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REGULAR SPLINE SOLUTIONS TO A TWO-POINT BOUNDARY VALUE PROBLEM

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INTRODUCTION

The past decade accounted for a vast literature of techniques and algorithms to solve numerically a variety of two-point boundary value problems. A rapid glance through any prominent journal of numerical analysis supports the opinion that the computer user is confronted with a wide, and even bewildering, choice of possibilities. Excluding the literature concerning 'shoting methods', and 'non-local approximants' (eg. Chebyshev series) the user enters the extensive field of ' local approximations ' encompassing all the documented finite difference and finite element schemes.

The subclass of finite element schemes has recently received the concerted attention of numerical analysts. In particular the user will be aware of the existence of projection methods (including collocation methods) and schemes derived from a variational formulation of the problem. This variational formulation was derived by Ciarlet, Schultz, and Varga [4] who established that the analytic solution of the boundary value problem strictly minimises a certain functional. Computational aspects and rates of convergence are considered in [4] [10] and [17] amongst others. The nomenclature 'projection' defines the underlying principle of projection methods. We project the problem into a finite dimensioned subspace of an appropriate Hilbert space by some technique, and derive the approximant to the remodelled problem. In particular we may view the Galerkin procedure as a specific example of a projection method. The Galerkin method is employed by Douglas and Dupont [6] , Wheeler [22] to investigate a class of linear two-point boundary value problems. A superconvergence result at the knots is established in [6]. Projection schemes of a collocation type for classes of non-linear equations are studied in [8],[9],[11] and [13]. Collocation methods require the spline approximant to satisfy the differential equation at certain internal points,

We illustrate the finite difference approach by mentioning the paper by Stepleman [18]. In particular for the two-point boundary value problem independent of the first derivative, and with disjoint boundary conditions, Stepleman notes that his method is the classical Numerov method.

Of fundamental significance is the structure of the approximant, or for finite differences, the structure of the difference formula. Independent of the approach employed the numerical solution is dependent on a polynomial structure. The spline function spaces used in projection and variational formulations are piece—wise polynomials satisfying certain continuity constraints. Analogously, the finite difference schemes are polynomial based. For example the fourth order Numerov formula is derived by spanning the interval [0,2h] by a quartic polynomial, and collocating the values of the function and its second derivatives at the knots $x=0,h,2h$. Recently, much interest has surrounded the study of splines that are closer in structure to the function being approximated than the more conventional polynomial splines. The classes of regular splines defined in chapter 2 are but one example of an alternative structure.

The intrinsic characteristic of the class of regular splines of chapter 2 is a twice continuous differentiability throughout the region of application. Thus, it is to be expected that the collocation scheme based on regular splines is a generalisation of the cubic spline collocation scheme derived from the cubic spline's consistency relationship. For schemes utilising the cubic spline see [2],[3],[12], [13] amongst others. The method and results of Sakai [13] outlined in chapter 1 constitute the basis of the existence and convergence proof of chapter 4. Under the assumption ensuing the existence of a cubic spline collocation solution we establish an existence and convergence result for the regular spline collocation scheme of chapter 3. We note that these assumptions are arbitrary and others of identical consequence may be substituted. Finally, in Chapter 5, computational aspects of the regular spline collocation are discussed and numerical examples evaluated and compared.

1. A Two-Point Boundary Value Problem

The problem studied in the following chapters is

$$y''(x) = f(x,y(x)) \quad (1.1)$$

$$y(a) = y_a \quad (1.2)$$

$$y(b) = y_b \quad (1.3)$$

where y_a and y_a are constants and f(x,y) is twice continuously differentiable, with respect to x and y, in a region D of the (x,y) plane intercepted by two lines x=a and x=b.

To ensure that the solution, $\hat{y}(x)$, of (1.1) - (1.3), is unique in a subregion of D, we follow Urabe[19],[20], who introduces the concept of an 'isolated solution'. The solution, $\hat{y}(x)$, is called an isolated solution if and only if $\chi(b) \neq 0$ where $\chi(x)$ is the solution of the equation of first variation, namely

$$\chi''(a) = f_y(x, \hat{y}(x))\chi(x) \quad , \quad \chi(a) = 0 \quad and \quad \chi'(a) = 1$$

The terminology isolated solution is associated with a proposition by Urabe [20,pp 47] which states that, if $\chi(b) \neq 0$, there exists no other solution to (1.1)-(1.3) in U, where

$$U \equiv \{ (x, y) \,\big|\, | y(x) - \hat{y}(x) | < \zeta \ , \ a \leq x \leq b \} \subset D$$

and ζ, is an appropriately chosen positive constant.

We now introduce a cubic spline collocation solution, $\hat{u}(x)$, to (1.1) - (1.3) which is defined over a set of uniformly spaced knots $\{x_i\}_{i=1}^{m+1}$ where $a=x_1 < x_2 \ldots < x_{m+1} = b$, and $h = (b-a)/m$.

Sakai [13] uses a projection method and cubic spline functions to approximate the solution of the two point boundary value problem

$$y''(x) = g(x, y(x), y'(x)) \quad , \quad y(a) = y_a \quad , \quad y(b) = y_b$$

Furthermore, it is understood that for an equation independent of the first derivative $y'(x)$ the projection method reduces to a collocation method based on the consistency relationship for cubic splines [1,pp.284]. Thus, the result of Sakai [13,pp.361] is applicable to (1.1) - (1.3) and proves the existence of a unique cubic spline 'solution', $\hat{u}(x)$, in a region A_h,

$$A_h \equiv \{ (x, y) \,|\, | y(x) - \hat{y}(x) | < \gamma \quad , \quad a \le x \le b \} \subset U \qquad (1{\cdot}4)$$

where γ is some positive constant and h is sufficiently small, say $h < h_o$. Furthermore, $\hat{u}(x)$ satisfies the error bound

$$|| \hat{y}(x) - \hat{u}(x) ||_\infty < Nh^2 \qquad (1{\cdot}5)$$

where $|| \cdot ||_\infty$ is the usual supremun norm and N a positive constant independent of h.

Let us outline the construction of the cubic spline û.(x). Denote by $\underline{\hat{u}}$ the vector whose j^{th} element, $\underline{\hat{u}}_j$, is given by

$$\hat{u}_j = \hat{u}(x_j) \qquad j = 1,2,...., m+1 \qquad (1{\cdot}6)$$

The vector $\underline{\hat{u}}$ is determined as the solution of the non-linear system of equations $\underline{F}_s(\underline{u}) = \underline{o}$ where $\underline{F}_s(\underline{\hat{u}}) = \underline{o}$ is given by

$$\begin{aligned}
&\hat{u}_1 - y_a = 0 \\
&\frac{1}{6} f(x_{j-1}, \hat{u}_{j-1}) + \frac{2}{3} f(x_j, \hat{u}_j) + \frac{1}{6} f(x_{j+1}, \hat{u}_{j+1}) - \frac{1}{h^2}(\hat{u}_{j-1} - 2\hat{u}_j + \hat{u}_{j+1}) = 0 \\
&\qquad\qquad\qquad\qquad j = 2,3,\ldots,m \qquad (1.7) \\
&\hat{u}_{m+1} - y_b = 0.
\end{aligned}$$

The cubic spline, $\hat{u}(x)$, is determined uniquely over $[x_j, x_{j+1}]$, $j = 1,2, \ldots, m$, by the knot values $\hat{u}_j$, u_{j+1} and the knot values of the second derivative, namely $f(x_j, \hat{u}_j)$ and $f(x_{j+1}, \hat{u}_{j+1})$.

Finally, we note that the Jacobian, $J_s(\underline{u})$, of the system $\underline{F}_s(\underline{u})$ is non-singular when $\underline{u} = \underline{\hat{u}}$ and that $\| J_s^{-1}(\underline{\hat{u}})\|_\infty$ is bounded (see [14]).

