Envelope-Constrained H_∞ Filtering for Nonlinear Systems with Quantization Effects: The Finite Horizon Case ⋆

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Abstract

This paper is concerned with the envelope-constrained H_∞ filtering problem for a class of discrete nonlinear stochastic systems subject to quantization effects over a finite horizon. The system under investigation involves both deterministic and stochastic nonlinearities. The stochastic nonlinearity described by statistical means is quite general that includes several well-studied nonlinearities as its special cases. The output measurements are quantized by a logarithmic quantizer. Two performance indices, namely, the finite-horizon H_∞ specification and the envelope constraint criterion, are proposed to quantify the transient dynamics of the filtering errors over the specified time interval. The aim of the proposed problem is to construct a filter such that both the prespecified H_∞ requirement and the envelope constraint are guaranteed simultaneously over a finite horizon. By resorting to the recursive matrix inequality approach, sufficient conditions are established for the existence of the desired filters. A numerical example is finally proposed to demonstrate the effectiveness of the developed filtering scheme.

Key words: Nonlinear systems, H_∞ filtering, envelope constraint, quantization effects, finite-horizon filtering

1 Introduction

Due to the significance in control and signal processing, the nonlinear filtering problem has been attracting constant research interest in the past several decades. A number of approaches have been developed to deal with the filtering problem for nonlinear stochastic systems, among which some of the most widely used include but are not limited to Bayes filtering, particle filtering, extended Kalman filtering (EKF) and unscented Kalman filtering (UKF). The Bayes filter aims to, in a recursive fashion, estimate the hidden state by using the available measurements and the process model [9]. Based on the Bayesian theory in combination with the concept of sequential importance sampling, particle filtering is particularly useful in coping with nonlinear and/or non-Gaussian problems [4]. However, the high computational complexity largely hinders the utilization of particle filters. Another recursive filter that should be mentioned is the celebrated Kalman filter [12], which is in fact a linear version of Bayes filter for systems subject to Gaussian noises. As for nonlinear stochastic Gaussian systems, several invariants based on Kalman filters have been developed among which the most widely applied are EKF and UKF. EKF provides an approximation of an optimal estimate by linearizing the nonlinear system at the state estimates, which has been found wide applications in both theoretical research and engineering practice [7]. However, it is no longer applicable when the process/measurement models are highly nonlinear, which gives rise to the so-called unscented Kalman filtering. The UKF uses a deterministic sampling technique known as the unscented transform to pick a minimal set of sample points around the mean value and could give more accurately estimates than EKF especially for those highly nonlinear systems [20].

The past several decades have seen a surge of research interest on the H_∞ filtering problems for nonlinear systems and several effective approaches have been exploited to deal with filtering problems with the requested disturbance attenuation level, see e.g. [5, 7, 17–19, 21]. On another research frontier, networked control systems (NCSs) have attracted much attention owing to their clear application insights in a wide range of areas [27,28]. It has been recognized that, in the context of NCSs, the quantization effects stemming from analog-to-digital conversion processes are ubiquitous, which would probably lead to the deterioration of the system performance. In the NCS research, there are mainly two types of quantization models, namely, the uniform quantization [23] and the logarithmic quantization [8]. In particular, a sector-bound technique has been presented in [8] that is capable of coping with the logarithmic quantization

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issues conveniently, and such an elegant paradigm has then been quickly followed in the area, see e.g. [13,18].

The envelope-constrained filtering (ECF) algorithm has been stirring some research interest in the past few decades. The main idea of ECF algorithm is to confine the output of the filtering error (stimulated by a specified input) into a prescribed envelope. Such an envelope is determined by the desired output and tolerance band. The ECF technique has found successful applications in a variety of engineering branches ranging from signal processing to digital communications [2,3]. Up to now, several methodologies have been utilized in the literature to deal with the envelope-constrained filtering problems, see, e.g. [22,26]. It should be pointed out that almost all the results relevant to ECF have been concerned with the *linear time-invariant* systems. When it comes to *general nonlinear time-varying* systems, the corresponding envelope-constrained filtering problem has not been thoroughly investigated yet and this motivates us to shorten such a gap in the current study. It is, therefore, the main purpose of this paper to deal with the identified challenges by launching a major study on the so-called envelope-constrained H_∞ filtering problem.

The rest of this paper is organized as follows. Section 2 formulates the envelope-constrained H_∞ filtering problem for discrete-time nonlinear system subject to quantization effects. The main results are presented in Section 3 where sufficient conditions for solvability of the addressed filtering problem are given in terms of recursive linear matrix inequalities (RLMIs). Section 4 gives a numerical example and Section 5 outlines our conclusion.

2 Problem Formulation

Consider the following nonlinear system defined on the horizon $[0, N]$:

$$\begin{cases} x_{k+1} = f(x_k) + g(x_k) + B_k w_k \\ y_k = h(x_k) + D_k v_k \\ z_k = L_k x_k \end{cases} \tag{1}$$

where $x_k \in \mathbb{R}^{n_x}$, $y_k \in \mathbb{R}^{n_y}$ and $z_k \in \mathbb{R}^{n_z}$ represent, respectively, the system state, the measurement output and the signal to be estimated. $w_k \in l_2([0, N]; \mathbb{R}^{n_w})$ and $v_k \in l_2([0, N]; \mathbb{R}^{n_v})$ are the disturbance inputs. B_k, D_k and L_k are known time-varying matrices with appropriate dimensions.

The deterministic nonlinearities $f(x_k)$ and $h(x_k)$ are known and analytic everywhere over the finite horizon $[0, N]$. On the other hand, the stochastic nonlinearity $g(x_k)$ is assumed to have the following first moment for all x_k:

$$\mathbb{E}\{g(x_k)|x_k\} = 0 \tag{2}$$

with the covariance given by

$$\begin{aligned} \mathbb{E}\{g(x_k)g^{\mathrm{T}}(x_j)|x_k\} &= 0, \quad k \neq j \\ \mathbb{E}\{g(x_k)g^{\mathrm{T}}(x_k)|x_k\} &= \sum_{l=1}^{q} \varrho_{l,k}\varrho_{l,k}^{\mathrm{T}}\big(x_k^{\mathrm{T}}\Upsilon_{l,k}x_k\big) \end{aligned} \tag{3}$$

where $\varrho_{l,k}$ and $\Upsilon_{l,k} \geq 0$ $(l = 1, 2, \ldots, q)$ are, respectively, known column vectors and matrices with compatible dimensions.

