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ISOTHERM MIGRATION ALONG ORTHOGONAL FLOW LINES IN TWO DIMENSIONS

by

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ABSTRACT

A novel approach to the solution of transient heat flow problems in two dimensions is described. The movements of isotherms along orthogonal flow lines are tracked in successive small intervals of time by solving a locally one-dimensional IMM form of radial heat equation. The determination of the new orientation of the orthogonal system at the end of each time interval is based on geometrical considerations.

The method is applied to the moving boundary problem presented by the solidification of a square prism of fluid and the results compared with those obtained by the enthalpy method.

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1. INTRODUCTION

When a temperature gradient exists in an isotropic medium flow of heat always takes place along the normal to an isotherm. Thus in a two-dimensional region the flow lines comprise a set of curves orthogonal to the set of isotherms. Traditionally the partial differential equation of heat flow and its solutions express the temperature distribution throughout a given region and the way in which it changes with time. In analogous fluid flow problems this is what is called the Eulerian viewpoint. Recently the heat flow equations have been written in a form which concentrates attention on the movement of the isotherms, Dix and Cizek [1] ,Crank and Phahle [2]. The Isotherm Migration Method (IMM) tracks the movements through the medium of individual points on the isotherms. Thus it is analogous to the Lagrangian formulation of problems in fluid flow. Several authors have explored the computational advantages of using various curvilinear coordinate systems mainly in fluid flow problems in arbitrarily shaped regions.

One group of papers is concerned with the generation of a curvilinear coordinate system with coordinate lines coincident with all boundaries in the physical problem. The advantage of the IMM formulation of the type of problem discussed in this paper is that it yields just such a coordinate system. A general method of generating an appropriate grid is to let the curvilinear coordinates be solutions of subsidiary elliptic partial differential equations in the physical plane with Dirichlet conditions on all the boundaries. The curvilinear system is not necessarily orthogonal but its advocates claim that the freedom for arbitrary spacing of the grid lines around the boundary is more important than orthogonality. Following a pioneer paper by Winslow [3] a succession of authors have proposed various ways of using a curvilinear grid. A useful list of references and an extension to multiconnected regions with any number of arbitrarily shaped bodies or holes is given by Thompson, Thames and Mastin [4]. Oberkampf [5] discusses some useful generalised mapping functions.

In the two-dimensional version of the IMM described by Crank and Gupta [6] the movements of isotherms in the y direction along equally spaced lines of constant x, are computed.
This method can be regarded as a particular case of the use of curvilinear coordinates in which one of the new coordinates is identified with temperature, u, and the other is the x coordinate in the original physical plane. The original y coordinate becomes the new dependent variable to be computed on the rectangular u, x grid. Thus some, but not all of the advantages of a general curvilinear grid are reaped in this IMM approach without the need to solve subsidiary elliptic equations. The same idea is used by Boadway [7] in solving Laplace's equation in irrotational flow.
The present paper is closer in concept to the approach of Potter and Tuttle [8] who refer to the advantages of maintaining an orthogonal grid when a two-dimensional problem is formulated in Lagrangian terms. They describe a method of constructing generalised discrete orthogonal coordinates, suitable for digital computation and which may be applied at each timestep to the Lagrangian solution of multidimensional initial value problems. We describe here a simpler method of solving two-dimensional heat flow problems using the orthogonal system of isotherms and flow lines. The motions of points on the isotherms along the orthogonal flow lines are computed. The solution proceeds in small steps Δt in time and for each interval Δt the partial differential equation to be solved contains only one independent variable other than time t. The changing shape and orientation of the orthogonal system are catered for by a geometric procedure.
As an example the method is used to study the solidification of a square prism of fluid.