The non-zero elements of $J_s(\underline{\hat{u}})$ are

$$\begin{aligned}
&J_{s(1,1)} = 1 \\
&J_{s(j,j-1)} = -\frac{1}{h^2} + \frac{1}{6} f_y(x_{j-1}, \hat{u}_{j-1}) && j = 2,3,\ldots, m \\
&J_{s(j,j)} = \frac{2}{h^2} + \frac{2}{3} f_y(x_j, \hat{u}_j) && j = 2,3,\ldots, m \qquad (1.8) \\
&J_{s(j,j+1)} = -\frac{1}{h^2} + \frac{1}{6} f_y(x_{j+1}, \hat{u}_{j+1}) && j = 2,3,\ldots, m \\
&J_{s(m+1,m+1)} = 1
\end{aligned}$$

2. Regular Splines

Regular splines, as employed in the following context, were introduced by Werner [21]. Let $a = z_1 < z_2 \ldots < z_{n+1} = b$, and consider functions $\hat{t}_\ell(x,c,d)$, defined for $x \in [z_\ell, z_{\ell+1}]$, depending on two parameters c,d where c and d are in certain prescribed intervals, for example □or □$_{\dagger}$. The functions, $\hat{t}_\ell$, are subject to conditions defined below.

The set of equally spaced knots $\{x_i\}_{i=1}^{m+1}$ is specified such that each z_ℓ, $\ell = 1,2,\ldots, n+1$, coincides with a knot x_p for some $p \in \{i\}_{i=1}^{m+1}$. Hence each interval I_p, where $I_p \equiv [x_p, x_{p+1}]$, is contained in exactly one interval $[z_\ell, z_{\ell+1}]$. The notation $t_j(x,c,d)$ is used to denote the restriction of $\hat{t}\ell$ (x,c,d) to I_j. In this way the functions $t_j(x,c,d)$ are well defined when the mesh of knots is refined.

For a given set of knots $\{x_i\}_{i=1}^{m+1}$ and classes of functions $\{\hat{t}_\ell\}_{\ell=1}^{n}$ which are twice continuously differentiable with respect to x, and continuous with respect to c and d, we define a spline by

$$\eta(x_1, x_2, \ldots, x_{m+1}; x) = \{u(x)|\ u(x) \in C^2[a,b],\ u\big|_{I_j} \equiv p_j + t_j(x, c_j, d_j),$$

$$p_j \in \pi \quad \text{and} \quad j = 1, 2, \ldots, m\} \tag{2.1}$$

where π is the set of linear polynomials.

In the context of this paper we need the following assumptions on the classes of functions $t_j(x,c,d)$:

(A1) The classes $t_j(x,c,d)$ shall be regular.

ie. any two functions of the same class either coincide or the difference of their second derivatives have at most one zero in I_j.

This assumption ensures that the functions $t_j(x,c_j,d_j)$ j=,2,...,m can be parameterised in terms of $t''_j(x_j)$ and $t''_j(x_{j+1})$ To explain (A1) in greater detail it is necessary to quote a theorem by Werner [21], namely

'If the family of splines is made up of regular functions then the interpolating spline is unique.'

Thus regularity is a sufficient condition for uniqueness of the spline interpolating the data produced by the collocation method of chapter 3, always assuming that the data is within the range of the spline.

If the classes of functions $t_j(x,c,d)$ are parameterised as above we adopt the notation

$$t_j \equiv t_j(xj,\, x_{j+1},\, M_j,\, M_{j+1};\, x)$$

Where

$$\frac{d^2}{dx^2}\, t_j(x_j,\, x_{j+1},\, M_j,\, M_{j+1};\, x_i = M_i \qquad \text{for } i = j \text{ and } j+1$$

(A2) The functions t_j are smooth .

ie. the functions t_i are four times continuously differen— tiable with respect to x, and these derivatives are twice continuously differentiable with respect to M_j and M_{j+1}.

(A3) The functions t_j are 4-bounded „

ie. the fourth derivative, $\overline{t_j^{iv}}$, of t_j with respect to x shall be a twice continuously differentiable function of $\frac{1}{h}(M_{j+1} - M_j)$ and either M_j or M_{j+1}.

The assumption (A3) is motivated by the fact that only two parameters are needed to control the behaviour of the second and higher derivatives of the spline. It would be unwise to use bounds on M_j and M_{j+1}, as when $h \to O$ the two parameters become increasingly identified with each other. However M_j and $\frac{1}{h}(M_{j+1}, - M_j)$ are well defined as $h \to O$.

Examples of admissible classes of functions t_j are now given, cf.[16,pp.176].

Example 1 $\quad t_j = c(d-x+x_j.)^k \quad k \neq 0,l,2$

$$\text{eg. } d = \frac{h}{1-\left(\frac{M_{j+1}}{M_j}\right)^{\frac{1}{k-2}}} \quad \text{and} \quad c = \frac{M_j\, d^{2-k}}{k(k-1)} \quad \text{whenever} \quad M_j \neq 0$$

The above classes of functions yield splines of various structures for different values of k. For any $k < 0$ we have a rational spline. The standard cubic spline is derived by allocating to k the value three. The condition for (Al) - (A3) to hold is given by

$$\frac{M_j}{M_{j+1}} \in IR_+ \quad , \quad M_j \neq M_{j+1}$$

unless $k = 2n+l$ where n is any positive integer. For the latter cases (Al) — (A3) hold unconditionally.

Example 2 $\quad t_j = c\ e^{d(1+x-x_j)}$

$$\text{where} \quad d = \frac{1}{h} \log \left(\frac{M_{j+1}}{M_j}\right) \quad \text{and} \quad c = \frac{M_j}{d^2 e^d}$$

Once again, the necessary and sufficient condition for (Al) - (A3) to hold is

$$\frac{M_j}{M_{j+1}} \in IR_+, \; M_j \neq M_{j+1}$$

Example 3 $\qquad t_j = c \log(d - x + x_j)$

where $\qquad d = \dfrac{h}{1 - \left(\dfrac{M_j}{M_{j+1}}\right)^{\frac{1}{2}}}$ and $c = -M_j d^2$

For this class of function the conditions (Al) - (A3) are satisfied whenever

$$0 < \frac{M_j}{M_{j+1}} < 1$$

<u>Example 4</u> $\qquad t_i = c \sin(\mu (x - x_i) + d) \qquad \mu \neq O$

eg. $\qquad d = \cot^{-1}\left\{\dfrac{1}{\sin(\mu h)}\left(\dfrac{M_{j+1}}{M_j} - \cos(\mu h)\right)\right\}$

and $\quad \mu^2 c = -M_j \operatorname{cosec} d \qquad$ whenever $M_i \neq o$.

Functions of this class satisfy (Al) - (A3) unconditionally.

Following Schaback [16] and Werner [21] we define the difference operators

$$\Delta^1(x_1, x_2)g(x) = \frac{g(x_2) - g(x_1)}{x_2 - x_1}$$

$$\Delta^2(x_1, x_2, x_3)g(x) = \frac{1}{x_3 - x_1}\left[\Delta^1(x_2, x_3)g(x) - \Delta^1(x_1, x_2)g(x)\right]$$

where x_1, x_2, x_3 are piecewise disjoint and $g(x) \in C\,[a,b]$.

If the function $g(x)$ is differentiable we may allow x_1 and x_2

3. A Regular Spline Collocation Method

In this chapter we derive a collocation method that yields a regular spline as an approximate solution to the problem (1.1) - (1.3).

