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A NEW CLASS OF MATRICES WITH POSITIVE INVERSES.

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A B S T R A C T

It is well known that irreducibly diagonally-dominant matrices with positive diagonal and non-positive off-diagonal elements have positive inverses. A whole class of symmetric circulant and symmetric quindiagonal Toeplitz matrices with positive inverses which do not satisfy the above conditions is found.

Introduction

A condition that the zeros of a certain polynomial are positive implies that the corresponding quindiagonal matrix has a positive inverse. A similar condition means that the related circulant with five entries per row has a positive inverse. The quindiagonal matrix is analysed by expressing the first row of its inverse as the solution of a finite difference equation and showing that this solution is positive. A result of Trench (1964) may then "be used to prove that for symmetric quindiagonal Toeplitz matrices, the complete inverse is positive if the first row of the inverse is positive. The behaviour of the first row of the inverse for very large orders is considered and examples of this new class for matrices are given.

Main Theorems

Theorem 1. (Proof in section 3)

If the real numbers a and b, b non-zero, are chosen so that the symmetric polynomial

$$bx^4 + ax^3 + x^2 + ax + b \qquad (1)$$

has real positive zeros, then the inverse of the symmetric quindiagonal Toeplitz matrix,

$$T_n = \begin{pmatrix} 1 & a & b & & & & \\ a & 1 & a & b & & \text{\Large 0} & \\ b & a & 1 & a & \ddots & & \\ & b & a & \ddots & \ddots & \ddots & b \\ & & \ddots & \ddots & \ddots & \ddots & a \\ & \text{\Large 0} & & \ddots & \ddots & \ddots & \\ & & & & b & a & 1 \end{pmatrix}_{nxn},$$

consists of positive elements for all order n.

The values of a and b for which the polynomial (l) has real positive zeros may be described as follows. Suppose the real zeror. of polynomial (l) are r_1 , l/r_1 , r_2 and l/r_2 where r_1 and r_2 are positive, then the values a and b may be expressed as

$$a = -\frac{(r_1 + 1/r_1) + (r_2 + 1/r_2)}{(r_1 + 1/r_1) + (r_2 + 1/r_2) + 2} \ ,$$

and

$$b = \frac{1}{(r_1 + 1/r_1) + (r_2 + 1/r_2) + 2} \ .$$

It is easy to verify that $-2/3 < a < 0$ and $0 < b \leq 1/6$.

The relation $a + b \geq ½$ follows from the inequality

$(r_1 + l/r_1 - 2)(r_2 + l/r_2 - 2) \geq 0$ and $a^2 + 8b^2 - 4b \geq 0$ is the condition that r_1 and r_2 are real. The above inequalities describe a region, R, in the a - b plane bounded by the three curves $b = 0$, $a + b = -\frac{1}{2}$ and $a^2 + 8b^2 - 4b = 0$ with intersection points (0,0),(-1/2,0) and (- 2/3, 1/6).

Theorem 2.

If the real numbers a and b, b non-zero, are chosen such that the symmetric polynomial

$$bx^4 + ax^3 + x^2 + ax + b \qquad (l)$$

has real positive zeros, not equal to one, then the inverse of the symmetric circulant matrix

$$C_n = \text{circ}\ (l,a,b,0\ \ldots\ ,0,b,a)_{nxn} \qquad (3)$$

consists of positive entries for all orders n.

Proof

The matrix C_n , in (3), can be factored into the product of two matrices,

$$C_n = [(r_1 + 1/r_1)I - (P + P^{-1})][(r_2 + 1/r_2)I - (P + P^{-1})]$$, where P is the permutation matrix circ $(0,1,0,\ldots,0)_{nxn}$ and r_1 and r_2 are zeros of (l) not equal to 1. However, both factors are irreducibly diagonally dominant and have positive inverses, Varga (1962), and consequently, the inverse of the matrix Cn is positive.

The above method of proof may be used for a more general theorem concerning circulant matrices, Meek (1973).

Theorem 3. (Trench(1964)).
If T_n is the n^{th} order, $n \geq 3$, symmetric Toeplitz matrix in (2) with $T_n^{-1} > 0$ .and the first row of the matrix T_{n+1}^{-1} is positive, then the matrix $T_{n+1}^{-1} > 0$.