In this paper, the quantization effects are taken into consideration. Denote the quantizer as

$$\sigma(\cdot) \triangleq \Big[\, \sigma_1(\cdot)\ \sigma_2(\cdot)\ \cdots\ \sigma_{n_y}(\cdot) \,\Big]$$

which is symmetric, i.e., $\sigma_j(-y) = -\sigma_j(y)$ $(j = 1, 2, \ldots, n_y)$. The quantizer is assumed to be logarithmic type and the process of the quantization is described by

$$\sigma(y_k) = \Big[\, \sigma_1(y_k^{(1)})\ \sigma_2(y_k^{(2)})\ \cdots\ \sigma_{n_y}(y_k^{(n_y)}) \,\Big]^{\mathrm{T}} \tag{4}$$

where $y_k^{(j)}$ $(j = 1, 2, \ldots, n_y)$ denotes the j-th entry of the vector y_k. For each $\sigma(\cdot)$, the set of quantization level is described by

$$\begin{aligned} \mathscr{U}_j = &\big\{ \pm\hat{\mu}_i^{(j)},\ \hat{\mu}_i^{(j)} = \chi_j^i \hat{\mu}_0^{(j)},\ i = 0, \pm 1, \pm 2, \ldots \big\} \cup \big\{0\big\}, \\ &0 < \chi_j < 1,\ \hat{\mu}_0^{(j)} > 0. \end{aligned} \tag{5}$$

where χ_j $(j = 1, 2, \ldots, n_y)$ is the quantization density. Each of the quantization level corresponds to a segment such that the quantizer maps the whole segment to this quantization level. According to [8], the associated quantizer is defined as follows:

$$\sigma(y_k^{(j)}) = \begin{cases} \hat{\mu}_i^{(j)}, \dfrac{1+\chi_j}{2}\hat{\mu}_i^{(j)} \leq y_k^{(j)} \leq \dfrac{1+\chi_j}{2\chi_j}\hat{\mu}_i^{(j)} \\ 0, \qquad\qquad y_k^{(j)} = 0 \\ -\sigma(-y_k^{(j)}), \qquad y_k^{(j)} < 0 \end{cases} \tag{6}$$

Consequently, it can be easily seen from the above definition (6) that the following inequality holds:

$$\big(\sigma(y_k) - G_1 y_k\big)^{\mathrm{T}}\big(\sigma(y_k) - G_2 y_k\big) \leq 0 \tag{7}$$

where $G_1 \triangleq \mathrm{diag}_{n_y}\{2\chi_j/(1+\chi_j)\}$ and $G_2 \triangleq \mathrm{diag}_{n_y}\{2/(1+\chi_j)\}$. Since $0 < \chi_j < 1$, it is obvious that $0 \leq G_1 < I \leq G_2$. Then, $\sigma(y_k)$ can be decomposed as follows:

$$\sigma(y_k) = G_1 y_k + \varphi(y_k) \tag{8}$$

where $\varphi(y_k)$ is a nonlinear vector-valued function which, from (7), satisfies

$$\varphi^{\mathrm{T}}(y_k)\big(\varphi(y_k) - G y_k\big) \leq 0 \tag{9}$$

with G being defined as $G \triangleq G_2 - G_1$.

Definition 1 *[6] A bounded ellipsoid $\mathscr{E}(c, P, n)$ of $\mathbb{R}^n$ with a nonempty interior in the mean square sense can be defined by*

$$\mathscr{E}(c, P, n) \triangleq \{x \in \mathbb{R}^n : \mathbb{E}\{(x - c)^{\mathrm{T}} P^{-1} (x - c)\} \leq 1\}$$

where $c \in \mathbb{R}^n$ is the center of $\mathscr{E}(c, P, n)$ and $P > 0$ is a positive definite matrix.

In this paper, the filter to be designed is of the following form:

$$\hat{x}_{k+1} = F_k \hat{x}_k + H_k \sigma(y_k), \quad \hat{x}_0 = 0. \tag{10}$$

Denote $e_k \triangleq x_k - \hat{x}_k$ and $\tilde{z}_k \triangleq z_k - \hat{z}_k$. Subtracting (10) from (1) and taking (8) into account, we obtain the following filtering error system:

$$\begin{cases} e_{k+1} = f(x_k) + g(x_k) + B_k w_k - F_k \hat{x}_k \\ \qquad\quad - H_k G_1 h(x_k) - H_k \varphi(y_k) - H_k G_1 D_k v_k \\ \tilde{z}_k = L_k e_k \end{cases} \tag{11}$$

By defining

$$\Phi_k \triangleq \frac{\partial f(x)}{\partial x}\Big|_{x=\hat{x}_k},\ \Psi_k \triangleq \frac{\partial h(x)}{\partial x}\Big|_{x=\hat{x}_k},$$

and utilizing the Taylor series expansion formula, we linearize the nonlinear functions $f(x_k)$ and $h(x_k)$ around the state estimate $\hat{x}_k$ as follows:

$$f(x_k) = f(\hat{x}_k) + \Phi_k(x_k - \hat{x}_k) + L_1\Delta_1(x_k - \hat{x}_k) \quad (12)$$

$$h(x_k) = h(\hat{x}_k) + \Psi_k(x_k - \hat{x}_k) + L_2\Delta_2(x_k - \hat{x}_k) \quad (13)$$

where $L_1 \in \mathbb{R}^{n_x \times n_{l_1}}$ and $L_2 \in \mathbb{R}^{n_y \times n_{l_2}}$ are known scaling matrices, and $\Delta_1 \in \mathbb{R}^{n_{l_1} \times n_x}$ and $\Delta_2 \in \mathbb{R}^{n_{l_2} \times n_x}$ are unknown matrices such that $\|\Delta_1\| \leq 1$ and $\|\Delta_2\| \leq 1$.

Taking (12) and (13) into consideration, we reformulate the filtering error system (11) as

$$\begin{cases} e_{k+1} = f(\hat{x}_k) - F_k\hat{x}_k - H_kG_1h(\hat{x}_k) \\ \qquad\quad + (\Phi_k + L_1\Delta_1 - H_kG_1\Psi_k - H_kG_1L_2\Delta_2)e_k \\ \qquad\quad + E_k\xi_k - H_k\varphi(y_k) + g(x_k) \\ \tilde{z}_k = L_ke_k \end{cases} \quad (14)$$

where $\xi_k \triangleq [w_k^{\mathrm{T}}\ v_k^{\mathrm{T}}]^{\mathrm{T}}$ and $E_k \triangleq [B_k \quad -H_kG_1D_k]$.