From (2.1) any regular spline, u(x), satisfying (A l) can be expressed by

$$u(x)\Big|_{I_j} = a_j + b_j x + t_j(x) \qquad j = 1,2,\ldots, m$$

where the parameters are still undetermined. Following Werner [21] the linear parameters a_j and b_j may be determined in terms of $u(x_j)$ and $u(x_{j+1})$ giving

$$u(x)\Big|_{I_j} = u(x_j) - t_j(x_j) + \Big(u(x_{j+1}) - u(x_j) + t_j(x_j) - t_j(x_{j+1})\Big)\left(\frac{x - x_j}{h}\right) + t_j(x) \qquad j = 1,2,\ldots, m \tag{3.1}$$

The function u(x) and its second derivative are continuous for $x \in [a,b]$. Hence the conditions

$$u'(x_{j+1})\Big|_{I_j} = u'(x_{j+1})\Big|_{I_{j+1}} \qquad j = 1,2,\ldots m-1 \tag{3.2}$$

are necessary and sufficient for $u(x) \in C^2[a,b]$. The expression (3.2) applied to (3.1) yields a relationship analogous to the consistency relationship of cubic splines, namely

$$p(x_j, x_{j-1}, M_j, M_{j-1}) + P(x_j, x_{j+1}, m_j, m_{j+1}) = 2\Delta^2(x_{j-1}, x_j, x_{j+1})u(x) \qquad j = 2,3,\ldots,m \tag{3.3}$$

A collocation, method is derived by fitting the equation (3·1) to the problem (1·1)—(1·3) at the set of knots $\{x_j\}_{j=1}^{m+1}$. Setting $u_j = u(x_j)$ this can be written as

$$M_j = f(x_j,u_j) \qquad j - 1,2,...,m+1 \qquad (3{\cdot}4)$$

Equations (3·3) and (3·4), when combined, yield a non—linear system of equations, $F(\underline{u}) = \underline{o}$, from which the knot values of the regular spline collocation solution are calculated. This system of m-1 equations in m-1 unknowns is given by

$$\begin{aligned}
& u_1 - y_a = 0 \\
& p(x_j, x_{j-1}, f(x_j, u_j), f(x_{j-1}, u_{j-1}) \\
& \quad + p(x_j, x_{j+1}, f(x_j, u_j)f(x_{j+1}, u_{j+1})) \\
& \quad - \frac{1}{h^2}(u_{j-1}, -2u_j + u_{j+1}) = 0 \qquad j = 2,...., m \\
& u_{m+1} - y_b = 0
\end{aligned} \qquad (3.5)$$

Let us assume that the system of equations (3.5) has a unique solution $\underline{u}^*$. The vector $\underline{u}^*$ yields values at the knots, which, combined with (3.1) and (3.4), construct a regular spline approximant, $y^*(x)$, to $\hat{y}(x)$.

4. An Existence Theorem

In the following we use results from Urabe [l9,pp.l23] and Sakai [13,14]. Firstly, we quote a proposition by Urabe as its implementation establishes the required theorem.

Proposition

Let $\underline{F}(\underline{\alpha}) = \underline{o}$ be a given real system of equations where $\underline{\alpha}$ and $F(\underline{\alpha})$ are the vectors of the same dimension and $\underline{F}(\underline{\alpha})$ is a continuously differentiable function of a defined in some region Ω of $\underline{\alpha}$. Assume that $\underline{F}(\underline{\alpha})=\underline{0}$ has an approximate solution $\underline{\alpha} = \underline{\alpha}$ for which the determinant of the Jacobian matrix $J(\underline{\alpha})$ of $\underline{F}(\underline{\alpha})$ with respect to $\underline{\alpha}$ does not vanish at $\underline{\alpha} = \hat{\underline{\alpha}}$, and there is a positive constant ρ and a non-negative constant $\kappa < 1$, such that

(i) $\Omega_\rho \equiv \{\underline{\alpha} \mid \|\underline{\alpha} - \hat{\underline{\alpha}}\|_\infty \le \rho\} \subset \Omega$

(ii) $\| J(\underline{\alpha}) - J(\hat{\underline{\alpha}})\|_\infty \le \kappa / M'$ for any $\underline{\alpha} \in \Omega_\rho$

(iii) $\dfrac{M'r}{1-\kappa} < \rho$

where r and M' are positive constants such that $\|\underline{F}(\hat{\underline{\alpha}})\|_\infty \le r$, and $\| J^{-1}(\hat{\underline{\alpha}})\|_\infty \le M'$.

Then the system $\underline{F}(\underline{\alpha}) = \underline{0}$ has one and only one solution $\underline{\alpha} = \underline{\alpha}^*$ in $\Omega\rho$ and

$$\|\underline{\alpha}^* - \hat{\underline{\alpha}}\|_\infty \le \frac{M'r}{1-\kappa}$$

In our implementation of the above proposition we take $\hat{\underline{u}}$ as the approximate solution to the system of equations $\underline{F}(\underline{u}) = \underline{O}$, given by (3.5). Our first intention is to construct a region Ω_h, ($\hat{\underline{u}}$) over which $\underline{F}(\underline{u})$is well-defined and differentiable.

Let the set $\Omega_h(\underline{\hat{u}}) \subset IR^{m+1}$ be defined by

$$\Omega_h(\underline{\hat{u}}) \equiv \{\underline{u} \| u_j - \hat{u}(x_j)| < ch, \ j = 2,3,\ldots, m, \ u_1 = y_a, u_{m+1} = y_b\} \tag{4·1}$$

where u_j is the j^{th} component of the vector $\underline{u} \in IR^{m+1}$, and c is a positive constant independent of h. Also, it is necessary to define the region V_h by

$$v_h \equiv \{(x, y) | \| y(x) - \hat{u}(x) \|_\infty < \gamma - Nh^2, x \in [a, b]\} \subset A_h \tag{4·2}$$

for any $h < h_o$, where the constants N and γ are explained in (1.4) and (1.5). Furthermore, let h be sufficiently-small, say $h < h_1 \leq h_o$, such that the points

$$(\eta, u_{j+1}), \ (x_j, \alpha) \ \varepsilon \ V_h \quad \text{for any } \underline{u} \ \varepsilon \ \Omega_h(\underline{\hat{u}}) \tag{4·3}$$

where $x_j \leq \quad \leq x_{j+1}$ and α lies between u_j and u_{j+1}, $j = 1,2,,..,m$. The above restriction on h ensures that, at the points indicated in (4·3), f(x,y) is twice continuously differentiable with respect to x and y and all these derivatives are bounded.

For simplicity of notation we adopt

$$M. = f(x_J, \hat{u}(x)) \quad \text{and} \quad M_J. = f(x_J, u_J.).$$

In the latter $u_J.$ is the j^{th} component of an arbitrary vector $\underline{u} \ \varepsilon \ \Omega_h \ (\underline{\hat{u}})$, $h < h_1$. The expression (4·3) immediately yields the boundedness of $M_j.$ and $M_j.$, and further that

$$\hat{M}_j - M_j | \leq f_y(x_j, \alpha) \| \hat{u}(x_j) - u_j | \leq f_y | ch$$

for some a between $\hat{u}(x_j)$ and u_j $j = 2,3,\ldots,m$.

In the examples of chapter 2, stipulations guaranteeing the fulfilment of (A1)-(A3) imposed restrictions on the values M_j for various classes of functions t_j Consequently, we require one further assumption, namely

(A4) The functions t_j,J=l,2,...,m, are admissible.

ie. the admissible values for $\{M_j\}_{j=1}^{m+1}$ include the set m where

$$m \equiv \{\underline{M} \mid |M_j - \hat{M}_j| < L\,ch,\ j = 2,3,....,m,\ M_1 = f(a, y_a),\ M_{m+1} = f(b, y_b)\}$$

whenever $h < h_2$, for some $h_2 \leq h_1$ .The constant L is a bound on f_y (•, •) over U, and $M_j = (\underline{M})_j$,

The above assumption is a theoretical restriction on the choice of classes of functions $\{t_i\}_{i=1}^{m}$ and dictates that the admissible values for $\{M_i\}_{i=1}^{m}$ must include a small interval containing the values $\{\hat{M}_i\}_{i=1}^{m}$ Hence, from (1.5), M_j .has $f(x_j, \hat{y}(x_j))$ as an admissible value whenever h is sufficiently small, and thus the choice of classes $\{t_j\}_{j=1}^{m}$ is well—defined as $h \to O$.
The following lemmas are essential to our proof

Lemma (3)

Let u(x) be the function derived by a combination of (3·1) and (3.4) where the knot values, $u(x_J)$ are replaced by the $_j$th component of an arbitrary vector $\underline{u}\ \varepsilon\ \Omega_h$ $(\underline{\hat{u}})$. Then for h sufficiently small $u(x)|I_J$ is 4-bounded.
Proof.

