, separating a high reservoir from a lower one. The upper surface of the water within the dam has to be determined as part of the solution.

Successive authors have approached the problem by solving a sequence of fixed boundary problems corresponding to successive, iteratively-computed positions of the free boundary. Relaxation methods, finite differences and finite elements have all been used to execute the solution. Key references are to be found in Cryer (1976), Aitchison (1972; 1977), and Furzeland (1977; 1979).

More recently, Aitchison (1977) and others referred to in Furzeland (1977;1979) have avoided the iterations by using the Baiocchi transformation (1972) to reformulate the problem as a variational inequality over a fixed domain.

The present paper transforms the region within the dam contained partly by the free boundary into a domain with fixed, known boundaries by interchanging the dependent variable with one of the independent, space variables.

This idea is well-tried in fluid flow problems and has recently been applied to moving boundary problems in heat flow (Crank & Phahle, 1973; Crank & Gupta, 1975; Crank & Crowley, 1978, 1979).

In the transformed plane it is possible to solve the dam problem, for example, by an iterative algorithm which approximates within each single iterative loop both the solution of the partial differential- equation and the position of the free boundary.

2. The Seepage Problem

The mathematical formulation of the problem depicted in Figure 1, in terms of the velocity potential ϕ, is

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \text{ in ABCDE}, \qquad (1)$$

$$\phi = 1, \; x = 0 \text{ on AB}, \qquad (2)$$

$$\phi = \phi_d, \; x = L \text{ on CD}, \qquad (3)$$

$$\partial\phi/\partial n = 0, \; y = 0 \text{ on BC}, \qquad (4)$$

$$\phi \;\; y, \; x = L \text{ on DE} \qquad (5)$$

$$\phi = y, \; \partial\phi/\partial n = 0 \text{ on AE}, \qquad (6)$$

where n is the outward normal on AE.

The double condition on AE is needed here as in all free boundary problems in order to determine the position of AE as well as to solve the differential equation (1).

We now replace $\phi(x,y)$, the usual dependent variable, by $x = x(\phi, y)$ as the new dependent variable. It is easy to see that the partial differential equation (1) becomes

$$\frac{\partial^2 \phi}{\partial y^2} - \frac{\partial^2 x}{\partial \phi^2} = \left(\frac{\partial x}{\partial \phi}\right)^{-3} = 0 \qquad (7)$$

In general, on a boundary $y = g(x)$ we have

$$\phi_n = \frac{g'\,\phi_x - \phi_y}{\left[1 + (g')^2\right]^{\frac{1}{2}}}, \qquad g'=dy/dx, \tag{8}$$

where, for example, $\phi_n = \partial\phi/\partial n$ n and n is the outward normal to $y = g(x)$.

Provided $g' \neq \infty$, we have on the free boundary AE

$$g'\frac{\partial\phi}{\partial x} - \frac{\partial\phi}{\partial y} = 0, \quad y = \phi . \tag{9}$$

But $(\partial\phi/\partial y)_x = -(\partial x/\partial y)/(\partial x/\partial\phi)$ so that

$$g' + \left(\frac{\partial x}{\partial y}\right)_\phi = 0, \quad y = h . \tag{10}$$

On the impervious foundation BC we have

$$\frac{\partial x}{\partial y} = 0, \quad y = 0 . \tag{11}$$

The other boundary conditions are

$$x = 0, \quad \phi = 1; \quad x = L, \quad \phi = \phi_d, \qquad 0 \leq y \leq d ,. \tag{12}$$

$$\text{and} \quad x = L, \quad \phi = y \quad \text{on DE.} \tag{13}$$

Thus, we now wish to solve (7) subject to conditions (10) to (13) inclusive in the region B'A'E'D'C in Figure 2 in the (y, ϕ) plane. All the boundaries are fixed. The original free boundary AE has become the known straight boundary A'E', $y = \phi$. What we do not know, however, is the position of E' on A'D' corresponding to the separation point E in Fig. 1, at which the boundary condition on A'D' changes from $x = L$ to the condition (9).