Proof

The matrix T_{n+1}^{-1} may be partitioned in the form

$$T_{n+1}^{-1} = \begin{bmatrix} t & W_n^T \\ W_n & T_n^{-1} + W_n W_n^T / t \end{bmatrix}_{n+1 \times n+1}$$

If the first row of $T_{n+1}^{-1} > 0$ · is positive, then both t and the lxn Vector W_n^T are positive. However, thus $T_n^{-1} > 0$, all of the elements of the matrix T_{n+1}^{-1} are positive.

3. Proof of Theorem 1.

The first row of the matrix T_n^{-1} is shown positive by solving a difference equation so that theorem 3 is an induction step in the proof of theorem 1. It is easy to verify that $T_3^{-1} > 0$ when a and b are chosen so that (a,b) is in region R.

The difference equation for the elements of the first row of the matrix T_n^{-1} is

$$bD_{r-2} + aD_{r-1}, + D_r + aD_{r+1} + bD_{r+2} = e_r, r = 1,2,\dots,n, \qquad (4)$$

with end conditions $D_{-1} = D_0 = D_{n+1} = D_{n+2} = 0$, where

$$e_r = \begin{cases} 1 & r = 1 \\ 0 & r = 2,3,\dots\dots,n. \end{cases}$$

A particular solution to equation (4) is the function

$$H_r = \begin{cases} 1/b & r = -1 \\ 0 & r = 0,1,\dots,n+2, \end{cases}$$

while the general solution falls into four cases, depending upon the zeros of the polynomial (l). The more general case of positive real distinct zeros not equal to 1 will be discussed first.

The zeros of the polynomial (l) are positive, distinct and not equal to 1, if and only if a and b are such that the point (a,b) lies in the interior of the region R. For convenience take $r_1 > r_2 > 1$ to be zeros of (l), then the general solution to the difference equation (4) is of the form

$$D_r = Ar_1^{-r} + Br_1^{-r} + Cr_2^{r} + Dr_2^{-r} H_r, r = -1,0,\dots\dots, n+2.$$

If the function f(r,n) is defined

$$f(r,n) = s_r t_1 + s_1 t_r + s_{n+2} t_{n-r+1} - s_{n-r+2} t_{n+1} - s_{n+1} t_{n-r+2} + s_{n-r+1} t_{n+2} \ , \qquad (5)$$

where $s_\omega \equiv r_1^{\omega} - r_1^{-\omega}$ and $t_\omega \equiv r_2^{\omega} - r_2^{-\omega}$, then the solution satisfying the end condition is

$$D_r = f(r,n)/ (b\ f(l,n+l)) + H_r \quad , r = -1,0,\dots,n+2. \qquad (6)$$

As n becomes large, D_r approaches the function

$$\lim_{n \to \infty} D_r = \begin{cases} 1/(b r_1, r_2) & t = 0 \\ 0 & 0 < t \le 1 \end{cases} ,$$

where $r \equiv (n-l)t+l$.

The substitution $r_1 = e^{(\theta+\psi)}$, $r_2 = e^{(\theta+\psi)}$, where $\theta > \psi > 0$, transforms the function f(r,n) in (5) into

$$f(r,n) = l6 \sinh(n+l)\theta \ \sinh(n-r+2)\theta \ \sinh r\psi \ \sinh\psi$$
$$- \ l6 \sinh(n+l)\psi \ \sinh(n-r+2)\psi \ \sinh r\theta \ \sinh\theta$$

which is positive for $r = 1,2,\dots,n$ since both

$$\sinh(n+l)\theta \ \sinh r\psi \ - \ \sinh(n+l)\psi \ \sinh r\theta$$

and

$$\sinh(n-r+2)\theta \ \sinh\psi \ - \ \sinh(n-r+2)\psi \ \sinh e$$

are positive r = 1,2,... ,n, (see the appendix) - Now tooth b and f(r,n) are positive, thus D_r , r= 1,2,...,n in equation (6) is positive.