Before giving the main objective of this paper, for the brevity of later presentation, we denote

$$\Lambda_i \triangleq [\,\underbrace{0\ \cdots\ 0}_{i-1}\ 1\ \underbrace{0\ \cdots\ 0}_{n_z-i}\,]. \quad (15)$$

This paper aims to design filter (10) such that the following requirements are achieved simultaneously:

R1) (H_∞ specification) Given $\gamma > 0$ and $\Xi > 0$, the output $\tilde{z}_k$ of the filtering error system (14) satisfies

$$\sum_{k=1}^{N} \mathbb{E}\left\{\|\tilde{z}_k\|^2\right\} \leq \gamma^2 \sum_{k=1}^{N} \|\xi_k\|^2 + \gamma^2 e_0^{\mathrm{T}}\Xi e_0 \quad (16)$$

for any nonzero $\xi_k \neq 0$.

R2) (Envelope constraint) Given the following input

$$\xi_k^\circ = \begin{cases} 1, & k = 0 \\ 0, & 1 \leq k \leq N \end{cases} \quad (17)$$

under the zero-initial condition, the corresponding output $\tilde{z}_k^\circ$ of filtering error system (14) satisfies

$$\Lambda_i(\psi_k - \beta_k) \leq \mathbb{E}\{\Lambda_i\tilde{z}_k^\circ\} \leq \Lambda_i(\psi_k + \beta_k) \quad (18)$$

where the sequences of vectors $\{\psi_k\}_{0\leq k\leq N}$ and $\{\beta_k\}_{0\leq k\leq N}$ represent, respectively, the desired output and the tolerance band.

3 Main Results

3.1 H_∞ requirement

For the brevity of later presentation, we denote

$$\eta_k \triangleq \begin{bmatrix} 1 & e_k^{\mathrm{T}} & x_k^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}},\ \varphi_k \triangleq \varphi(y_k),\ \mathcal{G} \triangleq \begin{bmatrix} 0 & I & I \end{bmatrix}^{\mathrm{T}},$$

$$\mathcal{H}_k \triangleq \begin{bmatrix} 0 & -H_k^{\mathrm{T}} & 0 \end{bmatrix}^{\mathrm{T}},\ \Gamma \triangleq \begin{bmatrix} 0 & 0 & I \end{bmatrix},\ \mathcal{L}_k \triangleq \begin{bmatrix} 0 & L_k & 0 \end{bmatrix},$$

$$\tilde{f}_k \triangleq f(\hat{x}_k) - F_k\hat{x}_k - H_kG_1h(\hat{x}_k),$$

$$\tilde{\Phi}_k \triangleq \Phi_k + L_1\Delta_1 - H_kG_1\Psi_k - H_kG_1L_2\Delta_2,$$

$$\mathcal{A}_k \triangleq \begin{bmatrix} 1 & 0 & 0 \\ \tilde{f}_k & \tilde{\Phi}_k & 0 \\ f(\hat{x}_k) & \Phi_k + L_1\Delta_1 & 0 \end{bmatrix},\ \mathcal{E}_k \triangleq \begin{bmatrix} 0 & 0 \\ B_k & -H_kD_k \\ B_k & 0 \end{bmatrix}.$$

Accordingly, from (1) and (14), we have:

$$\begin{cases} \eta_{k+1} = \mathcal{A}_k\eta_k + \mathcal{E}_k\xi_k + \mathcal{G}g(\Gamma\eta_k) + \mathcal{H}_k\varphi_k, \\ \tilde{z}_k = \mathcal{L}_k\eta_k. \end{cases} \quad (19)$$

Theorem 1 *Given $\gamma > 0$, $\Xi > 0$ and $\{F_k, H_k\}_{0\leq k\leq N}$. For system (14), the H_∞ performance index is guaranteed if there exist a sequence of positive definite matrices $\{P_k\}_{0\leq k\leq N+1}$ with $P_0 - \gamma^2\bar{\Xi} \leq 0$ ($\bar{\Xi} = \mathrm{diag}\{0, \Xi, 0\}$), and a sequence of positive scalars $\{\varepsilon_k\}_{0\leq k\leq N}$ such that*

$$\Omega_k - \varepsilon_k \begin{bmatrix} \Pi_{11} & \Pi_{21}^{\mathrm{T}} \\ \Pi_{21} & 2I_{n_z} - GL_2L_2^{\mathrm{T}}G \end{bmatrix} \leq 0 \quad (20)$$

where

$$\Omega_k \triangleq \begin{bmatrix} \bar{\Omega}_k + \mathcal{L}_k^{\mathrm{T}}\mathcal{L}_k & \mathcal{A}_k^{\mathrm{T}}P_{k+1}\mathcal{E}_k & \mathcal{A}_k^{\mathrm{T}}P_{k+1}\mathcal{H}_k \\ * & \mathcal{E}_k^{\mathrm{T}}P_{k+1}\mathcal{E}_k - \gamma^2 I_{n_\xi} & \mathcal{E}_k^{\mathrm{T}}P_{k+1}\mathcal{H}_k \\ * & * & \mathcal{H}_k^{\mathrm{T}}P_{k+1}\mathcal{H}_k \end{bmatrix}, \quad (21)$$

$$\Pi_{11} \triangleq \mathrm{diag}\{0, 0, -I_{n_x}, 0\},$$

$$\Pi_{21} \triangleq \begin{bmatrix} -Gh(\hat{x}_k) & -G\Psi_k & 0 & -G\tilde{D}_k \end{bmatrix}.$$

Proof: See Appendix A.