We cover the region with a mesh of spacings $\delta\phi$, δy, and denote $x(i\delta\phi,\ j\delta y)$ by $x_{i,j}$. Equation (7) can be approximated in part by

$$\frac{\partial^2\phi}{\partial y^2} - 8\delta\phi \frac{(x_{i-1,j} - 2x_{i,j} + x_{i+1,j})}{(x_{i+1,j} - X_{i-1,j})^3} = 0, \qquad (14)$$

In order to approximate $\partial^2\phi/\partial y^2$ at the point $(i\delta\phi, j\delta y)$ we need three values of ϕ on the lines corresponding to equally-spaced values of y, y_{j-1}, y_j, y_{j+1}, but chosen such that $x = x_{i,j}$ at each point. We interpolate linearly on each of the mesh lines $y = y_{j+1}$ and $y = y_{j-1}$ as illustrated in Figure 3. Thus on the $y = y_{j+1}$ line we obtain

$$\phi(x_{i,j}) = \frac{\phi_i(x_{i,j} - x_{i+j,j+1}) + \phi_{i+1}(x_{i,j+1} - x_{i,j}),}{x_{i,j+1} - x_{i+1,j+1}} \qquad (15)$$

and similarly on $y = y_{j-1}$.

Substitution of the resulting two values of ϕ together with ϕ_i itself into the usual finite-difference replacement for $\partial^2\phi/\phi y^2$ in (14) yields an approximation to (7) for the typical internal point $(i\delta\phi, j\delta y)$. If we collect together terms in $x_{i,j}$ we can write the difference equation in the simplest iterative form

$$\left\{\frac{\phi_i - \phi_{i+1}}{x^n_{i,j+1} - x^n_{i+1,j+1}} + \frac{\phi_i - \phi_{i-1}}{x^n_{i,j-1} - x^n_{i-1,j-1}} + \frac{16\delta\phi(\delta y)^2}{(x^n_{i+1,j} - x^n_{i-1,j})^3} + \right\} x^{n+1}_{i,j} =$$

$$= \frac{\phi_i x^n_{i+1,j+1} - \phi_{i+1} x^n_{i,j+1}}{x^n_{i,j+1} - x^n_{i+1,j+1}} + 2\phi_i + \frac{\phi_i x^n_{i-1,j-1} - \phi_{i-1} x^n_{i,j-1}}{x^n_{i,j-1} - x^n_{i-1,j-1}} +$$

$$+ \frac{8\delta\phi(\delta y)^2\ (x^n_{i+1,j} + x^n_{i-1,j})}{(x^n_{i+1,j} + x^n_{i-1,j})^3}$$

where $x^n_{i,j}$ is the n^{th} iterate of $x_{i,j}$. (16)

Correspondingly, on the boundary $y = \phi$ we use either $x = L$ or

$$x_{i,j}^{n+1} = x_{i,j-1}^{n} - \frac{2(\delta y)^2}{x_{i+1,j+1}^{n} - x_{i-1,j-1}^{n}} \qquad (17)$$

from (10).

Application of (16) to points on the lower boundary, $y = 0$, i.e. $j = 0$, introduces .fictitious points on the line $j = -1$, one step outside the region; these can be eliminated from (16) since (11) implies $x_{i,-1} = x_{i,1}$ for all i.

3. The iterative cycle

We start by assuming the separation point E' to be at one of the mesh points on A' D' in Figure 2. Then we know that $x = L$ along C' D' and D' E' and also that $x = 0$ on B'A'. For every other mesh point within the region and on the remaining parts of the boundary we have derived an equation. We carry out one iterative cycle by sweeping along successive j-lines from left to right in Figure 2 in the order j =0, 1, 2, ... where $y = j = 0$ is the lower boundary. The new values of $x_{i,j}$ are retained for use in the next cycle subject to the proviso that on the boundary A'D' we take the new value $x_{i,i}^{n+1}$ to be

$$x_{i,i}^{n+1} = \min\left(L,\ x_{i,i}^{n+1}\right),$$

since we know that $x_{i,j} \leq L$. We proceed with successive loops, iterating values of the solution and the position of the separation point E' along A 'D' in the same loop by using (18). The iteration proceeds until the difference between successive iterates at each point of the mesh is less than some prescribed amount. The highest mesh point on the boundary A 'D' (Fig. 2) at which $x = L$ is the best approximation to the separation

point that can be obtained directly from the set of finite-difference equations (16) and (17). In general, however, the true separation point will lie between two neighbouring mesh points and the finite-difference solution itself will be least accurate near the separation point.
In particular, this solution will not satisfy the condition that the gradient of the free boundary dx/dy, should approach zero at the separation point. We therefore fit a "parabolic tail" to the finite-difference approximation to the free boundary.

