LINEAR-QUADRATIC TERM STRUCTURE MODELS FOR NEGATIVE EURO AREA YIELDS

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Abstract

Four factor linear-quadratic models (LQTSM) fit negative Euro yields well, as short yields can be negative, but not the longest yields. LQTSM outperform four factor quadratic models that permit negative yields, which in turn outperform affine Gaussian models.

Key words: linear-quadratic term structure models, quadratic models, discrete time, negative yields, Extended Kalman Filter.

JEL classification: G12; G13.

1 Introduction

Until few years ago negative nominal Government bond yields were almost unknown and economic theory ruled them out, since "investors can always hold their cash". Affine Gaussian term structure models (AGTSM) were regularly criticised because they permitted negative bond yields. After the 2008 crisis, academic research even concentrated on term structure models with a zero lower bound, i.e. models that could rule out negative yields while at the same time matching very low observed yields. However as of early 2017 negative yields have been observed for extended periods in Japan, Euroland and elsewhere, which somewhat vindicates AGTSM. Now we need models that can match at least slightly negative yields. In this spirit this paper tests linear-quadratic term structure models (LQTSM) whereby the short rate may turn negative, but not the longest yields, since the central tendency of the factors driving the short

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rate is a quadratic non-negative function of Gaussian factors. This paper also considers quadratic term structure models (QTSM) that do permit negative yields. All tests use four factor models and AAA-rated euro area Government bond yields of maturities up to 30 years.

The evidence shows that LQTSM perform much better than AGTSM and much better than QTSM that rule out negative yields. LQTSM can match the moderately negative yields for maturities up to ten years and at the same time the higher positive yields observed for the longest maturities up to thirty years. Also specifications of QTSM that permit negative yields perform better than AGTSM and better than "classic" QTSM that rule out negative yields, but slightly worse than LQTSM.

This paper tests a LQTSM that builds on Realdon (2011, 2016), who proposed discrete time versions of the continuous time linear-quadratic pricing model of Cheng and Scaillet (2007). The next section presents the pricing models and another section illustrates their empirical performance.

2 Discrete time linear-quadratic term structure models (LQTSM)

This section presents LQTSM in discrete time. Discrete time implies Gaussian conditional transition density for the factors, which helps model estimation through Kalman Filters. LQTSM encompass linear Gaussian models and quadratic models as special cases. Let $V_{n,t}$ be the time t value of a zero coupon bond with n trading days to maturity, thus the bond matures on trading day $t+n$. Each time step is equal to $\Delta = \frac{1}{261}$ as we observe about 261 trading days per year. r_t is the continuously compounded risk-free interest rate for the trading day $[t, t+1]$, therefore $V_{1,t} = e^{-\Delta \cdot r_t}$ and $r_t = \frac{-\ln V_{1,t}}{\Delta}$. Following Realdon (2011) we further assume

$$r_t = \beta' \mathbf{x}_t + \mathbf{x}_t' \mathbf{\Psi} \mathbf{x}_t + \delta' \mathbf{y}_t \tag{1}$$

$$\mathbf{x}_t = (x_{1,t}, .., x_{m,t})', \quad \mathbf{y}_t = (y_{1,t}, .., y_{p,t})' \tag{2}$$

$$\mathbf{x}_{t+1} - \mathbf{x}_t = \phi (\mu - \mathbf{x}_t) + \mathbf{\Sigma} \xi_{t+1}^{\mathbb{Q}} \tag{3}$$

$$\mathbf{x}_{t+1} - \mathbf{x}_t = \phi^* (\mu^* - \mathbf{x}_t) + \mathbf{\Sigma} \xi_{t+1} \tag{4}$$

$$\mathbf{y}_{t+1} - \mathbf{y}_t = \phi_y (\mu_y + diag (\mathbf{x}_t' \mathbf{L}_j \mathbf{x}_t) \cdot \mathbf{1}_{p \times 1} - \mathbf{y}_t) + (\mathbf{\Sigma}_{yx}, \mathbf{\Sigma}_y) \begin{pmatrix} \xi_{t+1}^{\mathbb{Q}} \\ \xi_{y,t+1}^{\mathbb{Q}} \end{pmatrix} \tag{5}$$