2. IMM FORMULATION OF HEAT FLOW IN CYLINDRICAL COORDINATES

In non-dimensional form the partial differential equation describing heat flow in cylindrical coordinates (r,θ) may be written

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} \qquad (1)$$

where u denotes temperature and the thermal properties are

assumed constant. The radial coordinate r is measured from an origin which remains fixed as time t changes. Following Dix and Cizek [1] the change of dependent variable $u(r,\theta,t)$ to $r(u,\theta,t)$ leads to the equation

$$\frac{\partial r}{\partial t} = \left(\frac{\partial^2 r}{\partial u^2}\right) \left(\frac{\partial r}{\partial u}\right)^2 - \frac{1}{r} - \frac{1}{r^2} \left(\frac{\partial r}{\partial u}\right) \frac{\partial^2 u}{\partial \theta^2}. \qquad (2)$$

This radial form of the TMM equation allows the movements of isotherms to be calculated instead of the more conventional variations of temperature at fixed points.

Consider now a general heat flow problem in two space dimensions. There will be a family of isotherms and an associated family of flow lines, orthogonal to the isotherms. In an isotropic medium, any point on an isotherm moves along the flow lines, normal to the isotherm at that point. Heat flow is everywhere normal to the isotherms and never across flow lines. Provided we confine attention to a small segment of an isotherm for a short interval of time, we can regard the isotherm element as part of a cylindrical system and identify the coordinate r in (2) as the local radius of curvature of the isotherm measured from the local centre of curvature assumed fixed in its position at time t. Equation (2) yields the velocity dr/dt of the selected element of the isotherm along the normal to itself.

Because the general system is distorting and rotating, both the centre of curvature for the element of the isotherm considered and the curvature itself may change with time as well as from point to point in the system. The flow lines will not strictly be radial lines of constant θ and the local isotherms will not be exactly concentric circular arcs. But because r in (2) is chosen to be along the local normal, the term $\partial^2 u/\partial\theta^2$ will in general be small but non-zero. In this paper we approximate it to zero in order to calculate the movement of a point on the isotherm in the direction of its normal in the time interval Δt, i.e. we solve the equation

$$\frac{\partial r}{\partial t} = \left(\frac{\partial^2 r}{\partial u^2}\right) \Big/ \left(\frac{\partial r}{\partial u}\right)^2 - \frac{1}{r}. \qquad (3)$$

We then carry out a separate calculation of the new direction of the normal, at the end of the same interval Δt.

We shall describe a geometrical derivation of the normal direction before proceeding with a numerical solution of (3).

3. CURVATURE AND NORMAL DIRECTION

Fig.1. shows a section of an isotherm which we approximate by a circular arc ABC that is concave downwards and to which we assign a positive curvature. The tangents at the midpoints P,Q of each of the arcs AB, BC are parallel to the corresponding chords AB, BC. Thus the change in the direction of the tangent along the arc PBQ is given by $\psi_m - \psi_{m+1}$ where the ψ's are the angles made by the perpendicular bisectors of the chords AB and BC respectively with the x axis as in Fig.1. The arc length PBQ may be approximated by $\frac{1}{2}(s_{m+1} + s_m)$ where s_m denotes the length of the chord AB, labelled chord m. Then the radius of curvature r_m at the point $B(x_m, y_m)$ may be written

$$r_m = \left(\frac{\partial s}{\partial \psi}\right)_m = \frac{1}{2}\left(\frac{s_m + s_{m+1}}{\psi_m - \psi_{m+1}}\right). \qquad (4)$$

Regarding BQ as a circular arc of radius $|r_m|$ and length s_{m+1} we approximate the angle marked γ_m on Fig.1 as $\frac{1}{2}s_{m+1}/|r_m|$ and hence the direction of the radius at B, normal to the isotherm is specified by the angle θ_m where

$$\theta_m = \psi_{m+1} + \gamma_m \,\mathrm{sign}(r_m) = \psi_{m+1} + \frac{s_{m+1}}{2r_m}. \qquad (5)$$

4. NUMERICAL METHOD

Consider three adjacent isotherms as in Fig. 2. on which the temperatures are $(j-1)\,\delta u$, $j\delta u$ and $(j+1)\,\delta u$ as indicated and ABC are three points whose coordinates are known on isotherm $j\delta u$.