4. Improved position of separation point

Suppose the true position of the separation point is denoted by $E'(x = L, y = y_s)$, (Fig. 2) where $y_s < y_j$, and (j,j) is the lowest mesh point on the free boundary for which $x < L$ in the finite-difference solution.
Since $dx/dy = 0$ at $E'(x = L, y = y_s)$ we write near E', for $x \leq L$,

$$x = L - A(y - y_s)^2 \;, \tag{19}$$

where A is a constant to be determined. Then

$$x_{j,j} = L - A(y_j - y_s)^2 \tag{20}$$

and

$$\left.\frac{dy}{dx}\right)_{y_j} = g'(y_j) = -\frac{1}{2A(y_j - y_s)} \tag{21}$$

But from (10) we have

$$g'(y_j) = -\left.\frac{\partial x}{\partial y}\right)_{y_j} = -\left(\frac{x_{j,j} - x_{i,j-1}}{\partial y}\right) \tag{22}$$

approximately

By inserting in (20), (21) and (22) the relevant values of x and y from the final stage of the iterative solution, we obtain an interpolated value for y_S. If $y_S < y_{j-1}$, (19) also yields a revised value for $x_{j-1,j-1}$.

In order to estimate the effect of the "parabolic tail" (19) on the finite-difference solution elsewhere in the domain, a further stage of the iterative process was performed using the value of $x_{j-1,j-1}$ and of g' at that point from the "parabolic tail". If a significant change was found in $x_{j,j}$ a new tail was fitted to update y_S.

Numerical results

In order to facilitate comparisons with the results quoted by Aitchison (1977) and Elliott (1976) we take L = 2/3, d = 1/6, H = 1 and h_d = 1/6.

Calculations have been carried out on an 18 x 18 mesh ($\delta\phi = \delta y$ = 0.0556) and 30 x 30 mesh ($\delta\phi = \delta y$ = 0.0333). In order to start the iterative process the separation point was first assumed to be at the point D' in Fig. 2, corresponding to D in Fig. 1. Starting at mesh points further along the line D'A' was later found to yield the same final results.

The initial values of x along the free boundary D'A' were linear interpolates between x - 2/3 at D' and x = 0 at A'. On each line, y = constant, initial values of x at internal mesh points were obtained by linear interpolation between the end values on C' D'A' and B' A'.

It has not been possible to carry out a formal study of convergence of the iterative process for solving the non-linear difference equations (16). Instead, Table 1 demonstrates the convergence of some of the numerical values obtained on the line y = 1/6 at. selected stages of the 18 x 18 mesh iterative solution. Essentially the same behaviour was found for the 30 x 30 mesh. The iteration process ceases when $\left|x_{ij}^{n+1} - x_{ij}^{n}\right| < 10^{-4}$ at all points of the mesh.

Fig. 4 shows an extract from the results obtained on the 18 x 18 grid and Fig. 5 shows all the mesh points in the region of the separation point.

In Table 2 the positions of the free boundary calculated by the present method using the 18 x 18 and 30 x 30 meshes are compared with corresponding results obtained by Aitchison (1977) and Elliott (1976).

Table 3 compares various values obtained for the y coordinate of the separation point E. It is worth recalling that Cryer (1976) considered the most reliable value to be 0.5297.

Conclusions

The primary aim of this paper is to demonstrate that the idea of interchanging the dependent with one of the independent variables can form the basis of a method which iterates simultaneously the position of the free boundary and the solution of the partial differential equation. The results compare favourably with those obtained recently by mathematically more sophisticated techniques. For a given mesh, this method locates the position of the free boundary more precisely than do fixed-domain methods based on a minimisation or similar formulation. We are conscious that we have used only the very simplest iteration algorithm and that the convergence would very probably be improved by using alternative algorithms. We have not felt it worthwhile to explore this aspect in relation to the already over-studied dam problem. Equally, the method could be developed using finite elements instead of finite differences.

