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The numerical solution of the wave equation at the first time step

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Abstract

Superimposing a uniformgrid on the space variable in the simple wave equation, for which boundary conditions, initial displacement and initial velocity are specified, leads to the numerical solution of a system of second order ordinary differential equations.

Before the numerical solution of this system can proceed, the displacement at the first time level must be estimated from the initial conditions. The accuracy of this starting value must be at least that of the method to be used at higher time levels.

In the present note, approximations to the displacement with second, fourth and sixth order accuracy are determined. The procedure followed can be used to obtain yet higher accuracy by choosing other approximations to the initial velocity.

Key Words and Phrases

Wave equation Second order system Starting values

Pade approximants Accuracy

In the numerical solution of the simple wave equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} \quad ; \qquad 0 < x < 1, \ t > 0 \tag{0}$$

with boundary conditions

$$u(0,t) = u(l, t) = 0 \ ; \ t > 0 \tag{1}$$

and initial conditions

$$u(x,0) = f(f) \quad ; \quad \frac{\partial u}{\partial t}(x,0) = g(x) \ , \tag{2}$$

replacing the space derivative with a suitable finite difference approximation, and applying equation (0) to N points on $0 < x < 1$ at time t, leads to a system of second order ordinary differential equations of the form

$$\frac{d^2 \underset{\sim}{U}(t)}{dt^2} = A\underset{\sim}{U}(t) \ . \tag{3}$$

In (3), the constant matrix A is of order N and depends on the finite difference approximation, and $\underset{\sim}{U}(t)$ is the vector of order N whose elements are the computed values of the solution at time t.

It was shown in [1] that the solution of system (3) satisfies the recurrence relation

$$\underset{\sim}{U}(t-\ell) - \{\exp(-\ell B) + \exp(\ell B)\} \underset{\sim}{U}(t) + \underset{\sim}{U}(t+\ell) = \underset{\sim}{o} \ , \tag{4}$$

where ℓ is a constant increment in t and B is a matrix of order N such that $B^2 = A$. Equation (4) is applied with $t = \ell, 2\ell ..$ and the accuracy of the computed solution $\underset{\sim}{U}$ depends on the chosen approximation to the exponential matrix functions $\exp(\pm\ell B)$, Choosing the (m,k) Pade approximant to $\exp(\ell B)$ results in an error $0(\ell^{m+k+l})$ for m+k odd and $0(\ell^{m+k+2})$ for m+k even ; the resulting algorithm is explicit for $m = 0$ and implicit for $m \neq 0$.

The order of the error can thus be made arbitrarily high (at least theoretically) by choosing m+k to be sufficiently high. The remaining difficulty then is to find a sufficiently accurate approximation to $\underset{\sim}{U}(\ell)$ which is not contained explicitly in the initial conditions. It is the purpose of the present note to derive approximations to $\underset{\sim}{U}(\ell)$, in terms of the initial vectors $\underset{\sim}{f}$ and $\underset{\sim}{g}$ with errors $0(\ell^3)$, $0(\ell^5)$ and $0(\ell^7)$. Errors of the same orders will then result in $\underset{\sim}{U}(t)$ for all $t = 2\ell, 3\ell, \ldots$ when equation (4) is used to compute the solution.

The derivations are as follows:

(a) Taking t = 0 in equation (4) leads to

$$\underset{\sim}{U}(-\ell) - \{\exp(\ell B) + \exp(\ell B)\}\underset{\sim}{f} + \underset{\sim}{U}(\ell) = \underset{\sim}{o} \tag{5}$$

and using the approximation

$$\underset{\sim}{g} = \{\underset{\sim}{U}(\ell) - \underset{\sim}{U}(-\ell)\}/2\ell + 0(\ell^2) \tag{6}$$

in (5) gives [1]

$$\underset{\sim}{U}(\ell) = (I + \frac{1}{2}\ell^2 A)\underset{\sim}{f} + \ell\underset{\sim}{g} + 0(\ell^3) \quad , \tag{7}$$

where the first three terms of the Taylor expansion have been used to approximate the exponential function. Equation (7) gives $\underset{\sim}{U}(\ell)$ explicitly to second order accuracy,

(b) Writing equation (4) with t = 0 with a double increment 2ℓ gives

$$\underset{\sim}{U}(-2\ell) - \{\exp(2\ell B) + \exp(2\ell B)\}\underset{\sim}{f} + \underset{\sim}{U}(2\ell) = \underset{\sim}{0} \tag{8}$$

Combining equations (5) and (8) by subtracting 8*(5) from (8) and using the approximation

$$\underset{\sim}{g} = \{\underset{\sim}{U}(-2\ell) - 8\underset{\sim}{U}(-\ell) + 8\underset{\sim}{U}(\ell) - \underset{\sim}{U}(2\ell)\}/12\ell + 0(\ell^4) \tag{9}$$

gives

$$\underset{\sim}{U}(2\ell) - 8\underset{\sim}{U}(\ell) = \frac{1}{2}\{\exp(-2\ell B) - \exp(2\ell B) - 8\exp(-\ell B) - 8\exp(\ell B)\}\underset{\sim}{f} - 6\ell\underset{\sim}{g} + 0(\ell^5) \ . \tag{10}$$