The points G and F are found by the intersections of the radius r_m at B with the chords approximating the isotherms $(j-1)\,\delta u$ and

$(j+1)\delta u$. Let $n_m = |r_m|$ be the distance of B from the centre of curvature of the arc ABC and let n_m^+ and n_m^- denote the distances of F and G from this same centre of curvature respectively. Then

$$\left(\frac{\partial n}{\partial u}\right)_B = \frac{n_m^+ - n_m^-}{2\delta u} \tag{6}$$

and

$$\left(\frac{\partial^2 n}{\partial u^2}\right)_B = \frac{n_m^+ - 2n_m + n_m^-}{(\delta u)^2} \quad . \tag{7}$$

Thus if we denote by $\Delta n_{j,m}$ the movement of the point m on the isotherm $j\delta u$ along the normal in a small interval Δt, we may replace (4) explicitly by

$$\Delta n_{j,m} = 4\Delta t \frac{\left(n_m^+ - 2n_m + n_m^-\right)}{\left(n_m^+ - n_m^-\right)^2} - \frac{\Delta t}{n_m}. \tag{8}$$

If the coordinates of the point m on the isotherm $j\delta u$ at time $i\Delta t$ are denoted by $x_{j,m}^i, y_{j,m}^i$, then the new coordinates at time $(i+1)\Delta t$ are

$$x_{j,m}^{i+1} = x_{j,m}^i + \Delta n_{j,m}^i \cos\theta_{j,m}^i \ , \tag{9}$$

$$y_{j,m}^{i+1} = y_{j,m}^i + \Delta n_{j,m}^i \sin\theta_{j,m}^i . \tag{10}$$

where $\Delta n_{j,m}^i$ is given by (8) and $\theta_{j,m}^i$ by (5).

Thus if the initial data in a two-dimensional heat flow problem are given as the coordinates of a set of points on each of a number of isotherms we can use the above numerical method to advance each point in a succession of small time steps Δt. Points on and near the boundary will need special treatment depending on the nature of the boundary conditions.

5. MOVING BOUNDARIES

In many problems of practical importance, one or more conditions are specified on boundaries which move through the medium. They include problems of melting and freezing in which a moving interface separates the liquid from the solid phase. The temperature on the interface remains constant at the melting temperature u_0. A second boundary condition expresses the heat balance at the interface and determines its velocity. If the phase-change surface is $S(\underline{r},t) = 0$ these conditions may be expressed as

$$u = u_O \text{ on } S, \tag{11}$$

and

$$L\frac{\partial S}{\partial t} = \nabla u.\nabla S\Big|_+ - \nabla u.\nabla S\Big|_- \tag{12}$$

Where $\Big|_+\Big(\Big|_-\Big)$ denotes the limit as S is approached from the liquid (solid) phase.

Problems of this general kind are described as Stefan problems [9]. Early work was largely confined to problems in one space dimension but recently more attention has been given to two and three space dimensions [10] [11] [12] [13]. The IMM method is essentially concerned with the tracking of isotherms through a medium and if the phase-change boundary is itself an isothermal surface, as it commonly is, no special problems arise in calculating its motion except the necessary conversion of the melting condition to IMM form. If n is the local radial coordinate, measuring distance from the centre of curvature of S, (12) becomes just

$$L\frac{\partial n}{\partial t} = \left[\frac{\partial n}{\partial u}\right]^{-1}\Bigg|_- - \left[\frac{\partial n}{\partial u}\right]^{-1}\Bigg|_+ \tag{13}$$

on the isotherm $u = u_0$. (13) is then used in place of (3) to calculate the movement of the freezing front $u = u_0$. One version of IMM has been described and applied to a two-dimensional Stefan problem [6]. We now illustrate the use of the present version by applying it to the solidification of a prism of fluid.