Next, for presentation simplicity, we denote

$$\bar{L}_1 \triangleq \begin{bmatrix} 0 & L_1^{\mathrm{T}} & L_1^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}},\ \bar{L}_2 \triangleq \begin{bmatrix} 0 & -L_2^{\mathrm{T}}G_1^{\mathrm{T}}H_k^{\mathrm{T}} & 0 \end{bmatrix}^{\mathrm{T}},$$

$$\mathcal{I} \triangleq \begin{bmatrix} 0 & I_{n_{l_1}} & 0 \end{bmatrix},\ \tilde{\mathcal{I}} \triangleq \begin{bmatrix} \mathcal{I} & 0 & 0 & 0 \end{bmatrix},\ \tilde{L}_1 \triangleq \begin{bmatrix} 0 & 0 & 0 & \bar{L}_1^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}},$$

$$\tilde{L}_2 \triangleq \begin{bmatrix} 0 & 0 & 0 & \bar{L}_2^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}},\ \tilde{L} \triangleq \begin{bmatrix} \epsilon_{1,k}\tilde{L}_1 & \epsilon_{2,k}\tilde{L}_2 & \tilde{\mathcal{I}}^{\mathrm{T}} & \tilde{\mathcal{I}}^{\mathrm{T}} \end{bmatrix},$$

$$J \triangleq \mathrm{diag}\{\epsilon_{1,k}I_{n_{l_1}}, \epsilon_{2,k}I_{n_{l_2}}, \epsilon_{1,k}I_{n_x}, \epsilon_{2,k}I_{n_x}\},$$

$$\bar{\mathcal{A}}_k \triangleq \begin{bmatrix} 1 & 0 & 0 \\ f(\hat{x}_k) - F_k\hat{x}_k - H_kG_1h(\hat{x}_k) & \Phi_k - H_kG_1\Psi_k & 0 \\ f(\hat{x}_k) & \Phi_k & 0 \end{bmatrix},$$

$$\tilde{\Omega}_k \triangleq \begin{bmatrix} \sum_{l=1}^{q}\Gamma^{\mathrm{T}}\Upsilon_{l,k}\Gamma\alpha_{l,k} - P_k + \mathcal{L}_k^{\mathrm{T}}\mathcal{L}_k & 0 & 0 \\ * & -\gamma^2 I_{n_\xi} & 0 \\ * & * & 0 \end{bmatrix},$$

$$\bar{\mathscr{A}}_k \triangleq \begin{bmatrix} \bar{\mathcal{A}}_k & \mathcal{E}_k & \mathcal{H}_k \end{bmatrix},\ \tilde{\Pi}_k \triangleq \begin{bmatrix} -\varepsilon_k\Pi_k + \tilde{\Omega}_k & \bar{\mathscr{A}}_k^{\mathrm{T}} \\ \bar{\mathscr{A}}_k & -P_{k+1}^{-1} \end{bmatrix}.$$

Theorem 2 *Given $\gamma > 0$, $\Xi > 0$ and $\{F_k, H_k\}_{0\leq k\leq N}$ be given. For filtering error system (14), the H_∞ performance index is guaranteed if there exist a sequence of positive definite matrices $\{P_k\}_{0\leq k\leq N+1}$ with $P_0 - \gamma^2\bar{\Xi} \leq 0$, sequences of positive scalars $\{\varepsilon_k, \epsilon_{1,k}, \epsilon_{2,k}\}_{0\leq k\leq N}$, and sequences of positive scalars $\{\alpha_{l,k}\}_{0\leq k\leq N}$ ($l = 1, 2, \ldots, q$) such that*

$$\begin{bmatrix} -\alpha_{l,k} & \varrho_{l,k}^{\mathrm{T}}\mathcal{G}^{\mathrm{T}} \\ * & -P_{k+1}^{-1} \end{bmatrix} \leq 0, \qquad \begin{bmatrix} \tilde{\Pi}_k & \tilde{L} \\ * & -J \end{bmatrix} \leq 0. \quad (22)$$

Proof: See Appendix B.

3.2 Envelope constraint

Lemma 1 *Given $\{F_k, H_k\}_{0\le k\le N}$. If there exist a sequence of positive definite matrices $\{Q_k\}_{0\le k\le N+1}$, sequences of positive scalars $\{\tau_{1,k},\tau_{2,k},\tau_{3,k},\tau_{4,k},\tau_{5,k}\}_{0\le k\le N}$ satisfying the following recursive linear matrix inequality:*

$$\begin{bmatrix} -M_k & \Sigma_k^{\mathrm{T}} \\ \Sigma_k & -Q_{k+1} \end{bmatrix} \le 0 \tag{23}$$

where

$$\tilde{\Psi}_{21} \triangleq \Big[-G\big(h(\hat{x}_k)+\tilde{D}_k\xi_k^\circ\big) \ -G\Psi_k\Theta_k \ 0 \ -GL_2 \Big],$$

$$\tilde{\Psi}_k \triangleq \mathrm{diag}\Big\{ \begin{bmatrix} 0 & \tilde{\Psi}_{21}^{\mathrm{T}} \\ \tilde{\Psi}_{21} & 2I_{n_y} \end{bmatrix}, 0\Big\},$$

$$\tilde{\Lambda}_k \triangleq \mathrm{diag}\Big\{ \begin{bmatrix} \tilde{\Lambda}_k^{(11)} & \tilde{\Lambda}_k^{(12)} \\ \tilde{\Lambda}_k^{(21)} & \tilde{\Lambda}_k^{(22)} \end{bmatrix}, 0,0,0, I_{n_x}\Big\},$$

$$\tilde{\Lambda}_k^{(11)} \triangleq -\hat{x}_k^{\mathrm{T}} \sum_{l=1}^{q} \varrho_{l,k}\varrho_{l,k}^{\mathrm{T}} \Upsilon_{l,k}\hat{x}_k,$$

$$\tilde{\Lambda}_k^{(12)} \triangleq -\hat{x}_k^{\mathrm{T}} \sum_{l=1}^{q} \varrho_{l,k}\varrho_{l,k}^{\mathrm{T}} \Upsilon_{l,k}\Theta_k,$$

$$\tilde{\Lambda}_k^{(21)} \triangleq -\Theta_k^{\mathrm{T}} \sum_{l=1}^{q} \varrho_{l,k}\varrho_{l,k}^{\mathrm{T}} \Upsilon_{l,k}\hat{x}_k,$$

$$\tilde{\Lambda}_k^{(22)} \triangleq -\Theta_k^{\mathrm{T}} \sum_{l=1}^{q} \varrho_{l,k}\varrho_{l,k}^{\mathrm{T}} \Upsilon_{l,k}\Theta_k,$$

$$\begin{aligned} M_k \triangleq{}& \tau_{4,k}\tilde{\Psi}_k + \tau_{5,k}\tilde{\Lambda}_k + \mathrm{diag}\{1-\tau_{1,k}, \tau_{1,k}I_\vartheta \\ &- (\tau_{2,k}+\tau_{3,k})\Theta_k^{\mathrm{T}}\Theta_k, \tau_{2,k}I_{n_{l_1}}, \tau_{3,k}I_{n_{l_2}}, 0, 0\}, \end{aligned}$$