By (A.4) and (4.3) the function u(x) is well—defined over $\Omega_h(\underline{\hat{u}})$ whenever $h < h_2$. The assumption (A3) informs us that $u(x)\big|I_j$ is 4-bounded if M_j and $\frac{1}{h}(M_{j+1} - M_j)$ are bounded.

We note that, whenever $h < h_2$, $M_j = f(x_j, u_j)$ is bounded by (4.3), and

$$M_{j+1} - M_j = \frac{1}{h}(f(x_{j+1}, u_{j+1}) - f(x_j, u_j))$$
$$= \frac{\partial f}{\partial x}\Big|_{(\eta,\, u_{j+1})} + \frac{(u_{j+1} - u_j)}{h}\frac{\partial f}{\partial y}\Big|_{(x_j,\, \alpha)} \qquad (4\cdot4)$$

Where $x_j < \eta < x_{j+1}$ and α lies between u_j and u_{j+1}. The derivatives of (4·4) are bounded by (4·3) .Using the triangle rule, $\underline{u} \in \Omega_h(\underline{\hat{u}})$ and that $\hat{u}(x)$ is a cubic spline solution to (1·1)-(1·3) we have

$$|u_{j+1} - u_j| \le |u_{j+1} - \hat{u}_{j+1}| + |\hat{u}_{j+1} - \hat{u}_j| + |\hat{u}_j - u_j|$$
$$\le 2ch + h\,|\hat{u}'(\xi)| \qquad x_j < \xi < x_{j+1}.$$

Thus $\frac{1}{h}(M_{j+1} - M_j)$ is bounded for any $\underline{u} \in \Omega_h(\underline{\hat{u}})$ whenever $h < h_2$ and the proof is complete.

<u>Lemma 4</u>

Let $w(x)$ be four times continuously differentiable over I_j and be the cubic polynomial that interpolates the values $w(x_j)$, $w(x_{j+1})$, $w''(x_j)$ and $w''(x_{j+1})$ of $w(x)$ Then

$$\|w(x) - \pounds w(x)\|_\infty \le \frac{5h^4}{384} w^{\overline{\text{iv}}}(\xi) \quad x \in I_j \text{ where } x_j < \xi < x_{j+1}$$

Proof

It is easily seen that $\pounds w(x)$ can be expressed by

$$£w(x) = w(x_j)\frac{(x_{j+1} - x)}{h} + w(x_{j+1})\frac{(x - x_j)}{h}$$
$$+ \frac{1}{6}[2w''(x_j) + w''(x_{j+1})](x - x_j)(x - x_{j+1})$$
$$+ \frac{1}{6h}[w''(x_{j+1}) - w''(x_j)](x - x_j)^2(x - x_{j+1}) \qquad (4\cdot5)$$

The interpolation is exact for any $w(x) \varepsilon \pi_3$, the set of cubic polynomials. Applying the Peano kernel theorem, the interpolation error, Rw(x), is given by

$$Rw(x) = W(x) - £w(x) = \int_{x_j}^{x_{j+1}} k(t)w^{\overline{IV}}(t)dt$$

where $k(t) = R_x[(x-t)_+^3]/31$. The notation $R_x[.]$ implies an application of the functional R to a function of x, and

$$(x-t)_+ = \begin{cases}(x-t) & \text{if } x > t \\ 0 & \text{if } x \le t\end{cases} \quad k$$

Note that $Rw(x) \le w^{\overline{IV}}(\xi \int_{x_j}^{x_{j+1}} k(t)dt$ from Some ξ, $x_j < \xi < x_{j+1}$.

Evaluating the above integral yields a bound on Rw(x), and maximising this bound over I_j gives the desired result

Lemma 5

The non-linear system of equations (3·5) is well—defined, and differentiable with respect to $\underline{u}$, over Ω_h, $(\underline{\hat{u}})$ whenever h is sufficiently small.

Proof

Let $\underline{u}$ be an arbitrary vector in $\Omega_h(\underline{\hat{u}}) » h < h_2$, and u(x) be the function derived from (3·1) and (3·4)when the knot value

$u(x_j)$ is allocated the value of the j^{th} component of $\underline{u}$. Note that $u(x)$ coincides with a regular spline over each separate interval I_j but that $u'(x)$ is not necessarily continuous at the knots $\{x_j\}_{j=2}^{m}$ By lemma (3) $u(x)\big|_{I_j}$ is 4-bounded for $j=1,2,\ldots,m$.

we bound $\|u(x) - \hat{u}(x)\|_\infty$, x , I_j , by using

$$\|u(x)-\hat{u}(x)\|_\infty \leq \|u(x)- £u(x)\|_\infty + \|£u(x)- \hat{u}(x)\|_\infty \qquad (4{\cdot}6)$$

The first term on the right hand side of (4.6) may be bounded by using lemma (4), and the second term by using (4·5) whilst remembering that, as $\hat{u}(x)$ is a cubic spline, $£\hat{u}(x) \equiv \hat{u}(x)$, x e I_j, Hence,

$$\| u(x) - \hat{u}(x) \|_\infty < c[h^4 + \| \underline{u} - \underline{\hat{u}} \|_\infty],\ x \in I_j \qquad (4{\cdot}7)$$

for some constant C.

Finally, using (4.1) and taking the supremun of the bounds for j=l,2,...,m, it is easy to see that for $h < h_2$

$$\| u(x) - \hat{u}(x) \|_\infty = 0(h), \quad x \in [a, b].$$

Consequently, for h sufficiently small, say $h<h_3\underline{\leq}h$,we have

$$\|u(x) - \hat{u}(x)\|_\infty \leq \gamma - Nh^2.$$

and hence, for any $\underline{u}$ ε Ω_h $(\underline{\hat{u}})$, $h < h_3$ the function

$u(x)\ \varepsilon\ A_h$. Thus $\underline{F}(\underline{u})$ is well-defined and differentible with respect to $\underline{u}$ over $\Omega_h \hat{\underline{u}}$, $h < h_3$

Remember $\hat{\underline{U}}$ is an initial approximation to the exact solution, $\underline{u}^*$, of the non-linear system of equations, $\underline{F}(\underline{u})=\underline{0}$, defined by (3·5). Let us define $\bar{u}(x)$ to be the function derived from (3·1) and (3·4) when the knot values $\bar{u}(x_j.) = u(x_j)$ for J i=l,2,..,m+l. Note that $\bar{u}'(x)$ is not necessarily continuous at the knots unless $\hat{\underline{u}} = \underline{u^*}$. Set $\bar{u}^{\overline{iv}}(x)\,|_T \equiv \bar{u}_i^{\overline{iv}}(x)$. Now, by using lemma (3) we see that $\bar{u}(x)\ |\ I_j$ is 4-bounded.