6. EXAMPLE: SOLIDIFICATION OF A SQUARE PRISM OF FLUID

An infinitely long prism is initially filled with a fluid at the fusion temperature u=1. The temperature on its surface is subsequently maintained constant at u =0, below the fusion temperature, so that inwards solidification occurs. The prism extends between $-1 \le x \le 1$, $-1 \le y \le 1$. Assuming thermal properties to be constant, we require a solution of the equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial t^2} + \frac{\partial^2 u}{\partial y^2}, \tag{14}$$

subject to the boundary conditions

$$u = 0 \text{ on } (x-1)(y-1) = 0, \tag{15}$$

$$\partial u/\partial x = 0 \quad \text{on} \quad x = 0; \partial u/\partial y = 0 \text{ on } y = 0, \tag{16}$$

$$u = 1 \text{ and } L\partial S/\partial t = -\nabla u.\nabla S \Big|_{-} \text{ on } S(x,y,t) = 0. \tag{17}$$

In order to illustrate the method we shall start to apply it at a time t_o when an outer layer of the prism has already solidified and there is a temperature distribution in the solid. The coordinates of selected points on a number of isotherms at constant intervals of temperature were interpolated from a numerical solution obtained previously by the enthalpy method [13]. Because of symmetry about the diagonal y =x it is sufficient to work in the triangular region $0 \le x \le y, 0 \le y \le 1$.

We may apply the procedure of §4 to calculate new positions for all interior points at time $t_0 + \Delta t$.

Successive steps in the calculation are:

i) Evaluate at each point the direction of the normal to the isotherm and the local curvature using (4) and (5) at time $t = t_o$.

ii) Evaluate the normal movement of each point and hence its new coordinates at $t_o + \Delta t$ using (8), (9), (10).

iii) Proceed similarly from $t_o + \Delta t$ to $t_o + i\Delta t$ in successive time steps Δt.

The end points on the boundaries x = 0 and y =x need special consideration. A circle, with its centre on the axis at $(0, y_o)$ or on the diagonal at (a.a) as appropriate, is fitted through the end point and the next

point inside the region on each isotherm to find the curvatures. The angles $\theta_{j,O}$ on the y axis and $\theta_{j,M}$ say, on the diagonal are already known.

Thus on the y axis

$$x^2 + (y - y_o)^2 = r^2 \tag{18}$$

and we find from $(O, y_{j,o})$, $(x_{j,1}, y_{j,1})$ that

$$r = [(y_{j,0} - y_{j,1})^2 + x_{j,1}^2]/2(y_{j,o} - y_{j,1}) \tag{19}$$

with $\theta_{j,O} = \pi/2$. On the diagonal the use of

$$(x - a)^2 + (y - a)^2 = r^2 \tag{20}$$

yields

$$r = [(x_{j,M} - x_{j,M-1})^2 + (y_{j,M} - y_{j,M-1})^2]/2(x_{j,M} + y_{j,M} - x_{j,M-1} - y_{j,M-1}) \tag{21}$$

with $\theta_{j,M} = \pi/4$.

We use (19) or (21) in step (i) of the calculation instead of (4) and (5), and as appropriate in (8), (9) and (10).

RESULTS AND DISCUSSION

The computations were carried out as described in the previous sections. Each calculation was started at t=0.05 with isotherm positions obtained previously using the enthalpy method [13]. If the points on any isotherm moved so that they became very close, the midpoints of the line segments were taken as new points in such a way as to achieve a more even spacing. This procedure was necessary near the diagonal on the freezing front. In the calculation of Δn at the freezing front from (13) a three-point, one-sided derivative formula was used for $\partial n/\partial u$, as this gave increased accuracy near the diagonal.

The effects of varying the time step Δt, and the spacing, δu, of the isotherms, were investigated. For a given δu, as the time step was increased, persistent kinks appeared on the isotherms (which should be smooth curves as shown in Fig.3), indicating the onset of instability. Table 1 shows the values of the x-coordinates of the freezing front on the axis and on the diagonal at various times as calculated with $\delta u = 0.2$, 11 points on each isotherm, $\Delta t = 0.0001$ and $\Delta t = 0.0002$. The values are very close throughout. For $\Delta t = 0.0005$

Signs of instability appeared, although the results were similar except near the centre.