$$\Sigma_k \triangleq \Big[\Sigma_k^{(11)} \ \Sigma_k^{(12)} \ L_1 \ -H_kG_1L_2 \ -H_k \ I_{n_x} \Big],$$

$$\Sigma_k^{(11)} \triangleq f(\hat{x}_k) - F_k\hat{x}_k - H_kG_1h(\hat{x}_k) + E_k\xi_k^\circ,$$

$$\Sigma_k^{(12)} \triangleq (\Phi_k - H_kG_1\Psi_k)\Theta_k,$$

with Θ_k being a factorization of Q_k (i.e., $Q_k = \Theta_k\Theta_k^{\mathrm{T}}$), then $x_k \in \mathscr{E}(\hat{x}_k, Q_k, n_x)$ holds for all $k\in[0,N]$. In other words, the state estimation error e_k is confined in the ellipsoid $\mathscr{E}(0, Q_k, n_x)$ at each time step k over the finite horizon $[0,N]$.

Proof: See Appendix C.

Theorem 3 *Given $\{F_k, H_k\}_{0\le k\le N}$, $\{\psi_k\}_{0\le k\le N}$ and $\{\beta_k\}_{0\le k\le N}$. The envelope constraint defined in (18) is achieved if there exist a sequence of positive definite matrices $\{Q_k\}_{0\le k\le N+1}$ and sequences of positive scalars $\{\lambda_k,\tau_{1,k},\tau_{2,k},\tau_{3,k},\tau_{4,k},\tau_{5,k}\}_{0\le k\le N}$ satisfying*

$$\begin{bmatrix} -M_k & \Sigma_k^{\mathrm{T}} \\ \Sigma_k & -Q_{k+1} \end{bmatrix} \le 0, \tag{24}$$

$$\begin{bmatrix} -(\Lambda_i\beta_k)^2+\lambda_k & 0 & -\Lambda_i\psi_k \\ 0 & -\lambda_kI_\vartheta & \Theta_k^{\mathrm{T}}L_k^{\mathrm{T}}\Lambda_i^{\mathrm{T}} \\ -\Lambda_i\psi_k & \Lambda_iL_k\Theta_k & -1 \end{bmatrix} \le 0 \tag{25}$$

where Θ_k is a factorization of Q_k (i.e., $Q_k = \Theta_k\Theta_k^{\mathrm{T}}$).

Proof: See Appendix D.

3.3 Filter design

Theorem 4 *Given $\gamma > 0$, $\Xi > 0$, $\{\psi_k\}_{0\le k\le N}$ and $\{\beta_k\}_{0\le k\le N}$. The output estimation error $\tilde{z}_k$ satisfies simultaneously the prespecified H_∞ specification and envelope constraint if, under the initial condition $\{P_0 \le \gamma^2\Xi, Q_0 \ge 0\}$, there exist sequences of positive definite matrices $\{\mathcal{P}_k, Q_k\}_{0\le k\le N+1}$, sequences of real-valued matrices $\{F_k, H_k\}_{0\le k\le N}$, sequences of positive scalars $\{\varepsilon_k, \epsilon_{1,k}, \epsilon_{2,k}, \lambda_k, \tau_{1,k},\tau_{2,k},\tau_{3,k},\tau_{4,k},\tau_{5,k}\}_{0\le k\le N}$, sequences of positive scalars $\{\alpha_{l,k}\}_{0\le k\le N}$ $(l=1,2,\ldots,q)$ satisfying the following set of recursive linear matrix inequalities*

$$\begin{bmatrix} -\alpha_{l,k} & \varrho_{l,k}^{\mathrm{T}}\mathcal{G}^{\mathrm{T}} \\ * & -\mathcal{P}_{k+1} \end{bmatrix} \le 0, \quad \begin{bmatrix} \tilde{\Pi}_k & \tilde{L} \\ * & -J \end{bmatrix} \le 0, \tag{26}$$

$$\begin{bmatrix} -M_k & \Sigma_k^{\mathrm{T}} \\ \Sigma_k & -Q_{k+1} \end{bmatrix} \le 0, \tag{27}$$

$$\begin{bmatrix} -(\Lambda_i\beta_k)^2+\lambda_k & 0 & -\Lambda_i\psi_k \\ 0 & -\lambda_kI_\vartheta & \Theta_k^{\mathrm{T}}L_k^{\mathrm{T}}\Lambda_i^{\mathrm{T}} \\ -\Lambda_i\psi_k & \Lambda_iL_k\Theta_k & -1 \end{bmatrix} \le 0 \tag{28}$$

where the parameter P_{k+1} is updated recursively according to $P_{k+1} = \mathcal{P}_{k+1}^{-1}$, and the parameter Θ_k is computed iteratively by decomposing Q_k such that $Q_k = \Theta_k\Theta_k^{\mathrm{T}}$.

Proof: The proof follows directly from Theorems 1–3 and is therefore omitted here.

Remark 1 *So far, we have discussed the envelope-constrained H_∞ filtering problem for general nonlinear system subject to quantization effects. Within the established theoretical framework, our results can be extended to certain filtering problems with other performance requirements/constraints such as security performance [15], ellipsoidal constraints [16], consensus performance [14], communication cost [29] and dissipativity [25].*

4 Numerical Example

Consider the following univariate non-stationary growth model (UNGM) investigated in [11]:

$$\begin{cases} x_{k+1} = 0.5x_k + 25\dfrac{x_k}{1+x_k^2} \\ \qquad\quad + 8\cos(1.2*(k+1)) + g(x_k) + w_k \\ y_k = \dfrac{x_k^2}{20} + v_k \\ z_k = 0.2x_k \end{cases}$$

It is worth mentioning that in [11], w_k and v_k are assumed to obey Gaussian distribution with zero mean and known variances. However, in practical engineering, apart from the stochastic noises, sometimes systems might be corrupted by other different kinds of disturbances such as the energy bounded disturbances investigated in this paper. Therefore, in this paper, we assume $w_k = 0.5e^{-0.2k}\sin(k)$ and $v_k = 5\cos(2k)/(k+1)$,

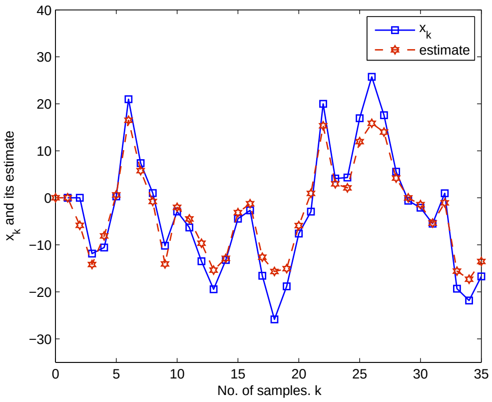

Fig. 1. The state x_k and its estimate.