TABLE 1

Convergence of values of 10^4x at selected internal points of an 18x18 mesh on y=1/6

Iteration \ ¢	4/18	8/18	12/18	17/18
0	6222	4444	2667	444
50	6374	4753	2833	465
250	6404	4995	3181	550
289	6405	4995	3186	552

TABLE 2

Comparison of values of 10^4x at chosen y-values on the free boundary.

y	(18 x 18) mesh	(30 x 30) mesh	Aitchison (24 x 24) mesh.	Linear element	Quadratic element
0.5330	6669	6667	6667	-	-
0.5333	6667	6667	6665	-	-
0.5667	6549	6562	6423	-	-
0.6000	6432	6458	6180	-	-
0.6333	6149	6120	5960	5957	5975
0.6667	5806	5787	5652	5631	5648
0.7000	5462	5448	5314	5283	5285
0.7333	5089	5064	4943	4908	4914
0.7667	4667	4647	4425	4494	4519
0.8000	4205	4192	4095	4072	4132
0.8333	3722	3695	3611	3583	3653
0.8667	3174	3151	3081	3050	3243
0.9000	2549	2549	2498	2440	2460
0.9333	1860	1871	1835	1735	1904
0.9667	-	1078	-	-	-

TABLE 3 COMPARISON OF SEPARATION POINTS

Mesh Size	y_s	Elliott	Aitchison
18 x 18	0.5338	-	Values in the range *
24 x 24	-	0.5338	
30 x 30	0.5337	-	0.5289 - 0.5426

* The range comes from fitting an analytic expression through r points of the numerical solution. For r = 13 to 16 inclusive y_s is effectively constant at $y_s = 0.5289$. Cryer (1976) quotes $y_s = 0.5297$ evaluated from the analytic solution by Polubarinova-Kochina (1962).

<u>References</u>

Aitchison, J.M., 1972, Proc. R. Soc. <u>A330</u>, 573-580.

Aitchison, J.M., 1977, J. Inst. Maths. Applics., <u>20</u>, 33-44.

Baiocchi, C., 1972, Ann. Mat. Pura Appl., <u>92</u>, 107-127.

Crank, J. and Phahle, R.D., 1973, Bull. Inst. Maths. Applics, <u>9</u>, 12-14.

Crank, J. and Gupta, R.S., 1975, Int. J. Heat Mass Transfer <u>18</u>, 1101-1107.

Crank, J. and Crowley, A.B., 1978, Int. J. Heat Mass Transfer <u>21</u>, 393-398.

Crank, J. and Crowley, A.B., 1979, Int. J. Heat Mass Transfer <u>22</u>, 1331-1337.

Cryer, C.W., 1976, MRC report 1657, Mathematics Research Center, University of Wisconsin.

Elliott, C., 1976, D.Phil, thesis, University of Oxford.

Furzeland, R.M., 1977, Technical Report TR76, Department of Mathematics, Brunel University.

Furzeland, R.M. , 1979, Bull. Inst. Maths. Applics. <u>15</u>, 172-176.

Polubarinova-Kochina, P.Ya. 1962, Theory of Ground Water Movement, Princeton University Press