Table 2 compares the values of the x-coordinates of the freezing front on the axis and the diagonal calculated with $\Delta t=0.0001$, $\delta u=0.1$ and $\delta u=0.2$. These are considerably affected by the value of δu, particularly as the freezing front approaches the centre. Those for $\delta u=0.2$ are more accurate, and are in good agreement with those obtained by the enthalpy method, as shown in Fig .4. This difference may be due to the errors in the geometrical procedure arising from the use of the intersections of the perpendiculars with the straight line segment approximations to the next isotherms rather than with the actual isotherms, in calculating $\partial^2 n/\partial u^2$. These errors are independent of δu, and hence proportionately less as δu, is increased, being perhaps of order of 2 -3% of the distance between the isotherms when $\delta u=0.2$. These errors are likely to be greatest for small values of n, such as occur near the diagonal, particularly as the freezing front approaches the centre. A typical computation for $\delta u=0.2$, $\Delta t=0.0002$ with 11 points on each isotherm took 72 seconds on CDC 7600.

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Table 1. Comparison of x-coordinate of the freezing front on axis and diagonal, calculated with $\delta u=0.2$, $\Delta t=0.0001$ and 0.0002.

	On axis			On diagonal	
t	$\Delta t=0.0001$	$\Delta t=0.0002$		$\Delta t=0.0001$	$\Delta t=0.0002$
0.05	0.775	0.775		0.732	0.732
0.1	0.675	0.675		0.623	0.623
0.15	0.600	0.600		0.541	0.541
0.2	0.535	0.535		0.471	0.471
0.25	0.477	0.477		0.409	0.409
0.3	0.422	0.422		0.352	0.353
0.35	0.368	0.368		0.300	0.301
0.4	0.314	0.315		0.250	0.252
0.45	0.259	0.261		0.202	0.204
0.5	0.201	0.204		0.153	0.156
0.55	0.138	0.143		0.102	0.108

Table 2. comparison of x-coordinate of the freezing front on axis and diagonal, calculated with $\delta t=0.0001$, $\delta u=0.1$ and 0.2

	On axis			On diagonal	
t	$\delta u=0.2$	$\delta u=0.1$		$\delta u=0.2$	$\delta u=0.1$
0.05	0.775	0.775		0.732	0.732
0.1	0.675	0.676		0.623	0.619
0.15	0.600	0.601		0.541	0.535
0.2	0.535	0.536		0.471	0.463
0.25	0.477	0.477		0.409	0.399
0.3	0.422	0.420		0.352	0.342
0.35	0.368	0.364		0.300	0.288
0.4	0.314	0.308		0.250	0.237
0.45	0.259	0.249		0.202	0.188
0.5	0.201	0.188		0.153	0.139
0.55	0.138	0.119		0.102	0.087

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CAPTIONS

Fig.1. Sketch illustrating the geometrical procedure used to determine the normal to each isotherm.

Fig.2. Sketch showing relative positions of isotherms.

Fig.3. Positions of isotherms having temperatures u=0.0(0.2)1.0 at t=0.5.

Fig.4. Graph of x coordinates of the freezing front on the axis and on the diagonal against time, compared with results obtained by the enthalpy method.