$$\mathbf{y}_{t+1} - \mathbf{y}_t = \phi_y^* \left(\mu_y^* + diag (\mathbf{x}_t' \mathbf{L}_j \mathbf{x}_t) \cdot \mathbf{1}_{p \times 1} - \mathbf{y}_t\right) + (\mathbf{\Sigma}_{yx}, \mathbf{\Sigma}_y) \begin{pmatrix} \xi_{t+1} \\ \xi_{y,t+1} \end{pmatrix} \tag{6}$$

$$\left(\xi_{t+1}^{\mathbb{Q}\prime}, \xi_{y,t+1}^{\mathbb{Q}\prime}\right)' \backsim N (\mathbf{0}_{m+p}, \mathbf{I}_{m+p}), \qquad (\xi_{t+1}', \xi_{y,t+1}')' \backsim N (\mathbf{0}_{m+p}, \mathbf{I}_{m+p}) \tag{7}$$

$$\xi_{t+1}^{\mathbb{Q}} = \left(\varepsilon_{1,t+1}^{\mathbb{Q}}, .., \varepsilon_{m,t+1}^{\mathbb{Q}}\right)', \quad \xi_{t+1} = (\varepsilon_{1,t+1}, .., \varepsilon_{m,t+1})' \tag{8}$$

$$\xi_{y,t+1}^{\mathbb{Q}} = \left(\varepsilon_{y,1,t+1}^{\mathbb{Q}}, .., \varepsilon_{y,p,t+1}^{\mathbb{Q}}\right)', \quad \xi_{y,t+1} = (\varepsilon_{y,1,t+1}, .., \varepsilon_{y,p,t+1})' \tag{9}$$

$$\mathbf{\Sigma} = \mathbf{S}\sqrt{\Delta},\ \mathbf{\Sigma}_{yx} = \mathbf{S}_{yx}\sqrt{\Delta},\ \mathbf{\Sigma}_y = \mathbf{S}_y \sqrt{\Delta} \tag{10}$$

$$\phi = \Delta \cdot \kappa,\ \phi^* = \Delta \cdot \kappa^*,\ \phi_y = \Delta \cdot \kappa_y,\ \phi_y^* = \Delta \cdot \kappa_y^*, \tag{11}$$

$$V_{n,t} = \exp (A_n + \mathbf{B}_n' \mathbf{x}_t + \mathbf{x}_t' \mathbf{C}_n \mathbf{x}_t + \mathbf{D}_n' \mathbf{y}_t) \tag{12}$$