The lemma (1), and the aforementioned 4-bound, yield for

$x\ \varepsilon\ 1_J$

$$\Delta^2(x_j, x_j, x_{j+1})\bar{u}(x) \equiv p(x_j, x_{j+1}, f(x_j, \hat{u}_j), f(x_{j+1}, \hat{u}_{j+1}))$$

$$= \frac{1}{3} f(x_j, \hat{u}_j) + \frac{1}{6} f(x_{j+1}, \hat{u}_{j+1}) + R(x_j + x_{j+1}, \bar{u}_j^{\overline{iv}}(x)) \tag{4·8}$$

where $R(x_j, x_{j+1}, \bar{u}^{\overline{IV}}(x)) = -\frac{h^2}{24} \bar{u}_j^{\overline{IV}}(\xi),\quad x_j < \xi < x_{j+1}$.

Using (1·7),(3·5) and (4·8) we determine $\underline{F}(\underline{u})$ to satisfy

$\hat{U}_1 - y_a = 0$

$$p(x_j, x_{j-1}, f(x_j, \hat{u}_j), f(x_{j-1}, \hat{u}_{j-1})) + p(x_j, x_{j+1}, f(x_j, \hat{u}_j), f(x_{j+1}, \hat{u}_{j+1}))$$

$$-\frac{1}{h^2}(\hat{u}_{j-1} - 2\hat{u}_j + \hat{u}_{j+1}) = 0(h^2) \qquad j = 2,3,\ldots, m$$

$\hat{U}_{m+1} - y_b = 0.$

Hence, for some positive constant R, and $h < h_3$

$$\| \underline{F}(\underline{\hat{u}}) \|_\infty < Rh^2 \tag{4·9}$$

To simplify the notation, we utilise corollary (2) to denote

$$R(x_j, x_{j+1}, \overline{u}_j^{IV}(x)) \equiv R(\hat{M}_j, \hat{M}_{j+1}), \qquad \hat{M}_j = f(x_j, \hat{u}_j) \quad \text{etc.}$$

$$\equiv \tilde{R}(\hat{u}., \hat{u}_{j+1})$$

and D_1, D_2 to be differential operators signifying differentiation with respect to the first, respectively second, variable of the above functions.

From (3·5) and (4·8) the hon—zero elements of the Jacobian, $J(\underline{\hat{u}})$,of the vector $\underline{F}(\underline{u})$ can be expressed as

$$J_{(1,1)} = 1$$

$$J_{(j,\ j-1)} = -\frac{1}{h^2} + \frac{1}{6} f_y(x_{j-1}, \hat{u}_{j-1}) + D_2 \tilde{R}(\hat{u}_j, \tilde{u}_{j-1})$$

$$J_{(j,\ j)} = \frac{2}{h^2} + \frac{2}{3} f_y(x_j, \hat{u}_j) + D_1(\tilde{R}(\hat{u}_j, \hat{u}_{j-1}) + \tilde{R}(\hat{u}_j, \hat{u}_{j+1}))$$

$$J_{(j,\ j+1} = -\frac{1}{h^2} + \frac{1}{6} f_y(x_{j+1}, \hat{u}_{j+1)} + D_2 \tilde{R}(\hat{u}_j, \hat{u}_{j+1}) \tag{4·10}$$

$$J_{(m+l,m+l)} = 1 \qquad\qquad 2,3\ldots\ldots,m$$

The corollary (2) and the 4-bound on $\bar{u}(x)|_{Ij}$ gives us that

$$D_p \tilde{R}(\bullet,\bullet) = 0(h^2) \qquad p = 1 \text{ and } 2 \tag{4·11}$$

A direct application of the expressions (1·8), (4·10) and (4•11) reveals that, for some positive constant T

$$\| J_S(\underline{\hat{u}}) - J(\underline{\hat{u}}) \|_\infty \le Th^2, h < h_3.$$

The existence and boundedness of $J^{-1}(\underline{\hat{u}})$ is achieved by applying the above bound, and a result from [7 ,pp. 365], to the expression

$$J^{-1}(\underline{\hat{u}}) = [I - J_S^{-1}(\underline{\hat{u}})(J_S(\underline{\hat{u}}) - J(\underline{\hat{u}}))]^{-1} J_S^{-1}(\underline{\hat{u}})$$

Thus, for h sufficiently small, say $h < h_4 \le h_3$
$J(\underline{\hat{u}})$ is nonsingular and

$$\| J^{-1}(\underline{\hat{u}}) \|_\infty \le M, \quad M > 0 \tag{4•12}$$

Next, we require a bound on $E \equiv J(\underline{u}) - J(\underline{\hat{u}})$ for any $\underline{u} \in \Omega_h(\underline{\hat{u}})$, $h < h_4$.Using (4•10), the non-zero elements of E may be written as

$$E_{(j,j-1)} = \frac{1}{6}(f_y(x_{j-1}, u_{j-1}) - f_y(x_{j-1}, \hat{u}_{j-1})) + D_2(\tilde{R}(u_j, u_{j-1}) - \tilde{R}(\hat{u}_j, \hat{u}_{j-1}))$$

$$E_{(j,j)} = \frac{2}{3}(f_y(x_j, u_j) - f_y(x_j, \hat{u}_j)) + D_1(\tilde{R}(u_j, u_{j-1}) - \tilde{R}(\hat{u}_j, \hat{u}_{j-1}))$$

$$+ D_1(\tilde{R}(u_j, u_{j+1}) - \tilde{R}(\hat{u}_j, \hat{u}_{j+1}))$$

$$E_{(j,j+1)} = \frac{1}{6}(f_y(x_{j+1}, u_{j+1}) - f_y(x_{j+1}, \hat{u}_{j+1})) + D_2(\tilde{R}(u_j, u_{j+1}) - \tilde{R}(\hat{u}_j, \hat{u}_{j+1}))$$

$$j=2,3 \ldots\ldots\ldots\ldots\ldots m \tag{4•13}$$

We illustrate the construction of the bound on (4•13) with reference to the element $E_{(j,j+1)}$. The line

$(x_{j+1}, \alpha u_{j+1} + (1-\alpha)\hat{u}_{j+1})$, $0 \le \alpha \le 1$, is contained in $V_h, h < h_3 \le h_1$,

and hence we can apply the mean value theorem to obtain

$$|f_y(x_{j+1}, u_{j+1}) - f_y(x_{j+1}, \hat{u}_{j+1})| \le C|u_{j+1} - \hat{u}_{j+1}| \qquad (4\cdot 14)$$

for some positive constant C ,

Note that $\tilde{R}(u_j, u_{j+1}) \equiv R(x_j, x_{j+1}, u^{\overline{IV}}(x))$ where the function u(x) is that of lemma (3) and hence $u(x)|I_j$ is 4-bounded. Now ,

$$D_2(\tilde{R}(u_j, u_{j+1}) - \tilde{R}(\hat{u}_j, u_{j+1})) \equiv \frac{\partial}{\partial u_{j+1}}\left[R(M_j, M_{j+1}) - R(\hat{M}_j, M_{j+1})\right]$$

$$\equiv \frac{\partial}{\partial u_{j+1}}\left[D_1 R(\alpha(\ _j + (1-\alpha)\hat{M}_j, M_{j+1})(M_j - \hat{M}_j)\right]$$

for some $\alpha, 0 < \alpha < 1$. It is necessary to prove the 4—boundedness of a spline over I_j when $M_j \equiv \alpha f(x_j, u_j) + (1-\alpha) f(x_j, \hat{u}_j)$, and $M_{j+1} \equiv f(x_{j+1}, u_{j+1})$. The bound on M_J or M_{J+1} is trivial, and

$$\frac{M_{j+1} - M_j}{h} = \frac{1}{h}\left[(1-\alpha)\ (M_{j+1} - \hat{M}_j) + \alpha\ (M_{j+1} - M_j)\right]$$

is readily bounded by similar arguments to those used to bound (4·4)
We may now consult corollary (2) and deduce that

$$|D_2(\tilde{R}(u_j, u_{j+1}) - \tilde{R}(\tilde{u}_j, u_{j+1}))| \le Ch^2 |u_j - \hat{u}_j| \quad ; \ c \ge 0. \ (4\cdot 15)$$

Analogously bounded expressions exist for the other terms of (4·13) and the combination of (4·13)-(4·15) achieves the desired result, namely

$$\| J(\underline{u}) - J(\underline{\hat{u}}) \|_\infty \le \frac{T}{C} \| \underline{u} - \underline{\hat{u}} \|_\infty < Th \qquad (4\cdot16)$$

for any $\underline{u} \varepsilon \Omega_h (\underline{\hat{u}})$, $h < h_4$ where T is a positive constant.