Fig.l

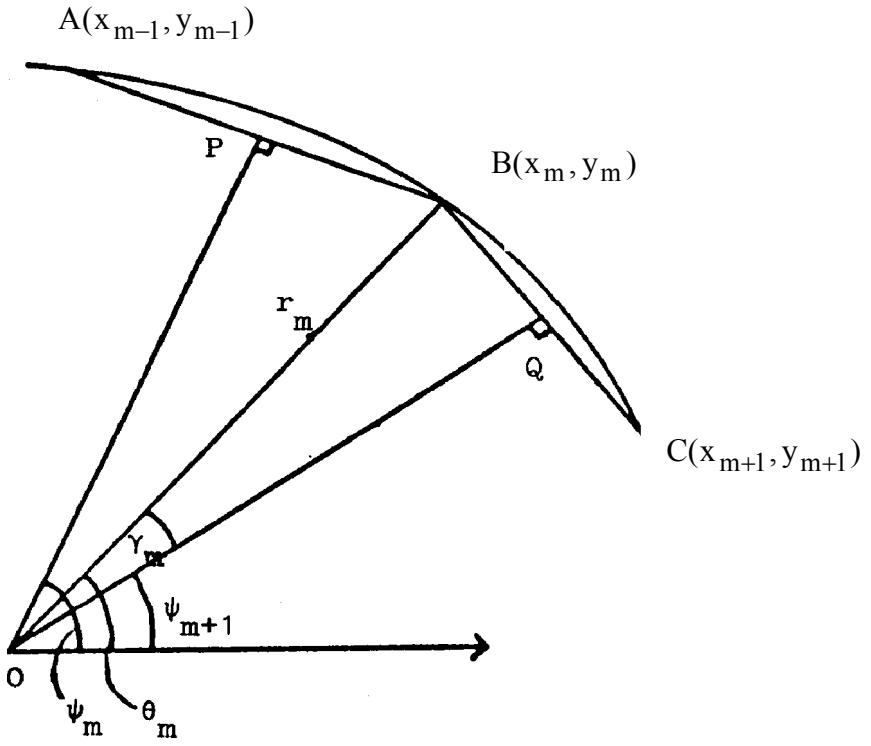
A(x m–1, y m–1)
P
B(x m, y m)
r m
Q
C(x m+1, y m+1)
γ m
ψ m+1
O
ψ m
θ m