where: $diag (\mathbf{x}_t' \mathbf{L}_j \mathbf{x}_t)$ is a diagonal matrix whose j-th diagonal entry is $\mathbf{x}_t' \mathbf{L}_j \mathbf{x}_t$; $\mathbf{L}_j$ is a $m \times m$ matrix and $\mathbf{1}_{p \times 1}$ is a $p \times 1$ matrix whose elements are all equal to 1; $\mathbf{x}_t, \mu, \mu^*, \xi_{t+1}^{\mathbb{Q}}, \xi_{t+1}, \mathbf{B}_n$ are $m \times 1$ vectors**;** $\mathbf{\Psi}, \phi, \phi^*, \kappa, \kappa^*, \mathbf{C}_n, \mathbf{\Sigma}, \mathbf{S}, \mathbf{L}_j$ are $m \times m$ matrices**;** r_t, A_n, A_0 are scalars; $\mathbf{\Sigma}_{yx}, \mathbf{S}_{yx}$ are are $p \times m$ matrixes; $\mathbf{y}_t, \mu_y, \varepsilon_{y,t+1}^{\mathbb{Q}}, \varepsilon_{y,t+1}, \mathbf{D}_n$ are $p \times 1$ vectors; $\phi_y, \phi_y^*, \kappa_y, \kappa_y^*, \mathbf{\Sigma}_y, \mathbf{S}_y$ are $p \times p$ matrixes; $N \left(\mathbf{0}_{(m+p) \times 1}, \mathbf{I}_{m+p}\right)$ denotes the multivariate normal density with mean $\mathbf{0}_{(m+p) \times 1}$ and covariance matrix $\mathbf{I}_{m+p}$**;** $\mathbf{0}_{(m+p) \times 1}$ is a $(m+p) \times 1$ matrix of zeros; $\mathbf{I}_{m+p}$ is the $(m+p) \times (m+p)$ identity matrix; $\varepsilon_{1,t+1}, .., \varepsilon_{m,t+1}, \varepsilon_{y,1,t+1}, .., \varepsilon_{y,p,t+1}$ and $\varepsilon_{1,t+1}^{\mathbb{Q}}, .., \varepsilon_{m,t+1}^{\mathbb{Q}}, \varepsilon_{y,1,t+1}^{\mathbb{Q}}, .., \varepsilon_{y,p,t+1}^{\mathbb{Q}}$ are scalar Gaussian random shocks respectively in the real and risk-neutral measures. The processes of the factors $\mathbf{x}$ and $\mathbf{y}$ are specified under both the real measure and the risk-neutral measure $\mathbb{Q}$. The discount bond value $V_{n,t}$ is exponential linear in $\mathbf{y}_t$ and exponential linear-quadratic in $\mathbf{x}_t$. This discrete time linear-quadratic model is a special case of Realdon (2011), whereby we can compute $A_n, \mathbf{B}_n, \mathbf{C}_n, \mathbf{D}_n$ appearing in 12 as

$$A_n = A_{n-1} + \mathbf{B}_{n-1}' \phi\mu + \mathbf{D}_{n-1}' \phi_y \mu_y + (\phi\mu)' \mathbf{C}_{n-1} \phi\mu + \ln \frac{|\gamma|}{abs\, |\mathbf{\Sigma}|} + \tag{13}$$

$$+ \frac{1}{2} \left(\mathbf{G}_{n-1} + \mathbf{D}_{n-1}' \mathbf{\Sigma}_{yx}' \mathbf{\Sigma}^{-1}\right) \gamma\gamma' \left(\mathbf{G}_{n-1} + \mathbf{D}_{n-1}' \mathbf{\Sigma}_{yx}' \mathbf{\Sigma}^{-1}\right)' + \frac{1}{2} \mathbf{D}_{n-1}' \Sigma_y \Sigma_y' \mathbf{D}_{n-1}$$

$$\mathbf{B}_n' = -\Delta\beta' + \mathbf{B}_{n-1}' (\mathbf{I}_m - \phi) + 2 (\phi\mu)' \mathbf{C}_{n-1} (\mathbf{I}_m - \phi) + \mathbf{D}_{n-1}' \phi_{yx} + 2 \left(\mathbf{G}_{n-1} + \mathbf{D}_{n-1}' \mathbf{\Sigma}_{yx}' \mathbf{\Sigma}^{-1}\right) \gamma\gamma' \mathbf{C}_{n-1}' (\mathbf{I}_m - \phi) \tag{14}$$

$$\mathbf{C}_n = -\Delta\mathbf{\Psi}+(\mathbf{I}_m-\phi)'\,\mathbf{C}_{n-1}\,(\mathbf{I}_m-\phi)+\sum_{j=1}^{p} D_{j,n-1}\cdot\mathbf{L}_j+2\,(\mathbf{I}_m-\phi)'\,\mathbf{C}_{n-1}\gamma\gamma'\mathbf{C}'_{n-1}\,(\mathbf{I}_m-\phi) \tag{15}$$