Collecting together the relevant results we may now apply Urabe's proposition. Note that, for h sufficiently small, lemma (5), (4·9), (4·12) and (4·16) satisfy the prerequisite conditions over $\Omega_h (\underline{\hat{u}})$.

ie $$\| J(\underline{u}) - J(\underline{\hat{u}}) \|_\infty \le Th < \frac{k}{M}, \qquad 0 < k < 1$$

and $$\frac{MRh^2}{1 - k} \le ch$$

Therefore, we have proved the existence of the vector $\underline{u}^*$ such that $\underline{F}(\underline{u}^*) = \underline{0}$, and

$$\| \underline{\hat{u}} - \underline{u}^* \|_\infty \le \frac{MRh^2}{1 - k} \qquad (4\cdot17)$$

To determine a global convergence result we use (1·5), lemma (4), (4·7), (4·17) and the bound

$$\|y^*(x) - \hat{y}(x)\|_\infty \le \|y^*(x) - \pounds y^*\|_\infty + \| \pounds y^*(x) - \hat{u}(x)\|$$

$$+ \| \hat{u}(x) - \hat{y}(x)\|_\infty, \qquad x \in I_j.$$

to prove that

$$\| y^*(x) - \hat{y}(x) \|_\infty = \sup \{\| y^*(x) - \hat{y}(x) \|_\infty, \quad x \in I_j; \ j = 1,2, \ldots, m\}$$

$$= 0(h^2).$$

The results of this chapter are summarised in the following theorem;

<u>Theorem</u>

Let the classes of functions $\{t_j\}_{j=1}^{m}$ satisfy the conditions (A1)-(A4), then in a sufficiently small neighbourhood of the isolated solution, $\hat{y}(x)$, of the problem (1·1)-(1·3) there exists a unique regular spline solution $y^*(x)$ such that

(i) The knot values $\{y^*(x_j)\}_{j=1}^{m+1}$ satisfy the determining system of equations $\underline{F}(\underline{u}) = \underline{0}$.

(ii) $$\| \hat{y}(x) - y^*(x) \|_\infty = 0(h^2) \text{ as } h \to 0.$$

5. <u>Computational Aspects and Examples</u>

The selection of optimum classes of functions $\{\hat{t}_i\}_{i=1}^{n}$ is of fundamental importance. This process relies heavily on a preconceived notion of the analytic solution from the formulation of the problem. However, for a particular type of equation, we can be considerably influenced by the structure of the function $f(x,y(x))$ or by predetermining a characteristic of the analytic solution. A possibility is to asssume a power series expansion for $\hat{y}(x)$

ie $$\hat{y}(x) = (x - a_1)^\alpha \; (a_2 + a_3(x - a_1) + \ldots\ldots\ldots) \qquad (5\cdot1)$$

The exponent, α, is determined by substituting (5·1) into equation (1·1) and equating the least exponents of $(x-a_1.)$ on either side of the equation. Consequently, a feasible solution may incorporate the function

$$\hat{t}_i(x, c, d) = c(x - d)^{\alpha}, \alpha \neq 1,2.$$

Flexibility of application is an important feature of regular splines and different classes may be deployed over consecutive intervals $[z_{i-1}, z_i],[z_i, z_{i+1}]$. Computationally, this is facilitated by expressing (3.5) in a simplified form. For an arbitrary regular spline, $u(x)$, defined over I_j we have by lemma (1) that

$$\Delta^2(x_j, x_j, x_{j+1})u(x) = \frac{M_j}{3} + \frac{M_{j+1}}{6} + A_j. \qquad (5{\cdot}2)$$

With predetermined expressions for $\{A_i\}_{i=1}^{m}$ the terms (3·5) and (5·2) yield a computationally versatile system of equations, namely

$$U_i = y_a$$

$$\frac{1}{6}f(x_{j-1}, u_{j-1}) + \frac{2}{3}f(x_j, u_j) + \frac{1}{6}f(x_{j+1}, u_{j+1}) - \frac{1}{h^2}(u_{j-1} - 2u_j + u_{j+1})$$

$$+ A_{j-1} + A_j = 0 \quad i=2,3,\ldots,m$$

$$u_{m+1} = y_b \qquad (5{\cdot}3)$$

The expressions A_j for the examples of chapter 2 are

Example 1

$$A_j = \frac{M_{j+1}}{k(k-1)}\left[a^2 - 2a + 1 - \frac{k(k-1)}{6}\right] - \frac{M_j}{k(k-1)}\left[a^2 - ka + \frac{k(k-1)}{3}\right]$$

where $a = \dfrac{1}{1 - \left[\dfrac{M_{j+1}}{M_j}\right]^{\frac{1}{k-2}}}$

When $K = 3$, the cubic spline, $A_j = 0$.

Example 2

$$A_j = \sum_{n=4}^{\infty} \left[\frac{6 - n^2 + n}{6.n!} \right] M_j \left[\log \left[\frac{M_{j+1}}{M_j} \right] \right]^{n-2}$$

Example 3.

$$A_j = - M_j \left[\frac{a^2}{2} \log \left[\frac{M_j}{M_{j+1}} \right] + a + \frac{1}{3} \right] - \frac{M_{j+1}}{6}$$

$$\text{where} \quad a = \frac{1}{1 - \left[\frac{M_j}{M_{j+1}} \right]^{\frac{1}{2}}}$$

<u>Example 4</u>

$$A_j = -M_j \sum_{n=1}^{\infty} (-1)^{n+1} (\mu h)^{2n} \left[\frac{1}{(2n + n)!} - \frac{1}{6.2n!} \right]$$

$$+ \left[\frac{M_j \cos \mu h - M_{j+1}}{\sin \mu h} \right] \sum_{n=1}^{\infty} (-1)^{n+1} (\mu h)^{2n+1} \left[\frac{1}{(2n + 3)!} - \frac{1}{6.(2n + 1)!} \right]$$

In examples 2 and A, a truncation of the infinite series for A_i is used in numerical work.

A change in class of spline is frequently necessitated by the nature of the solution. The examples of chapter 2 illustrate that some classes of regular splines are not defined for all values of the second derivative. A common occurrence is that the sign of the second derivative must remain constant throughout the region of application. Such a spline, t(x), is invalid in a small neighbourhood of any point, η, where $t''(\eta) = 0$. The spline t(x), is obviously a 'bad fit' to the analytic solution in the neighbourhood of the point $x = \eta$. Consequently, we require a criterion to determine the deployment of the spline $t(x)$.