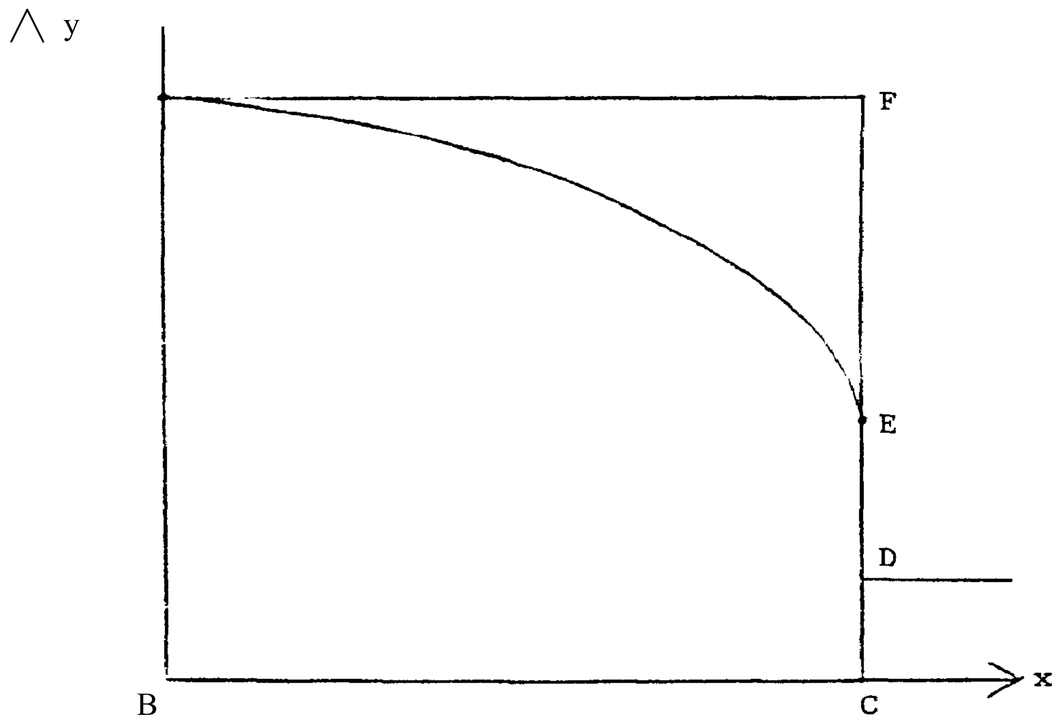


Figure 1. The dam problem.