$$\mathbf{D}_n = -\Delta\delta + (I-\phi_y)'\,\mathbf{D}_{n-1} \tag{16}$$

$$\mathbf{G}_{n-1} = \mathbf{B}'_{n-1} + 2\,(\phi\mu)'\,\mathbf{C}_{n-1}$$

$$\gamma = \left(\left(\mathbf{\Sigma\Sigma}'\right)^{-1} - 2\mathbf{C}_{n-1}\right)^{-1/2}$$

$$A_0 = 0, \quad \mathbf{B}_0 = \mathbf{0}_{m\times 1}, \quad \mathbf{C}_0 = \mathbf{0}_{m\times m}, \quad \mathbf{D}_0 = \mathbf{0}_{p\times 1}.$$

We compute $A_n, \mathbf{B}_n, \mathbf{C}_n, \mathbf{D}_n$ with 261 steps per year and $\Delta = 1/261$. The stochastic factors are latent. In this paper we focus on the following special cases.

2.1 The linear-quadratic model LQ2.2

LQ2.2 is such that $\delta = \mathbf{1}_{2\times 1}$**,** $\mathbf{\Psi} = \mathbf{0}_{2\times 2}$ and $\beta = \mathbf{0}_{2\times 1}$ so that $r_t = y_{1,t} + y_{2,t}$ and

$$\mathbf{S} = \begin{pmatrix} \sigma_1 & 0 \\ \rho_{12}\cdot\sigma_2 & \sqrt{1-\rho_{12}^2}\cdot\sigma_2 \end{pmatrix},\ \mu = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix},\ \mu^* = \begin{pmatrix} \mu_1^* \\ \mu_2^* \end{pmatrix},\ \kappa = \begin{pmatrix} \kappa_{1,1} & 0 \\ \kappa_{2,1} & \kappa_{2,2} \end{pmatrix},\ \kappa^* = \begin{pmatrix} \kappa_{1,1}^* & 0 \\ \kappa_{2,1}^* & \kappa_{2,2}^* \end{pmatrix}$$

$$\kappa_y = \begin{pmatrix} \kappa_{y_1} & 0 \\ 0 & \kappa_{y_2} \end{pmatrix},\ \mu_y = \mu_y^* = \begin{pmatrix} 0 \\ 0 \end{pmatrix},\ diag\left(\mathbf{x}_t'\mathbf{L}_j\mathbf{x}_t\right)\cdot\mathbf{1}_{2\times 1} = \begin{pmatrix} x_{1,t}^2 \\ x_{2,t}^2 \end{pmatrix}$$

$$(\mathbf{S}_{yx}, \mathbf{S}_y) = \begin{pmatrix} \sigma_{y_1}\left(\rho_{1,y_1},\ Q_{2,y_1},\ \sqrt{1-\rho_{1,y_1}^2 - Q_{2,y_1}^2}, 0\right) \\ \sigma_{y_2}\left(\rho_{1,y_2}, Q_{2,y_2}, Q_{y_1,y_2}\ , \sqrt{1-\rho_{1,y_2}^2 - Q_{2,y_2}^2 - Q_{y_1,y_2}^2}\right) \end{pmatrix},\ \begin{pmatrix} \xi_{t+1}^{\mathbb{Q}} \\ \xi_{y,t+1}^{\mathbb{Q}} \end{pmatrix} = \begin{pmatrix} \varepsilon_{1,t+1}^{\mathbb{Q}} \\ \varepsilon_{2,t+1}^{\mathbb{Q}} \\ \varepsilon_{y,1,t+1}^{\mathbb{Q}} \\ \varepsilon_{y,2,t+1}^{\mathbb{Q}} \end{pmatrix}$$

$$Q_{2,y_1} = \frac{\rho_{2,y_1} - \rho_{12}\cdot\rho_{1,y_1}}{\sqrt{1-\rho_{12}^2}},\ Q_{2,y_2} = \frac{\rho_{2,y_2} - \rho_{12}\cdot\rho_{1,y_2}}{\sqrt{1-\rho_{12}^2}},\ Q_{y_1,y_2} = \frac{\rho_{y_1,y_2} - \rho_{1,y_1}\rho_{1,y_2} - Q_{2,y_1}\cdot Q_{2,y_2}}{\sqrt{1-\rho_{1,y_1}^2 - Q_{2,y_1}^2}}.$$