Fig. 2

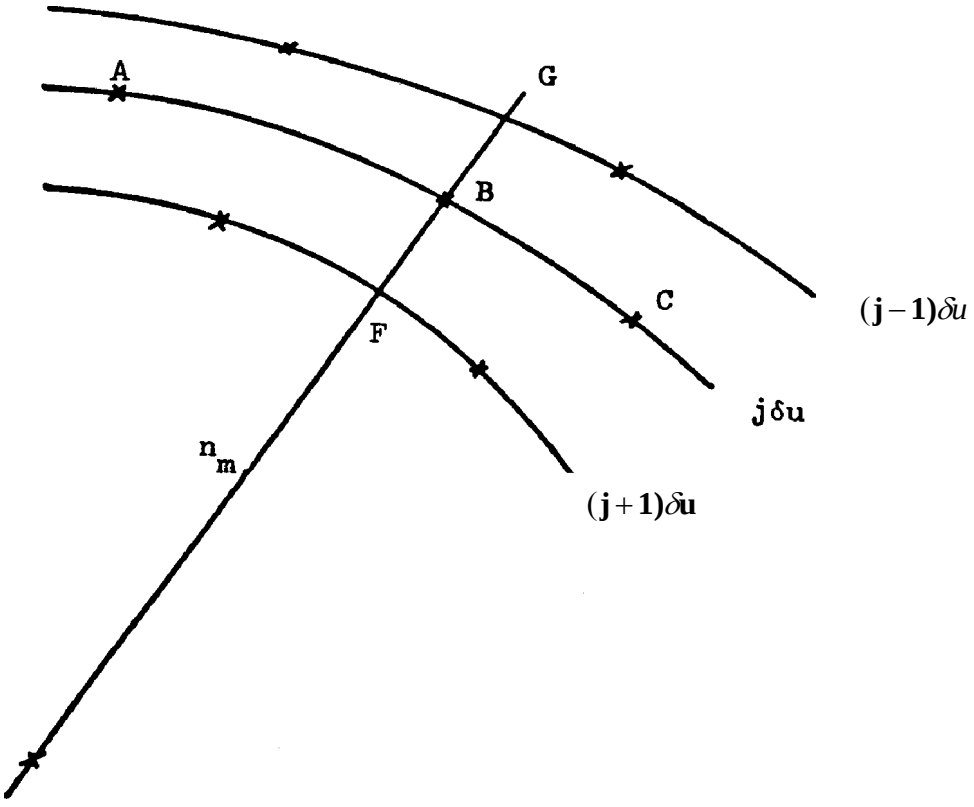
A
G
B
C
F
n m
(j – 1)δu
jδu
(j + 1)δu

Fig. 3

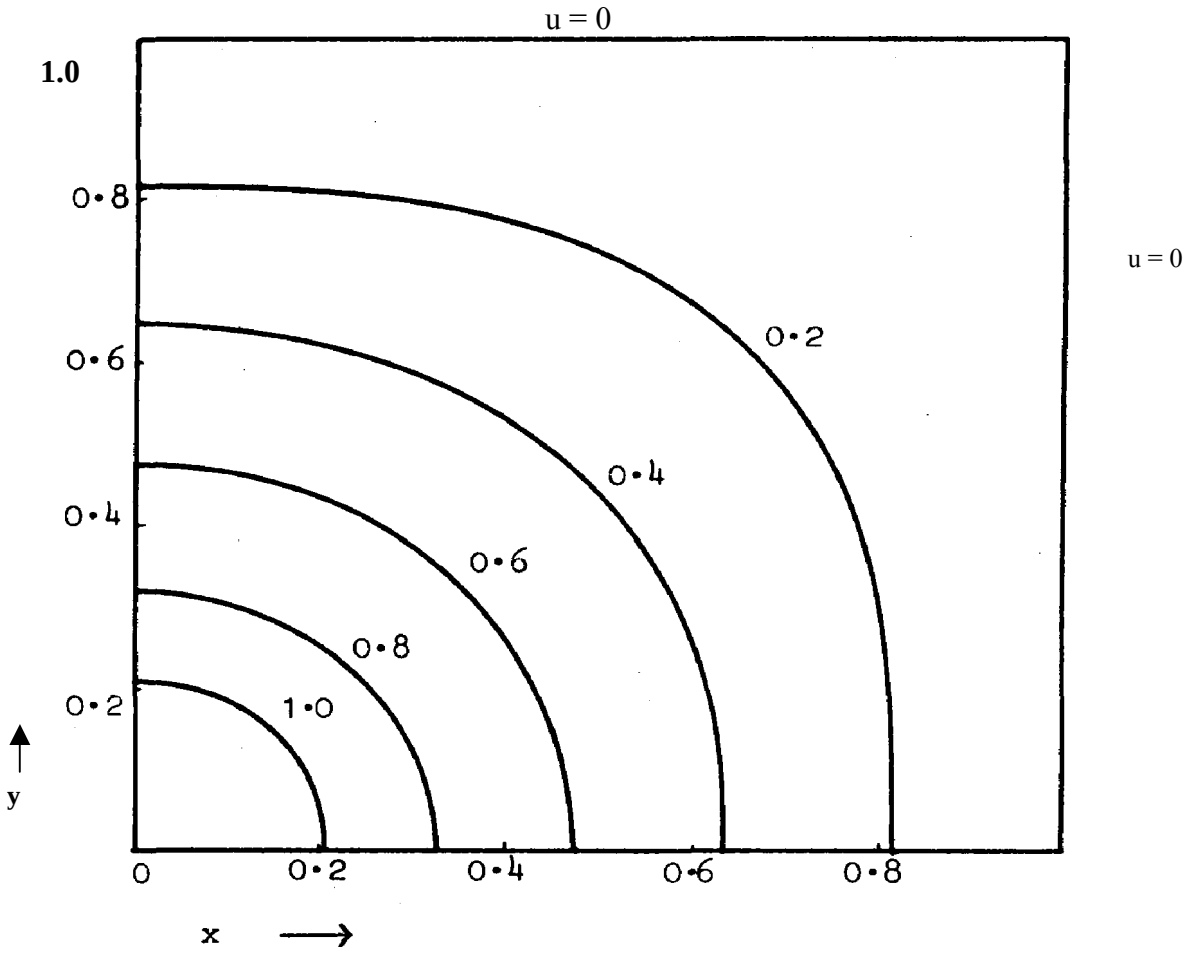

Fig. 4

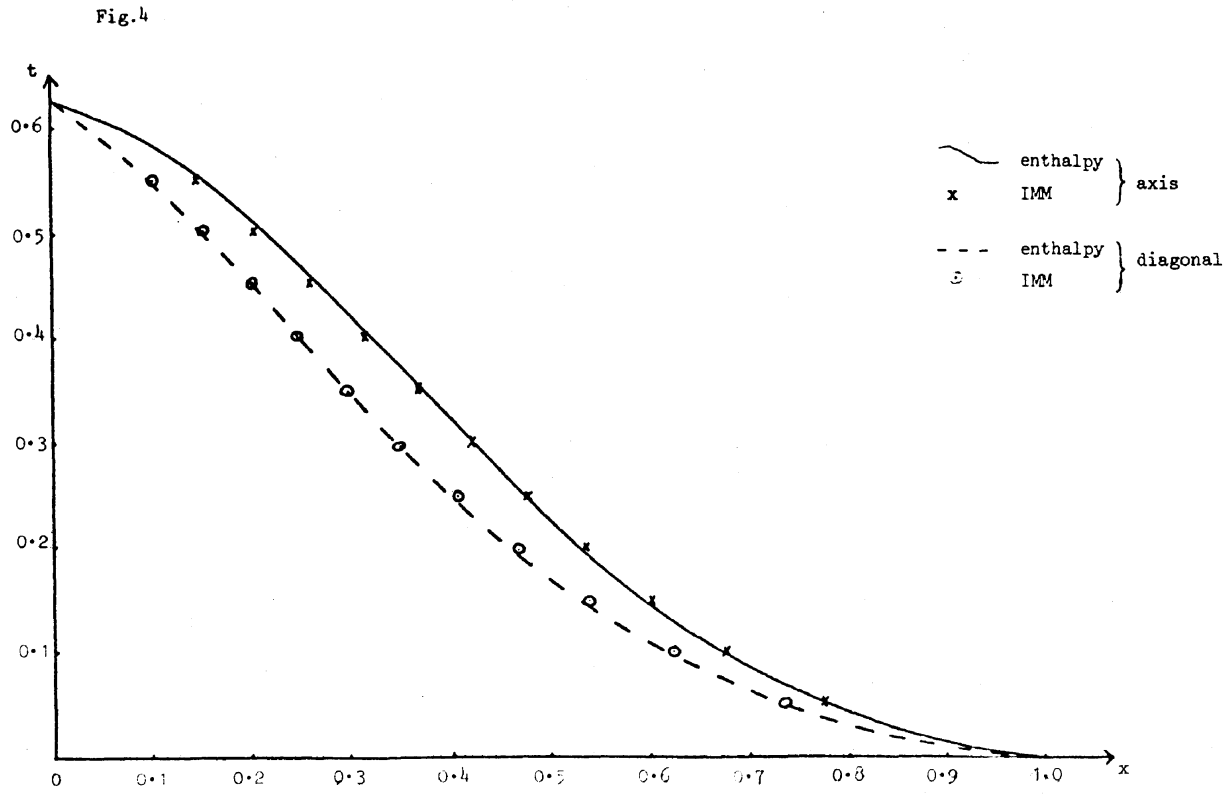
Fig.4
t
0·6
0·5
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0
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0·5
0·6
0·7
0·8
0·9
1·0
x
enthalpy
IMM
axis
enthalpy
IMM
diagonal