The results for the remaining three cases may now be summarized. The polynomial (l) may have one repeated zero, that is the zeros are r_1, $1/r_1$,l, l,$r_1 > 1$, and a+b = -1/2, or it may have two pairs of repeated zeros, that is the zeros are r_1 , $1/r_1$, r_1 ,$1/r_1$,$r_1 > 1$ and

$a^2 + 8b^2 - 4b = 0$, of it. may have all. Four zeros equal to 1, whence $a = -^2/_3$ $b = ^1/_6$. The solulion of the difference equation (4) is of the same form as in equation (6) with the function f(r,n) being defined

$$f(r,n) = s_{r+}rs_{1+}(n-r+1)s_{n+2}-(n+1)s_{n-r+2}-(n-r+2)s_{n+1}+^{(n+2)}s_{n-r+1},$$

$$f(r,n) = rs_{n+1}s_{n-r+2}-(n+1)(n-r+2)s_rs_1 ,$$

and

$$f(r,n) = r(n-r+l)(n-r+2),$$

respectively. As n becomes large, D_r, in these three cases approaches the functions

$$\lim_{n\to\infty} D_r = \begin{cases} 1/(br_1) & t = 0 \\ (1-t)/(b(r_1-1)) & 0 < t \le 1 \end{cases},$$

$$\lim_{n\to\infty} D_r = \begin{cases} 1/(br^2{}_1) & t = 0 \\ 0 & 0 < t \le 1 \end{cases},$$

and

$$\lim_{n\to\infty} D_r = \begin{cases} 6 & t = 0 \\ 6nt(1-t)^2 & 0 < t \le 1 \end{cases},$$

where $r=(n-1)t+1$.

The substitution $r_1 = e^{2\theta}$, $\theta > 0$, in the first case yields

$$f(r,n) = 8\sinh(n-r+2)\theta \ \sinh\theta[(n+l)\cosh(n+l)\theta \ \ \sinh r\theta - r\cosh r\theta \ \ \sinh(n+l)\theta]$$

$$+ 8\sinh(n+l)\theta \ \ \sinh r9\,[(n-r+2)\cosh(n-r+2)\theta \ \sinh\theta - \cosh\theta \ \sinh(n-r+2)\theta]$$

in which both of the terms are positive for each n and r=l,2,...,n (see the appendix). The substitution $r_1 = e^{\theta}$, $\theta > 0$, in the second case gives

f(r,n) = 4[rsinh(n+l)θ]:sinh(n-r+2)θ]-U[(n+l)sinhre3L(n-p+2)sinhe]

which is also positive for each n and r=l,2,— n (see the appendix). In the third case, f(r,n) is ohviously positive for each n and $r = 1,2,\ldots, n$.

4. <u>Examples</u>

The matrix arising from quintic polynomial spline interpolation on a uniform partition with non-periodic boundary conditions is the quindiagonal

$$A = \begin{pmatrix} 66 & 26 & 1 & & \text{\Large 0} \\ 26 & 66 & \ddots & \ddots & \\ 1 & \ddots & \ddots & \ddots & 1 \\ & \ddots & \ddots & \ddots & 26 \\ \text{\Large 0} & & 1 & 26 & 66 \end{pmatrix} ,$$

Ahlberg, Nilson and Walsh, p. 124 (1967). The related matrix DAD^T , where D is the diagonal matrix diag $(1,-1,\ldots,(-1)^{n-1})_{nxn}$, has a positive inverse since the zeros of

$$1/66\, z^4 - 26/66\, z3 + z^2 - 26/66\, z + 1/66$$

are real and positive

Hoskins and Ponzo (1972) have found another class of symmetric Toeplitz matrices with positive inverses which intersects with the class described here in the nxn matrix

$$\begin{pmatrix} 1 & -2/3 & 1/6 & & \text{\Large 0} \\ -2/3 & 1 & \ddots & \ddots & \\ 1/6 & \ddots & \ddots & \ddots & 1/6 \\ & \ddots & \ddots & \ddots & -2/3 \\ \text{\Large 0} & & 1/6 & -2/3 & 1 \end{pmatrix}$$

6. Appendix

Lemma 1.

The function (sinhmθ)m is greater than θ and monotone increasing
in m when $\theta > 0, m > 0$.

Proof

The result follows from a Maclaurin expansion of sinhmθ.

Lemma 2.

The function m cothmθ is greater than θ and is monotone increasing in m, $\theta > 0, m > 0$.

Proof

The derivative of m cothm9 with respect to m can be shown to be positive using lemma 1. for $\theta > 0$, $m > 0$ and $\lim_{m\to 0}$ (m cothmθ) = θ.

Lemma 3.

The function (sinhmθ)/(sinhmψ) is greater than θ/ψ and increases
with m, $\theta > 0$, $\psi > 0$, $m > 0$.

The derivative of (sinhmθ)/(sinhimψ) with respect to m can be shown to be positive using lemma 2. and $\lim_{m\to 0}$ (sinhmθ)/(sinhimy) = θ/ψ.

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