ρ_{12} is the conditional correlation between $x_{1,t+1}$ and $x_{2,t+1}$, ρ_{1,y_1} between $x_{1,t+1}$ and $y_{1,t+1}$, ρ_{2,y_1} between $x_{2,t+1}$ and $y_{1,t+1}$ and ρ_{y_1,y_2} has similar meaning. LQ2.2 is of interest since the factors $y_{1,t}$ and $y_{2,t}$ drive the short interest rate r_t, while longer term yields are driven by $x_{1,t}$ and $x_{2,t}$. Therefore short term and long term yields can move quite independently. $y_{1,t}$ tends to revert toward the level $x_{1,t}^2$ and $y_{2,t}$ tends to revert toward the level $x_{1,t}^2$, therefore long term yields tend to be positive when $\kappa_{y_1}, \kappa_{y_2} > 0$ and $\mu_{y_1}, \mu_{y_2} \geq 0$. In LQ2.2 we impose $\mu_y = \mu_y^* = \mathbf{0}_{2\times 1}, \mu_1 = \mu_2, \mu_1^* = \mu_2^*$ for the sake of parsimony and without much loss. The model can match very low and even negative short term yields and at the same time also positive long term yields. The main contribution of this paper is to provide evidence that LQ2.2 outperforms the other affine or quadratic models.

2.2 The quadratic models: Q4.0 and Q2.2

Q4.0 is a "classic" four factor quadratic model where $\delta = \mathbf{0}_{p\times 1}$, $\beta = \mathbf{0}_{4\times 1}$ and $\mathbf{\Psi} = \mathbf{I}_4$, so that $r_t = x_{1,t}^2 + x_{2,t}^2 + x_{3,t}^2 + x_{4,t}^2$. Q4.0 rules out negative yields. Q4.0 also assumes

$$\mathbf{S} = \begin{pmatrix} \sigma_1 & 0 & 0 & 0 \\ \rho_{12}\cdot\sigma_2 & \sqrt{1-\rho_{12}^2}\cdot\sigma_2 & 0 & 0 \\ \rho_{13}\cdot\sigma_3 & Q_{32}\cdot\sigma_3 & \sqrt{1-\rho_{13}^2-Q_{32}^2}\cdot\sigma_3 & 0 \\ \rho_{14}\cdot\sigma_4 & Q_{42}\cdot\sigma_4 & Q_{43}\cdot\sigma_4 & \sqrt{1-\rho_{14}^2-Q_{42}^2-Q_{43}^2}\cdot\sigma_4 \end{pmatrix}$$

$$Q_{32} = \frac{\rho_{32}-\rho_{12}\cdot\rho_{13}}{\sqrt{1-\rho_{12}^2}}, \quad Q_{42} = \frac{\rho_{42}-\rho_{12}\cdot\rho_{14}}{\sqrt{1-\rho_{12}^2}}, \quad Q_{43} = \frac{\rho_{43}-\rho_{13}\rho_{14}-Q_{32}\cdot Q_{42}}{\sqrt{1-\rho_{13}^2-Q_{32}^2}} \tag{17}$$

$$\mu = \begin{pmatrix} \mu_1 \\ \mu_1 \\ 0 \\ 0 \end{pmatrix}, \quad \mu^* = \begin{pmatrix} \mu_1^* \\ \mu_1^* \\ 0 \\ 0 \end{pmatrix}, \quad \kappa = \begin{pmatrix} \kappa_{1,1} & 0 & 0 & 0 \\ \kappa_{2,1} & \kappa_{2,2} & 0 & 0 \\ 0 & 0 & \kappa_{3,3} & 0 \\ 0 & 0 & 0 & \kappa_{4,4} \end{pmatrix}, \quad \kappa^* = \begin{pmatrix} \kappa_{1,1}^* & 0 & 0 & 0 \\ \kappa_{2,1}^* & \kappa_{2,2}^* & 0 & 0 \\ 0 & 0 & \kappa_{3,3}^* & 0 \\ 0 & 0 & 0 & \kappa_{4,4}^* \end{pmatrix}.$$