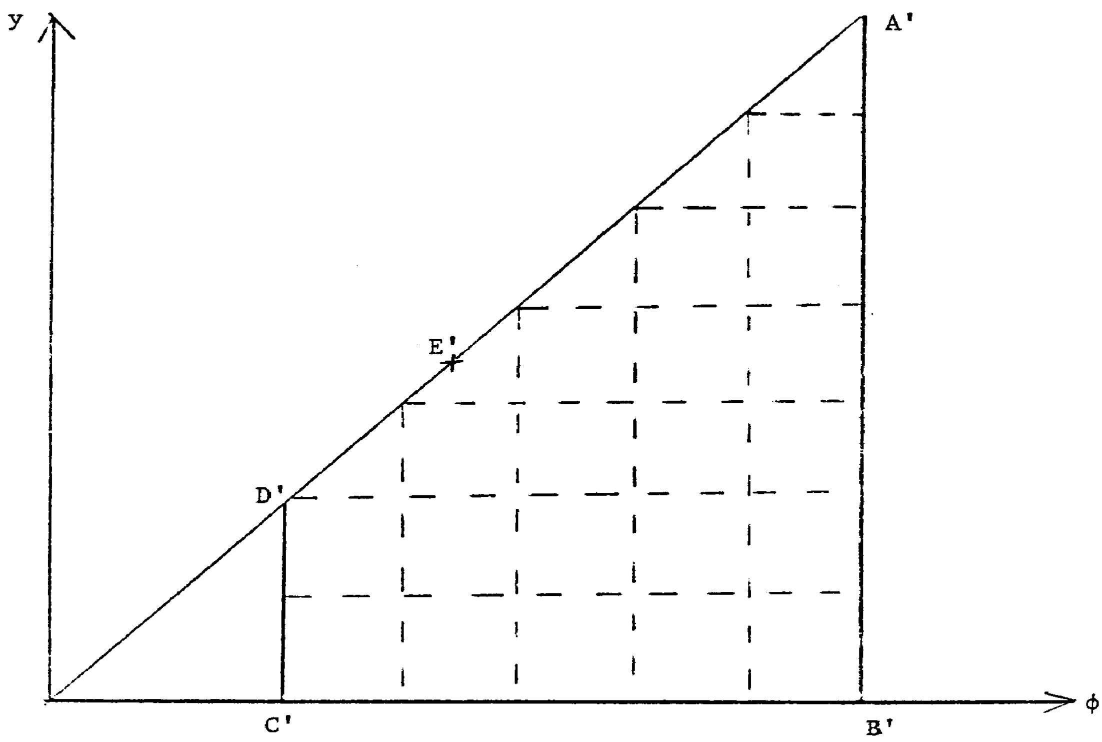
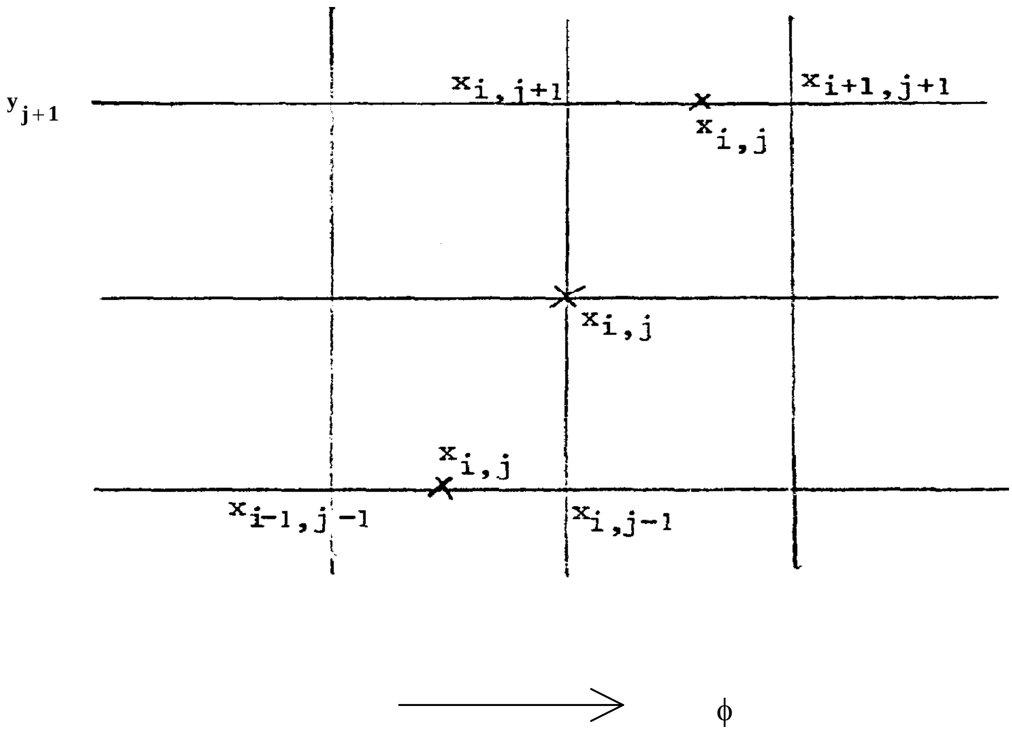


Figure 2. Transformed plane.



<u>Figure 3</u>

Interpolation for points at which $x = x_{i.j}$.