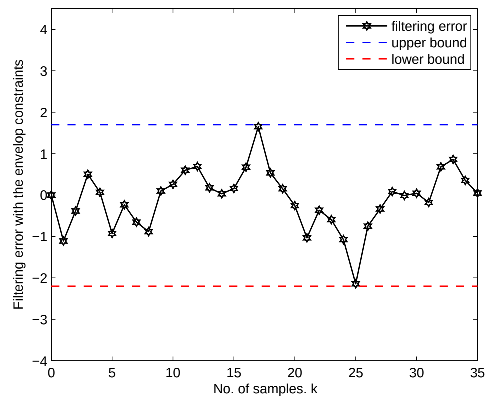

Fig. 2. The filtering error $\tilde{z}_k^{\circ}$ and its upper and lower bounds.

which, obviously, are two energy bounded disturbance sequences.

On the other hand, note that in [11], the item $g(x_k)$ does not exist. However, in many cases, the system may contaminate with the stochastic nonlinearities owing to a variety of reasons such as random failures and repairs of the components, changes in the interconnections of subsystems, and sudden environment changes. Thus, in order to better reflect the engineering reality and present a comprehensive model, we take into account the stochastic nonlinearity $g(x_k)$ with the form of $g(x_k) = 0.06\text{sign}[x_k]x_k\varsigma_k$ where ς_k is a Gaussian white sequences with unitary covariance. Then it can be easily checked that $g(x_k)$ satisfies

$$\begin{aligned}\mathbb{E}\{g(x_k)|x_k\} =& 0,\\ \mathbb{E}\{g(x_k)g^{\mathrm{T}}(x_k)|x_k\} =& 0.0036x_k^{\mathrm{T}}x_k.\end{aligned}$$

In this example, the parameters of the logarithmic quantizer $\sigma(\cdot)$ are taken as $\hat{\mu}_0 = 3$ and $\chi = 0.6$. Then, it can be obtained that $G_1 = 0.75$ and $G_2 = 1.25$.

To study the H_∞ performance, in the simulation, choose $\gamma = 1.5$ and $\Xi = 0.5$. Set the initial values by $P_0 = \text{diag}\{0, 0.5, 0\}$, $Q_0 = I_2$ and $\hat{x}_0 = 0$. According to Theorem 4, the presented time-varying LMIs can be solved recursively by utilizing Matlab software.

The simulation results are presented in Figs. 1–2. Fig. 1 depicts the trajectories of the state x_k and its estimate. It can be observed that all the estimate can track the true value with a satisfactory accuracy, which confirms that the proposed filtering algorithm performs quite well.

In order to investigate the envelope constraint criterion, with the given input ξ_k°, the desired output ζ_k and the tolerance band β_k are selected as $\psi_k = -0.25$ and $\beta_k = 1.95$. The simulation result is shown in Fig. 2, where it can be clearly seen that the filtering error output is constrained by the pre-specified bounds. Therefore, it can be summarized that the envelope constraint can be achieved by using the exploited recursive filtering algorithm.

5 Conclusion

In this paper, the envelope-constrained H_∞ filtering problem has been discussed for a class of discrete time-varying nonlinear system over a finite horizon. Both deterministic and stochastic nonlinearities are included in the system under consideration. The stochastic nonlinearity described by statistical means is quite general which could encompass certain frequently seen nonlinearities in the literature. An envelope-constrained performance index has been proposed to reflect the transient behavior of the filtering errors over the specified horizon. By means of recursive matrix inequalities approach, sufficient conditions of the existence of the desired filter have been established guaranteeing simultaneously the pre-specified H_∞ criterion and the envelope constraint. An illustrative example has been proposed to show the effectiveness and applicability of the presented filtering scheme.

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A Proof of Theorem 1

Proof: The proof can be conducted easily and is therefore omitted here due to the limitation of pages.

B Proof of Theorem 2

Proof: Based on Theorem 1, we only need to prove that the matrix inequalities (22) imply the matrix inequality (20). According to Schur Complement Lemma [1], the first inequality in (22) is true if and only if

$$\varrho_{l,k}^{\mathrm{T}} \mathcal{G}^{\mathrm{T}} P_{k+1} \mathcal{G} \varrho_{l,k} \le \alpha_{l,k}. \tag{B.1}$$

By utilizing the property of matrix trace, we obtain

$$\mathrm{tr}\big[\mathcal{G}\varrho_{l,k}\varrho_{l,k}^{\mathrm{T}}\mathcal{G}^{\mathrm{T}}P_{k+1}\big] \le \alpha_{l,k}. \tag{B.2}$$

Similarly, it follows from the Schur Complement Lemma [1] that the second linear matrix inequality (22) holds if and only if

$$\begin{bmatrix} -\varepsilon_k \Pi_k + \tilde{\Omega}_k & \bar{\mathscr{A}}_k^{\mathrm{T}} \\ \bar{\mathscr{A}}_k & -P_{k+1}^{-1} \end{bmatrix} + \tilde{L} J^{-1} \tilde{L}^{\mathrm{T}} \le 0, \tag{B.3}$$

which is equivalent to

$$\begin{bmatrix} -\varepsilon_k \Pi_k + \tilde{\Omega}_k & \mathscr{A}_k^{\mathrm{T}} \\ \mathscr{A}_k & -P_{k+1}^{-1} \end{bmatrix} \le 0 \tag{B.4}$$

where

$$\mathscr{A}_k \triangleq \begin{bmatrix} \mathcal{A}_k & \mathcal{E}_k & \mathcal{H}_k \end{bmatrix}.$$

It follows immediately from Schur Complement Lemma [1] and (B.4) that

$$-\varepsilon_k \Pi_k + \tilde{\Omega}_k + \mathscr{A}_k^{\mathrm{T}} P_{k+1} \mathscr{A}_k \le 0. \tag{B.5}$$

Subsequently, taking (B.2) into consideration, we obtain

$$-\varepsilon_k \Pi_k + \Omega_k \le 0. \tag{B.6}$$

Therefore, according to Theorem 1, we can conclude that the desired H_∞ requirement is achieved. The proof is now complete.