Now, writing $t = \ell$ in equation (4) gives

$$\underset{\sim}{U}(2\ell) = \text{fexp}(-\ell B) \;+\; \exp(\ell B) \;+ \exp(\ell B) \;\}\; \underset{\sim}{U}(\ell) \;-\underset{\sim}{f} \tag{11}$$

(11)and substituting (11) in (10) leads to

$$\begin{aligned}\{\exp(-\ell B) + \exp(\ell B) + 81\}\, \underset{\sim}{U}(\ell) &= \frac{1}{2}\{\exp(-2\ell B) - 8\exp(-\ell B) \\ &= 8\exp(\ell B) + 21\}\, \underset{\sim}{f} - 6\ell\, \underset{\sim}{g} + 0(\ell 5)\end{aligned} \tag{12}$$

Using the first five terms of the Taylor expansion to approximate the exponential matrix functions in equation (12) leads to

$$(I - \frac{1}{6}\ell^2 A - \frac{1}{2}\ell^4 A^2)\,\underset{\sim}{U}(\ell) \;-\; (I - \frac{1}{3}\ell^2 A - \frac{1}{2}\ell^4 A^2)\,\underset{\sim}{f} \;+\; \ell\,\underset{\sim}{g} \;+ 0(\ell^5)\;. \tag{13}$$

which may be solved implicitly to give $\underset{\sim}{U}(\ell)$ to fourth order accuracy,

(c) Writing equation (4) with t = 0 over a triple interval 3ℓ gives

$$\underset{\sim}{U}(-3\ell) \;-\; \{\exp(-3\ell B) \;+\; \exp(3\ell B)\,\}\underset{\sim}{f} \;+\; \underset{\sim}{U}(3\ell) \;=\; \underset{\sim}{0}\;. \tag{14}$$

Combining equations (5), (8) and (14) by subtracting (14) from 9*(8) and then subtracting 45*(5), and using the approximation

$$\underset{\sim}{g} \;=\; \{-\underset{\sim}{U}(-3\ell) + 9\,\underset{\sim}{U}(-2\ell) - 45\,\underset{\sim}{U}(-\ell) + 45\,\underset{\sim}{U}(\ell) - 9\,\underset{\sim}{U}(2\ell) + \underset{\sim}{U}(3\ell)\;\}/60\ell \;+\; 0(\ell^6) \tag{15}$$

gives

$$\begin{aligned}\underset{\sim}{U}(3\ell) \;-9\,\underset{\sim}{U}(2\ell) \;+\; 45\,\underset{\sim}{U}(\ell) \;=\; &\frac{1}{2}\{\exp(-3\ell B) + \exp(3\ell B) - 9\exp(-2\ell B) - 9\exp(2\ell B) \\ &+ 45\exp(-\ell B) + 45\exp(\ell B)\,\}\underset{\sim}{f} + 30\ell\,\underset{\sim}{g} \;+\; 0(\ell^7)\end{aligned} \tag{16}$$

Writing t = *2 ℓ* in equation (4) and using equation (11) gives

$$\begin{aligned}&[\,\{\exp(-\ell B) \;+\; \exp(\ell B)\;\}^2 \;-9\{\exp(-\ell B) + \exp(\ell B)\,\} \;+\; 441]\,\underset{\sim}{U}(\ell) \\ &= \frac{1}{2}\,\{\exp(-3\ell B) + \exp(3\ell B) - 9\exp(-2\ell B) \;-\; 9\exp(2\ell B) \\ &\quad + 47\exp(-\ell B) \;+\; 47\exp(\ell B) \;-\; 181\}\underset{\sim}{f} \;+\; 30\ell\,\underset{\sim}{g} \;+\; 0(\ell^7)\end{aligned} \tag{17}$$

Approximating the matrix exponential functions in (17) with the first seven terms of the Taylor expansion leads to

$$\begin{aligned}&(I - \frac{1}{6}\ell^2 A + \frac{7}{360}\ell^4 A^2 \;+\; \frac{11}{2160}\ell^2 A^3)\,\underset{\sim}{U}(\ell) \\ &= (I + \frac{1}{3}\ell^3 A \;-\; \frac{1}{45}\ell^4 A^2 \;+\; \frac{1}{108}\ell^6 A^3)\,\underset{\sim}{f} \;+\; \ell\,\underset{\sim}{g} \;+\; 0(\ell^7)\end{aligned} \tag{18}$$

which may be solved implicitly to give $\underset{\sim}{U}(\ell)$ to sixth order accuracy.

Reference

E.H. Twizell, An explicit difference method for the wave equation with extended stability range, BIT, 19(3) (1979), pp. 378.383.