To illustrate how a regular collocation scheme may be applied we investigate a hypothetical problem. Assume that $\hat{y}(x)$ has a singularity at $x = a_1$, $a_1 \notin [a, b]$, but is regular elsewhere, ie $\hat{y}(x)$ is given by (5·1) with appropriate constants $\{a_i\}_{i=2}^{\infty}$ The exponent, α, is determined as. previously stated and hence the splines to be incorporated in the solution include the rational spline,

$$t(x) = a_j + b_j\left(x - x_j\right) + \frac{c_j}{(d_j - x + x_j)^{\alpha}} \qquad x \in I_j$$

The effect of the singularity on $y(x)$, $x \in [a,b]$, whether significant or not, lies chiefly in the region of a boundary point. The scheme proposed is to apply the rational spline over $[a,a'],[b',b]$ and the cubic spline over $[a',b']$, for suitable a',b'. In this way the spline solution can ' fit ' the effect of the singularity and rid us of the necessity to use extremely small values of h if this effect is overwhelming (cf.Problem 1). The selection of a' and b' will be influenced by the function $f(x,y(x))$and its values at $x = a$ and $x=b$. A judicious choice for a',b' removes the obstacle associated with the sign of $\hat{y}''(x)$. The cubic spline is only one possibility for the interval $[a', b']$, and any class of splines that is defined unconditionally may be used instead (eg.Examples 1 and 4). Solution of the appropriate system of equations (5·3) will yield information to formulate and solve a refined system. Using this information it is possible to realize the character of $\hat{y}(x)$ by evaluating certain structural parameters, q_j, derived by a direct comparison of the supposed structure of the analytic solution to the corresponding regular spline approximant. We qualify this process by referring to Examples 1.4 of chapter 2. Assume that for $x \in [\bar{a}, \bar{b}] \subseteq [a, b]$, and constants e,f,g and p:

c f. <u>Example 1.</u>

$$\text{If } \hat{y}(x) \simeq e + fx + g(p - x)^k \qquad k \neq 1,2$$

then $q_j \simeq d_j + x_j$ for any j such that $I_j \subset [a', b']$.

Similarly, we have

<u>Example 2</u>

$$\hat{y}(x) \simeq e + fx + g\, e^{px} \qquad q_j \simeq d_j$$

<u>Example 3</u>

$$\hat{y}(x) \simeq e + fx + g \log (p - x) \qquad q_j \simeq d_j + x_j$$

<u>Example 4</u>

$$\hat{y}(x) \simeq e + fx + g \sin (\mu x + p) \qquad q_j \simeq d_j - \mu x_j$$

Returning to our hypothetical example, let us assume that the parameters $\{q_j\}_{j=1}^{r}$ of the rational spline are closely grouped but for every other class of splines the associated parameters, $\{q_j\}_{n=r+1}^{m}$ vary substantially. We decide that, for $x \in [a, x_{r+1}]$, the rational spline is a good 'fit' whilst the cubic spline is probably best for $x \in [x_{r+1}, b]$.

Numerical criteria implementing the above ideas, are as follows: Let $\{q_j\}_{j=1}^{m}$ be the values of the parameter for an arbitrary class of regular splines

(CI)

$$\text{Let } r_J = \left| \frac{q_{j+1} - q_j}{\tilde{q}_j} \right| \text{ where } \bar{q}_j = \max \{1, |q_j|\}$$

Then, if

$$|r_{p+1} - r_p| = \min_{1 \le j \le m-1} \{|r_{j+1} - r_j|\} \begin{cases} \le ch, \text{ the spline is application} \\ > ch, \text{ the spline is not application} \end{cases}$$

for some C $0 < C \le \frac{1}{2}$

Normalise the values $\{q_j\}_{j=1}^{m}$ by

$$\hat{q}_j = \begin{cases} q_j & \text{if } q_p \le 1 \\ \dfrac{q_j}{q_p} & \text{if } q_p > 1 \end{cases}$$

(C2)

Apply the spline over the intervals $[x_r, x_{p+1}]$, and $[x_p, x_s]$ where the integers r and s satisfy

$$|\hat{q}_j - \hat{q}_p| < \bullet\ 2 \qquad j = r, r+1, \dots, p$$

$$|\hat{q}_p - \hat{q}_p| < \bullet\ 2 \qquad j = p+1, p+2, \dots, s-1$$

The effect on the solution of the parameter C in (C1) will be discussed later.

We may now define a remodelled system of equations (5·3) based on the criteria (C1) and (C2). The solution of the first system of equations is an excellent initial value to the remodelled problem, and comparatively little extra effort is required to solve this additional iterative problem.

Four problems are evaluated by the above criteria. For comparative purposes the problems are also evaluated by the cubic spline collocation method and by Numerov's method. As previously stated, the latter is a fourth—order finite difference scheme. For simplicity of notation we define by E the absolute error, and E_r the relative error. The parameter C of (C1) is taken to be $C = \frac{1}{2}$.

<u>Problem 1</u>

$$y''(x) = \frac{2(y(x) + x^2)^3}{1.01^2} - 2$$

$$y(0) = 101, \quad y(l) = 0$$

$$\hat{y}(x) = \frac{1.01}{(x + .01)} - x^2$$

Table 1 The regular spline solution $y^*(x)$ uses the rational spline $K = -1$, and the cubic spline.

X	$y^*(x)$	E	E_r	h
.05	16.8358	4.97×10^{-3}	2.95×10^{-4}	. 1
.2	4.7728	3.32×10^{-3}	6.96×10^{-4}	. 1
.5	1.7331	2.68×10^{-3}	1.55×10^{-3}	. 1
.025	28.8576	1.08×10^{-3}	3.76×10^{-5}	.05
.02	4.7703	8.21×10^{-4}	1.72×10^{-4}	.05
.5	1.7309	5.41×10^{-4}	3.13×10^{-4}	.05

The values of the parameters $\{\hat{q}_j\}_{j=1}^{8}$, h = .1, are

$\hat{q}_1 = -0.00999$, $\hat{q}_2 = -0.00947$, $\hat{q}_3 = -0.00583$

$\hat{q}_4 = 0.00589$, $\hat{q}_5 = 0.0330$, $\hat{q}_6 = 0.0854$

$\hat{q}_7 = 0.175$, $\hat{q}_8 = 0.318$

The cubic spline and Numerov's solutions are too inaccurate to give a useful comparison with the above values of h.

Problem2 $y''(x) = \frac{(y(x) + x^4)^3}{50} - 12x^2, \qquad y(-\frac{1}{2}) = 19.9375, \qquad y(2) = -\frac{38}{3}$

$$\hat{y}(x) = \frac{10}{x+1} - x^4$$

The regular spline incorporated the rational spline, K = -1,the cubic spline and the polynomial spline, K= 4.

		Regular Spline		Cubic Spline		Numerov's Method	
x	h	E	E_r	E	E_r	E	E_r
-·3	0·1	$3{\cdot}90 \times 10^{-4}$	$2{\cdot}73 \times 10^{-5}$	$3{\cdot}90 \times 10^{-2}$	2.73×10^{-3}	$1{\cdot}23 \times 10^{-3}$	8.63×10^{-5}
0·2	0·1	$1{\cdot}72 \times 10^{-4}$	$2{\cdot}07 \times 10^{-5}$	$2{\cdot}31 \times 10^{-2}$	2.78×10^{-3}	$6{\cdot}11 \times 10^{-4}$	$7{\cdot}33 \times 10^{-5}$
310·8	0·1	$1{\cdot}05 \times 10^{-4}$	$2{\cdot}04 \times 10^{-5}$	$7{\cdot}039 \times 10^{-3}$	$1{\cdot}44 \times 10^{-3}$	$2{\cdot}67 \times 10^{-4}$	$5{\cdot}19 \times 10^{-5}$
1·5	0·1	$5{\cdot}46 \times 10^{-4}$	$5{\cdot}14 \times 10^{-4}$	$1{\cdot}36 \times 10^{-4}$	1.28×10^{-4}	$9{\cdot}10 \times 10^{-5}$	$8{\cdot}57 \times 10^{-5}$
- ·3	0·05	$1{\cdot}01 \times 10^{-4}$	$7{\cdot}04 \times 10^{-6}$	$9{\cdot}46 \times 10^{-3}$	6.63×10^{-4}	$7{\cdot}99 \times 10^{-5}$	$5{\cdot}60 \times 10^{-6}$
0·2	0·05	$4{\cdot}58 \times 10^{-5}$	$5{\cdot}50 \times 10^{-6}$	$5{\cdot}66 \times 10^{-3}$	6.79×10^{-4}	$3{\cdot}94 \times 10^{-5}$	$4{\cdot}73 \times 10^{-6}$
0·8	0·05	$2{\cdot}74 \times 10^{-5}$	$5{\cdot}33 \times 10^{-5}$	$1{\cdot}80 \times 10^{-3}$	3.49×10^{-4}	$1{\cdot}72 \times 10^{-5}$	$3{\cdot}34 \times 10^{-6}$
1·5	0·05	$1{\cdot}36 \times 10^{-4}$	$1{\cdot}28 \times 10^{-4}$	$5{\cdot}00 \times 10^{-5}$	4.70×10^{-5}	$5{\cdot}86 \times 10^{-6}$	$5{\cdot}51 \times 10^{-6}$