ρ_{12} is the conditional correlation between $x_{1,t+1}$ and $x_{2,t+1}$. ρ_{13}, ρ_{23}, ρ_{41}, ρ_{42}, ρ_{43} have similar meaning. $\sigma_1, \sigma_2, \sigma_3, \sigma_4$ are volatility parameters. The parameter restrictions in $\mu, \mu^*, \kappa, \kappa^*$ ensure that Q4.0 has a factor structure similar to LQ2.2 and the same number of "free" parameters as LQ2.2. Except for $\kappa_{2,1}, \kappa_{2,1}^* \neq 0$ Q4.0 corresponds to the quadratic model canonical form in Ahn, Dittmar and Gallant (2002), whereby $\mathbf{\Psi} = \mathbf{I}_4$, $\kappa\mu \geq 0$, $\kappa^*\mu^* \geq 0$, $\mathbf{S}$ is triangular while κ, κ^* are diagonal. The empirical evidence shows that Q4.0 clearly underperforms LQ2.2, i.e. a "classic" four factor quadratic model underperforms the four factor linear-quadratic model.

All else as in Q4.0, Q2.2 is another four factor quadratic model where $\delta = \mathbf{0}_{p\times 1}$, $\beta = (0,0,1,1)'$ and $\mathbf{\Psi} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, so that $r_t = x_{1,t}^2 + x_{2,t}^2 + x_{3,t} + x_{4,t}$ and r_t may turn negative. Q2.2 has the same parameter restrictions and the same number of parameters as Q4.0. The empirical evidence shows that Q2.2 clearly outperforms Q4.0, as the ruling out negative yields by Q4.0 becomes a shortcoming rather than a strength in recent years. However Q2.2 performs worse than LQ2.2. The four factor linear-quadratic model seems preferable even to the four factor quadratic model that permits negative yields.

2.3 The affine model A4

A4 is a four factor affine Gaussian model, which in unreported tests performed best across various four factor affine models we tested on our sample. A4 is such that $\delta = \mathbf{0}_{p\times 1}$, $\beta = (1,1,1,1)'$, $\mathbf{\Psi} = \mathbf{0}_{4\times 4}$ and

$r_t = x_{1,t} + x_{2,t} + x_{3,t} + x_{4,t}$

$$\mu = \begin{pmatrix} \mu_1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \mu^* = \begin{pmatrix} \mu_1^* \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \kappa = \begin{pmatrix} \kappa_1 & 0 & 0 & 0 \\ -\kappa_2 & \kappa_2 & 0 & 0 \\ 0 & -\kappa_3 & \kappa_3 & 0 \\ 0 & 0 & -\kappa_4 & \kappa_4 \end{pmatrix}, \quad \kappa^* = \begin{pmatrix} \kappa_1^* & 0 & 0 & 0 \\ -\kappa_2^* & \kappa_2^* & 0 & 0 \\ 0 & -\kappa_3^* & \kappa_3^* & 0 \\ 0 & 0 & -\kappa_4^* & \kappa_4^* \end{pmatrix}.$$

The empirical evidence below indeed shows that the four factor linear-quadratic model clearly outperforms also A4.

3 Empirical tests

The sample comprises daily yields for AAA-rated Euro area Government bonds for maturities from one year up to thirty years, i.e. thirty yields for each trading day. The yields cover $2,991$ trading days from 6-9-2004 to 12-5-2016 and can be downloaded from the European Central Bank's web-site. To compute such yields, the ECB employs a Svensson-type interpolation to market prices of AAA-rated Euro area Government bonds. In this paper the "in sample" period, used to estimate parameters, is from 6-9-2004 to 8-7-2010 and is made up of $1,496$ trading days. The "out of sample" period from 9-7-2010 to 12-5-2016 is made up $1,495$ trading days and is used to test each model after the model parameters are estimated "in sample". We also provide alternative "in sample" parameter estimates using the whole sample of $2,991$ days.