C Proof of Lemma 1

Proof: Applying the input ξ_k° defined in (17) to the filtering error system (14), we obtain the one-step-ahead state estimation error as follows:

$$\begin{aligned} e_{k+1} =& f(\hat{x}_k) - F_k \hat{x}_k - H_k G_1 h(\hat{x}_k) + E_k \xi_k^\circ - H_k \varphi_k \\ &+ (\Phi_k - H_k G_1 \Psi_k)(x_k - \hat{x}_k) + L_1 \Delta_1 (x_k - \hat{x}_k) \\ &- H_k G_1 L_2 \Delta_2 (x_k - \hat{x}_k) + g(x_k). \end{aligned} \tag{C.1}$$

In the following, we are going to prove the lemma by induction. Firstly, for $k = 0$, it can be known from $x_0 = 0$ and $\hat{x}_0 = 0$ that:

$$\mathbb{E}\left\{(x_0-\hat{x}_0)^{\mathrm{T}}Q_0^{-1}(x_0-\hat{x}_0)\right\}\leq 1 \tag{C.2}$$

where $Q_0>0$ is any given matrix. Thus, when $k=0$, $x_0\in\mathscr{E}(\hat{x}_0,Q_0,n_x)$ is satisfied.

Secondly, supposing that $x_k\in\mathscr{E}(\hat{x}_k,Q_k,n_x)$ is true at $k>0$, we shall demonstrate that the following holds:

$$\mathbb{E}\left\{(x_{k+1}-\hat{x}_{k+1})^{\mathrm{T}}Q_{k+1}^{-1}(x_{k+1}-\hat{x}_{k+1})\right\}\leq 1. \tag{C.3}$$

Since $x_k\in\mathscr{E}(\hat{x}_k,Q_k,n_x)$ is true, we have

$$\mathbb{E}\left\{(x_k-\hat{x}_k)^{\mathrm{T}}Q_k^{-1}(x_k-\hat{x}_k)\right\}\leq 1. \tag{C.4}$$

Consequently, it follows from [10] that there exists a vector $\vartheta_k\in\mathbb{R}^{n_\vartheta}$ ($\mathbb{E}\{\vartheta_k^{\mathrm{T}}\vartheta_k\}\leq 1$) such that

$$x_k=\hat{x}_k+\Theta_k\vartheta_k \tag{C.5}$$

where Θ_k is a factorization of Q_k, i.e., $Q_k=\Theta_k\Theta_k^{\mathrm{T}}$.

Substituting (C.5) into (C.1) yields

$$\begin{aligned}&x_{k+1}-\hat{x}_{k+1}\\=&f(\hat{x}_k)-F_k\hat{x}_k-H_kG_1h(\hat{x}_k)+E_k\xi_k^\circ\\&+(\Phi_k-H_kG_1\Psi_k)\Theta_k\vartheta_k+L_1\Delta_1\Theta_k\vartheta_k\\&-H_kG_1L_2\Delta_2\Theta_k\vartheta_k-H_k\varphi_k+g(x_k).\end{aligned} \tag{C.6}$$

Next, letting $\delta_{1,k}\triangleq\Delta_1\Theta_k\vartheta_k$, $\delta_{2,k}\triangleq\Delta_2\Theta_k\vartheta_k$ and denoting a vector as

$$\varpi_k\triangleq\begin{bmatrix}1 & \vartheta_k^{\mathrm{T}} & \delta_{1,k}^{\mathrm{T}} & \delta_{2,k}^{\mathrm{T}} & \varphi_k^{\mathrm{T}} & g^{\mathrm{T}}(x_k)\end{bmatrix}^{\mathrm{T}},$$

we can rewrite the dynamics of state estimation error (C.6) as follows:

$$x_{k+1}-\hat{x}_{k+1}=\Sigma_k\varpi_k. \tag{C.7}$$

On the other hand, the inequality $\mathbb{E}\{\vartheta_k^{\mathrm{T}}\vartheta_k\}\leq 1$ can be equivalently expressed in terms of vector ϖ_k as

$$\mathbb{E}\{\varpi_k^{\mathrm{T}}\mathcal{U}_1\varpi_k\}\leq 0 \tag{C.8}$$

where $\mathcal{U}_1\triangleq\mathrm{diag}\{-1,I_\vartheta,0,0,0,0\}$.

Noting $\delta_{1,k}=\Delta_1\Theta_k\vartheta_k$ and $\delta_{2,k}=\Delta_2\Theta_k\vartheta_k$, we can infer from $\|\Delta_1\|\leq 1$ and $\|\Delta_2\|\leq 1$ that

$$\delta_{1,k}^{\mathrm{T}}\delta_{1,k}=\vartheta_k^{\mathrm{T}}\Theta_k^{\mathrm{T}}\Delta_1^{\mathrm{T}}\Delta_1\Theta_k\vartheta_k\leq\vartheta_k^{\mathrm{T}}\Theta_k^{\mathrm{T}}\Theta_k\vartheta_k, \tag{C.9}$$
$$\delta_{2,k}^{\mathrm{T}}\delta_{2,k}=\vartheta_k^{\mathrm{T}}\Theta_k^{\mathrm{T}}\Delta_2^{\mathrm{T}}\Delta_2\Theta_k\vartheta_k\leq\vartheta_k^{\mathrm{T}}\Theta_k^{\mathrm{T}}\Theta_k\vartheta_k, \tag{C.10}$$

which can be rewritten by

$$\varpi_k^{\mathrm{T}}\mathcal{U}_2\varpi_k\leq 0, \tag{C.11}$$
$$\varpi_k^{\mathrm{T}}\mathcal{U}_3\varpi_k\leq 0, \tag{C.12}$$

where $\mathcal{U}_2\triangleq\mathrm{diag}\{0,-\Theta_k^{\mathrm{T}}\Theta_k,I_{n_{l_1}},0,0,0\}$ and $\mathcal{U}_3\triangleq\mathrm{diag}\{0,-\Theta_k^{\mathrm{T}}\Theta_k,0,I_{n_{l_2}},0,0\}$.