The parameters $\{\hat{q}_j\}_{j=1}^{5}$ of the rational spline with h = .1 , are

$$\hat{q}_1 = -.99798, \quad \hat{q}_2 = -1.00327, \quad \hat{q}_3 = -1.0123, \quad \hat{q}_4 = -1.0202, \quad \hat{q}_5 = -1.0143$$

<u>Problem 3</u> $\quad y''(x) = \frac{1}{5}(y(x) + x^2(x-1)^2)^2 - (12x^2 + 2 - 12x) \quad , (y(0) = 120, \quad y(1) = \frac{40}{3}$

$$\hat{y}(x) = \frac{30}{(x + \frac{1}{2})^2} - x^2(x-1)^2$$

The regular spline solution incorporated the rational spline, $\kappa = 2$, and the cubic spline.

		Regular Spline		Cubic Spline		Numerov's Method	
X	h	E	E_r	E	E_r	E	E_r
•2	•1	$9{\cdot}93 \times 10^{-4}$	$1{\cdot}62 \times 10^{-5}$	0·657	$1{\cdot}07 \times 10^{-2}$	$2{\cdot}90 \times 10^{-2}$	$4{\cdot}74 \times 10^{-4}$
•5	•1	$7{\cdot}96 \times 10^{-6}$	$2{\cdot}66 \times 10^{-7}$	0·360	$1{\cdot}20 \times 10^{-2}$	$1{\cdot}30 \times 10^{-2}$	$4{\cdot}34 \times 10^{-4}$
•8	•1	$5{\cdot}32 \times 10^{-4}$	$3{\cdot}00 \times 10^{-5}$	0·130	$7{\cdot}34 \times 10^{-3}$	$4{\cdot}27 \times 10^{-3}$	$2{\cdot}41 \times 10^{-4}$
•2	•05	2.50×10^{-4}	$4{\cdot}09 \times 10^{-6}$	0·157	$2{\cdot}56 \times 10^{-3}$	$1{\cdot}90 \times 10^{-3}$	$3{\cdot}10 \times 10^{-5}$
•5	•05	5.88×10^{-6}	$1{\cdot}96 \times 10^{-7}$	$8{\cdot}71 \times 10^{-2}$	$2{\cdot}91 \times 10^{-3}$	$8{\cdot}47 \times 10^{-4}$	$2{\cdot}83 \times 10^{-5}$
•8	•05	$1{\cdot}20 \times 10^{-4}$	$6{\cdot}75 \times 10^{-6}$	$3{\cdot}17 \times 10^{-2}$	$1{\cdot}79 \times 10^{-3}$	$2{\cdot}77 \times 10^{-4}$	$1{\cdot}57 \times 10^{-5}$

<u>Problem 4</u>

$$y''(x) = 16y - (\pi^2 + 16)\sin \pi x \quad , \quad y(1) = e^4 \quad , \quad y(2) = e^8$$

$$\hat{y}(x) = e^{4x} + \sin \pi x$$

The exponential spline comprised the regular spline solution.

		Regular Spline		Cubic Spline		Numerov 's Method	
X	h	E	E_r	E	E_r	E	E_r
1·2	·1	$2·05 \times 10^{-3}$	$1·69 \times 10^{-5}$	1·96	$1·62 \times 10^{-2}$	$1·54 \times 10^{-2}$	$1·28 \times 10^{-4}$
1.5	·1	$1·92 \times 10^{-3}$	$4·76 \times 10^{-6}$	5·25	$1·30 \times 10^{-2}$	$4·12 \times 10^{-2}$	$1·02 \times 10^{-4}$
1·8	·1	$4·27 \times 10^{-3}$	$3·19 \times 10^{-6}$	7·21	$5·38 \times 10^{-3}$	$5·64 \times 10^{-2}$	$4·2f \times 10^{-5}$
1·2	·05	$5·02 \times 10^{-3}$	$4·15 \times 10^{-6}$	0·485	$4·01 \times 10^{-3}$	$9·68 \times 10^{-4}$	$8·01 \times 10^{-6}$
1·5	·05	$4·84 \times 10^{-4}$	$1·20 \times 10^{-6}$	1·30	$3·23 \times 10^{-3}$	$2·59 \times 10^{-3}$	$6·44 \times 10^{-6}$
1·8	·05	$1·07 \times 10^{-3}$	$7·99 \times 10^{-7}$	1·78	$1·33 \times 10^{-3}$	$3·54 \times 10^{-3}$	$2·65 \times 10^{-6}$

Synopsis

The previous chapters generalise the well-established theory of the cubic spline collocation scheme to classes of regular spline collocation schemes. We have proved that the existence and convergence of the latter schemes is implied by that of the former. Consequently, we may now consider classes of schemes wherein, formally, only the cubic spline collocation option existed.

Numerically, the versatility of the proposed scheme is of major importance. The classes of splines utilised depends on the ingenuity of the user. These may include the examples of chapter 2 or a class derived from an intuitive idea of the dominent terms of the true solution. Corresponding to the classes employed, structural parameters will be evaluated and these may yield desirable information e.g. the location of a singularity. The numerical examples of chapter 5 illustrate the increased accuracy obtainable by a judicious application of regular splines compared with the cubic spline. Also, the results give a favourable comparison with Numerov's method, for the specified values of h. However, as $h \to O$, a fourth order method will converge faster than the second order collocation scheme and the comparison must favour the former. Yet, cf.problem 1, meaningful results may be obtained by the collocation scheme when the fourth order, polynomial based methods are inapplicable.

At this point we introduce the paper by Daniel and Swartz [5]. They derive a fourth-order, cubic spline scheme by collocating to a perturbed differential equation which is satisfied by the cubic spline interpolant of the true solution. The generalisation of their work to incorporate regular splines is a research possibility for the future.

We now consider the effect of varying the parameter C of (CI), chapter 5. Obviously as $C \to O$ the number of splines satisfying (CI) will decrease and may equal zero. However, a small value of C will ensure applicability of a spline that imitates the dominant structure of the true solution in a subregion of [a,b]. In particular $C = 1/16$ ensures applicability of the rational splines in problem 1 and 2, whilst $C = 1/100$ is sufficient for the rational spline in problem 1. Note that the evaluation of problem 2 by Numerov's method is perfectly acceptable, and hence, to detect problems to which regular splines are especially recommended, we suggest a value of $C = 1/30$. For comparison with the cubic spline collocation scheme the value $C = \frac{1}{2}$ is acceptable.

Let us conclude with the following comments. The regular spline collocation scheme is meaningful and interesting in itself, but note that the convergence is second order. Taking the parameter $C = \frac{1}{2}$ we achieve better results than those obtained by solely considering the cubic spline. However, if a polynomial based spline closely interpolates the true solution, without requiring excessively small values of h, it appears likely that a fourth order scheme is preferable. Therefore, an interesting possibility is the production of computer packages for the problem (1.1)-(1.3) involving the regular spline collocation scheme and some fourth order method. The collocation scheme may be applied, with $C = 1/30$, to remove the necessity of using excessively small values of h. Initially we employ the collocation scheme, to investigate the suitability of appropriate classes of regular splines, and then switch to the fourth order scheme if none are revealed.

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