Table 1 presents summary statistics of observed yields during the two periods. In the second period average yields and yield standard deviations are lower and even persistently negative short term yields are observed. Therefore we need term structure models that predict time varying yield volatilities, negative short term yields and higher longer term yields. LQTSM have these features.

[TABLE 1]

For estimating all models we employ Quasi-Maximum-Likelihood estimation through the Extended Kalman Filter (EKF) and use yields of all maturities from one year to thirty years. The quasi-log-likelihood function lk is maximised with the Nelder-Mead simplex method. h_j for $j = 1,..,30$ denotes the standard deviation of the observation errors for the j years' yield. As in Sarno, Schneider and Wagner (2016) we assume $h_j = e^{-\left(c_0 + j\cdot c_1 + j^2\cdot c_2\right)}$ where c_0, c_1, c_2 are constants to be estimated. To avoid arbitrary prior probability densities, the time $t = 0$ latent factors $\mathbf{x}_0$ and $\mathbf{y}_0$, which the Kalman Filter requires, are treated as parameters to be estimated.

Table 2 presents the estimation results. The BHHH estimator provides the estimates of the standard deviations of the parameter estimates. $RMSE_j$ is the root mean square error for the j year yield. The errors are the daily differences between observed yields and model predicted yields of any given maturity. For each model $Avg\ RMSE = \frac{\sum_{j=1}^{30} RMSE_j}{30}$, lk is the value of the log-likelihood function and AIC the value of the Akaike information criterion. $RMSE_j$ show that all models match short term yields, such as one-year yields, less well than yields of longer maturities, as is typical in the literature.

[TABLE 2]

With $AIC = -708.307$ for the first 1.496 days and $AIC = -1.396.223$ for the whole sample, LQ2.2 fits yields better than any other model, followed by the quadratic model that permits negative yields Q2.2 with $AIC = -703.234$ for the first 1.496 days and $AIC = -1.374.986$ for the whole sample. Unreported tests showed that LQ2.2 beats Q2.2 even as we relax some of the parameter constraints of Q2.2. The estimates of κ_{y_1} and κ_{y_2} for LQ2.2 are positive and significant, implying that $y_{1,t}$ and $y_{2,t}$ tend to revert respectively to $x_{1,t}^2$ and $x_{2,t}^2$ under the $\mathbb{Q}$ measure, as expected. The "classic" quadratic model Q4.0, which rules out negative yields, is the worst performer. With $Avg\ RMSE = 0.0059$ Q4.0 is especially disappointing in the out of sample period from 9-7-2010 to 12-5-2016, in which observed yields are lower or negative. The ruling out of negative yields is now a shortcoming, not a strength of quadratic models. The affine model A4 beats Q4.0 but performs clearly worse than LQ2.2 and Q2.2. Both AIC and $RMSE$ support the following ranking: first LQ2.2, second Q2.2, third A4 and fourth Q4.0. LQ2.2, Q2.2 and Q4.0 all have 29 parameters, while A4 has 27 parameters. LQ2.2 allows negative yields up ten years, but non-negative long term yields, such as 30 year yields, which is consistent with the data. In fact 30 year yields have never been negative. Linear quadratic models seem preferable even to quadratic models that permit negative yields.

4 Conclusion

This paper has tested discrete time linear-quadratic, affine and quadratic Gaussian term structure models using Euro area AAA rated Government bond yields. The linear-quadratic models outperform affine Gaussian models as well as "classic" quadratic models that rule out negative yields. Quadratic models that permit negative yields perform slightly worse than the linear-quadratic models, but much better than quadratic models that do rule out negative yields.

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