Likewise, we know from the inequality (9) that:

$$\varphi_k^{\mathrm{T}}\varphi_k-\varphi_k^{\mathrm{T}}Gy_k\leq 0\Longleftrightarrow\varpi_k^{\mathrm{T}}\tilde{\Psi}_k\varpi_k\leq 0. \tag{C.13}$$

Taking the statistical property of the stochastic nonlinearity $g(x_k)$ into consideration, we obtain

$$\begin{aligned}&\mathbb{E}\{g^{\mathrm{T}}(x_k)g(x_k)\}\\=&\hat{x}_k^{\mathrm{T}}\sum_{l=1}^{q}\varrho_{l,k}\varrho_{l,k}^{\mathrm{T}}\Upsilon_{l,k}\hat{x}_k+\vartheta_k^{\mathrm{T}}\Theta_k^{\mathrm{T}}\sum_{l=1}^{q}\varrho_{l,k}\varrho_{l,k}^{\mathrm{T}}\Upsilon_{l,k}\Theta_k\vartheta_k\\&+\hat{x}_k^{\mathrm{T}}\sum_{l=1}^{q}\varrho_{l,k}\varrho_{l,k}^{\mathrm{T}}\Upsilon_{l,k}\Theta_k\vartheta_k+\vartheta_k^{\mathrm{T}}\Theta_k^{\mathrm{T}}\sum_{l=1}^{q}\varrho_{l,k}\varrho_{l,k}^{\mathrm{T}}\Upsilon_{l,k}\hat{x}_k,\end{aligned}$$

which is equivalent to $\mathbb{E}\{\varpi_k^{\mathrm{T}}\tilde{\Lambda}_k\varpi_k\}=0$.

We are now in a position to demonstrate that the inequality (C.3) is true for the time instant $k>0$.

By means of Schur Complement Lemma [1], the inequality (23) is true if and only if

$$-M_k+\Sigma_k^{\mathrm{T}}Q_{k+1}^{-1}\Sigma_k\leq 0, \tag{C.14}$$

which implies

$$\begin{aligned}&\mathbb{E}\left\{\varpi_k^{\mathrm{T}}\Sigma_k^{\mathrm{T}}Q_{k+1}^{-1}\Sigma_k\varpi_k-\varpi_k^{\mathrm{T}}\mathrm{diag}\{1,0,0,0,0,0\}\varpi_k\right\}\\&-\tau_{1,k}\varpi_k^{\mathrm{T}}\mathcal{U}_1\varpi_k-\tau_{2,k}\varpi_k^{\mathrm{T}}\mathcal{U}_2\varpi_k-\tau_{3,k}\varpi_k^{\mathrm{T}}\mathcal{U}_3\varpi_k\\&-\tau_{4,k}\varpi_k^{\mathrm{T}}\tilde{\Psi}_k\varpi_k-\tau_{5,k}\varpi_k^{\mathrm{T}}\tilde{\Lambda}_k\varpi_k\leq 0.\end{aligned} \tag{C.15}$$

Accordingly, we know that

$$\mathbb{E}\left\{\varpi_k^{\mathrm{T}}\Sigma_k^{\mathrm{T}}Q_{k+1}^{-1}\Sigma_k\varpi_k\right\}\leq 1 \tag{C.16}$$

which indicates that

$$\mathbb{E}\left\{(x_{k+1}-\hat{x}_{k+1})^{\mathrm{T}}Q_{k+1}^{-1}(x_{k+1}-\hat{x}_{k+1})\right\}\leq 1. \tag{C.17}$$

Thus, the induction is now accomplished and we conclude that $x_k\in\mathscr{E}(\hat{x}_k,Q_k,n_x)$ holds for all $k\in[0,N]$. The proof is now complete.

D Proof of Theorem 3

Proof: First, according to Lemma 1, we can know directly from (24) that the one-step-ahead estimation error e_k belongs to the ellipsoid $\mathscr{E}(0,Q_k,n_x)$, and therefore there exists a random vector ϑ_k ($\mathbb{E}\{\vartheta_k^{\mathrm{T}}\vartheta_k\}\leq 1$) such that

$$e_k=\Theta_k\vartheta_k,\quad Q_k=\Theta_k\Theta_k^{\mathrm{T}}. \tag{D.1}$$

Next, it is easy to see that the inequality (18) holds if and only if

$$\left(\mathbb{E}\left\{\Lambda_i\tilde{z}_k^\circ\right\}-\Lambda_i\psi_k\right)^2\leq\left(\Lambda_i\beta_k\right)^2. \tag{D.2}$$

By defining a new vector $\tilde{\varpi}_k\triangleq\begin{bmatrix}1 & \mathbb{E}\{\vartheta_k^{\mathrm{T}}\}\end{bmatrix}^{\mathrm{T}}$, and taking into account the fact that $\mathbb{E}\{\vartheta_k^{\mathrm{T}}\}\mathbb{E}\{\vartheta_k\}\leq\mathbb{E}\{\vartheta_k^{\mathrm{T}}\vartheta_k\}\leq 1$, we acquire

$$\begin{aligned}&\left(\mathbb{E}\left\{\Lambda_i\tilde{z}_k^\circ\right\}-\Lambda_i\psi_k\right)^2-\left(\Lambda_i\beta_k\right)^2\\\leq&\left(\mathbb{E}\left\{\Lambda_i\tilde{z}_k^\circ\right\}-\Lambda_i\psi_k\right)^2-\left(\Lambda_i\beta_k\right)^2+\lambda_k-\lambda_k\mathbb{E}\{\vartheta_k^{\mathrm{T}}\}\mathbb{E}\{\vartheta_k\}\\=&\tilde{\varpi}_k^{\mathrm{T}}\Bigg(\begin{bmatrix}-\left(\Lambda_i\beta_k\right)^2+\lambda_k & 0\\ 0 & -\lambda_kI_\vartheta\end{bmatrix}\\&+\begin{bmatrix}-\Lambda_i\psi_k\\ \Theta_k^{\mathrm{T}}L_k^{\mathrm{T}}\Lambda_i^{\mathrm{T}}\end{bmatrix}\begin{bmatrix}-\Lambda_i\psi_k & \Lambda_iL_k\Theta_k\end{bmatrix}\Bigg)\tilde{\varpi}_k.\end{aligned} \tag{D.3}$$

Consequently, by means of Schur Complement Lemma [1], it can be readily known from the inequality (25) that $(\mathbb{E}\{\Lambda_i \tilde{z}_k^{\circ}\} - \Lambda_i \psi_k)^2 - (\Lambda_i \beta_k)^2 \leq 0$ holds for all $i \in \{1, 2, \ldots, n_z\}$. Therefore, the envelope constraint defined in (18) is guaranteed at each time step k over the horizon $[0, N]$. The proof is now complete.

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