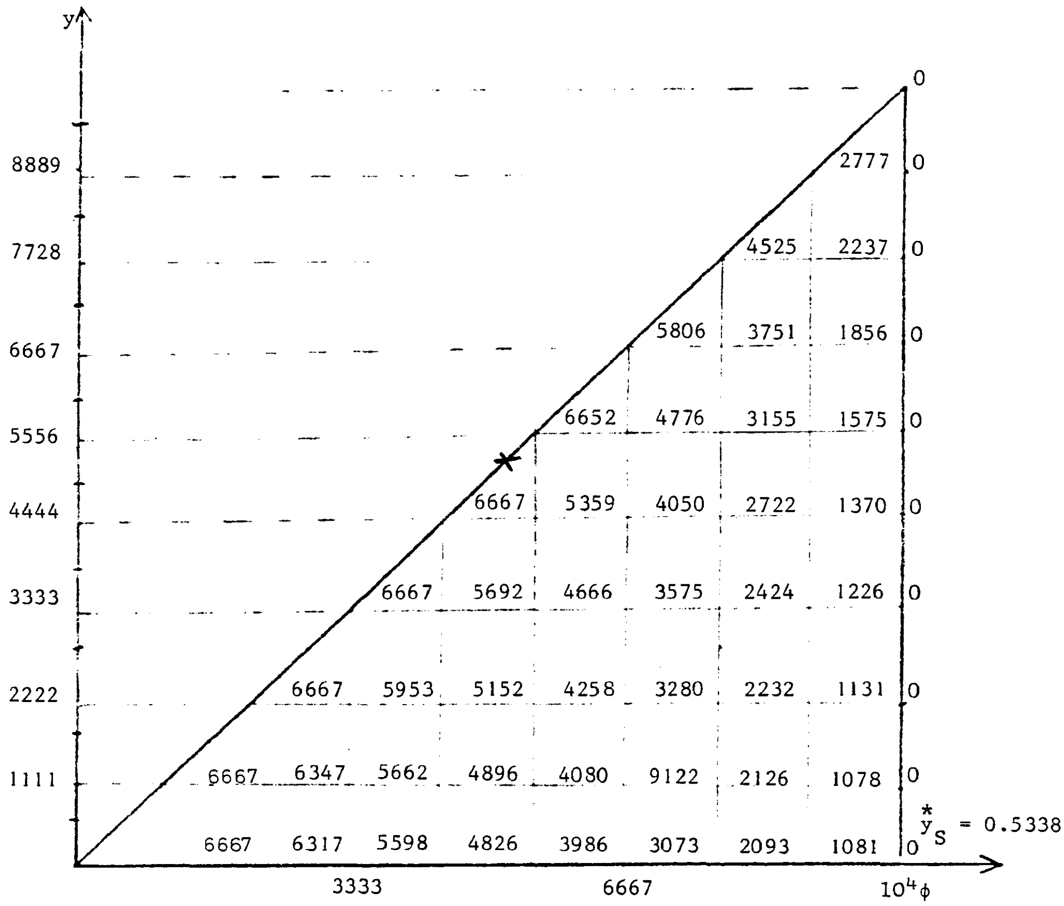


Figure 4. Selected values of 10^4x from 18 x 18 mesh.

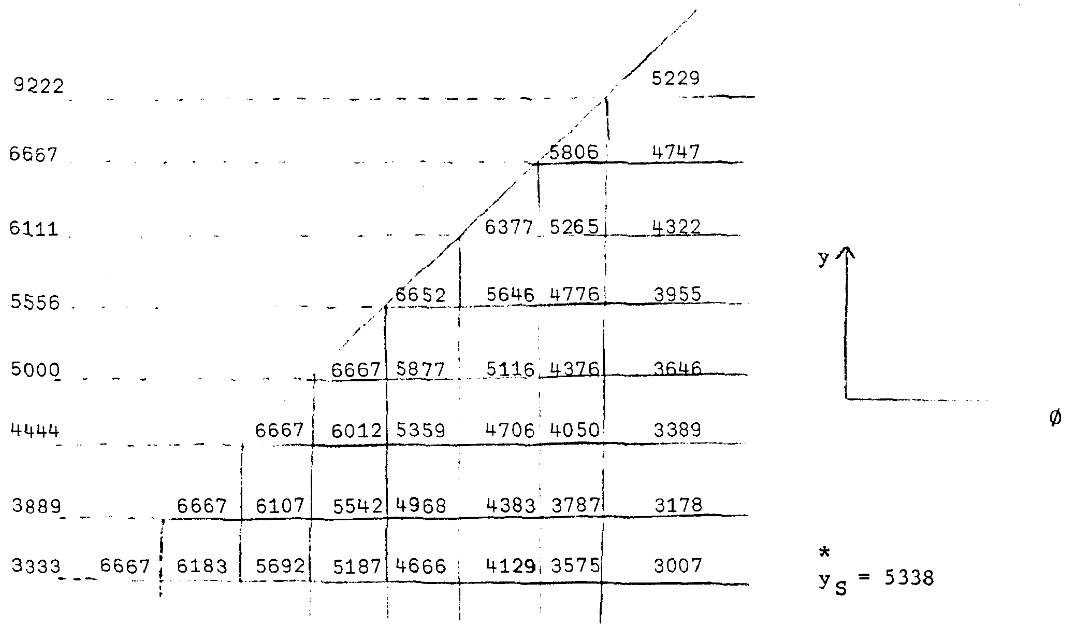


Figure 5. All values of 10^4x on section of 18 x 18 mesh. around separation point.