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CONTINUED FRACTIONS WHICH MATCH POWER SERIES EXPANSIONS AT TWO POINTS

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Summary Theoretical results are derived for constructing continued fractions which correspond, in some prescribed way, to given power series expansions at two points. In those cases where the two power series are expansions of the same function, new approximations to the function are obtained. Error estimates for such rational function approximations are developed in certain cases. Some examples are given.

1. Preliminary discussion and notation

Continued fractions which correspond to a single, given power series at a point have been much studied, [See for example Wall (1948), Perron (1950)]. In this paper we consider continued fractions which match power series expansions at two points.

Since for a rational function of the complex variable z, the power series expansions about the points $z = 0$ and $z = \infty$ are both available immediately by division, we consider analytic functions f(z) which possess power series expansions at these two points. Analogous results for expansions about two finite points can then be derived by applying a bilinear transformation to the variable z. (See example 3).

Accordingly we first consider the case where for $|z|$ small

$$f(z) = a_0 + a_1 z + a_2 z^2 + \dots \qquad (1.1)$$

and where for $|z|$ large

$$f(z) = -\left\{\frac{a_{-1}}{z} + \frac{a_{-2}}{z^2} + \frac{a_{-3}}{z^3} + \dots\dots\right\} \qquad (1.2)$$

We assume initially, that $a_0, a_{-1} \neq 0$ and deal with this restriction at a later stage. The a's may be complex but for most applications we take them to be real. The reason for the notation in (1.2) will become clear subsequently.

We further assume that f(z) is single valued and that on the positive real axis of z, f(z) has no singularities save possibly $z = 0$ and $z = \infty$. The equals signs in (1.1) and (1.2) are not strictly necessary in what follows, and these expansions may be regarded as asymptotic in character holding for some range of arg z which includes arg $z=0$.

We approximate to f(z) by rational function approximations of the form

$$\frac{\alpha_{m,o} + \alpha_{m,1} z + \ldots\ldots + \alpha_{m,m-1} z^{m-1}}{1 + ß_{m,1} z + \ldots\ldots + ß_{m,m} z^{m}} \equiv \quad f_{i,j}(z) \qquad \textbf{(1.3)}$$

for m = 1,2,3, ..., the coefficients α,/ß being constants independent of z. These 2m coefficients may be chosen such that when the rational function in (1.3) is expanded for |z| large, there is agreement with a certain number of terms in (1,1) and (1.2), totalling 2m in all. Consequently we denote the approximation (1.3) by $f_{i,j}(z)$ with i + j = 2m, where i, j are respectively, the number of terms of (1.1) and (1.2) which have been fitted. Thus

$$f_{i,j}(z) - f(z) = o(z^{1}, z^{-(j+1)}) \qquad (1.4)$$

where the symbol on the R.H.S. means that the L.H.S. is $o(z^i)$ for |z| small and is $o(z^{-(j+1)})$ for |z| large.

we denote a continued fraction by

$$\frac{p_1}{q_1} + \frac{p_2}{q_2} + \ldots\ldots\ldots + \frac{p_m}{q_m} + \ldots\ldots\ldots \qquad (1.5)$$

with m^{th} convergent P_m/Q_m, where P_m, Q_m both satisfy the 3-term recurrence relation

$$u_{m+1} = q_{m+1} u_m + p_{m+1} u_{m-1} \qquad (1.6)$$

for m = 1,2,3, ... with initial values $p_o = 0$, P_1 p_1 for the P_m's and $Q_o = 1$, $Q_1 = q_1$ for the Q_m's.

Defining Δ_{rs} by

$$\Delta_{rs} \equiv (p_r Q_s - p_s Q_r) \qquad \textbf{(1.7)}$$

we find from (1.6) that

$$p_{m+1} = - \Delta_{m,m+1}/\Delta_{m-1,m}\ , \ q_{m+1} = \Delta_{m-1,m+1}/\Delta_{m-1,m}, \tag{1.8}$$

for m = 1,2.3...

Taking

$$P_m(z) \equiv \alpha_{m,0} + \alpha_{m,1}z + \ldots \qquad + \alpha_{m,m-1}z^{m-1} \tag{1.9}$$

and

$$q_m(z) \equiv 1 + \beta_{m,1}z + \beta_{m,2}z^2 + \ldots \qquad + \beta_{m,m}z^m \tag{1.10}$$

we have

$$f_{i,j}(z) \equiv p_m(z)/Q_m(z) \tag{1.11}$$

with i + j = 2m, m =1,2,3 .....

Thus from a knowledge of the polynomials $P_m(z)$, $Q_m(z)$ we can determine, using (1.7) and (1.8), the elements p_m ,q_m of the continued fraction, where m = 1,2,3,,.

We consider the particular form of continued fraction

$$\frac{n_1}{1+d_1z} + \frac{n_2z}{1+d_2z} + \ldots.. + \frac{n_mz}{1+d_mz} + \ldots\ldots. \tag{1.12}$$

where n_m, d_m are constants independent of z, and assume that each $n_m \neq 0$ but that the d_m 's may be zero.

2. Construction of the continued fractions.

Case (i) Matching equal numbers of terms of each series

We consider the sequence of rational function approximations

$$f_{1,1}\ ,\ f_{2,2},\ \ldots,\ f_{m,m}, \text{-----------}$$

In the notation of (1.4)

$$f_{m,m}(z) - f(z) = 0\ (z^m, z^{-(m+1)}) \tag{2.1}$$

Using the expansions (1.1) and (1.2) for f(z) together with (2.1) the equations for the α's and ß 's are

$$\left.\begin{aligned}
\alpha_{m,0} &= a_0 \\
\alpha_{m,1} &= a_1 + a_0 ß_{m,1} \\
&\vdots \\
\alpha_{m,m-1} &= a_{m-1} + a_{m-2} + ß_{m,1} + \ldots\ldots \qquad + a_0 ß_{m,m-1}
\end{aligned}\right\} \quad (2.2)$$

and

$$\left.\begin{aligned}
-\alpha_{m,m-1} &= a_{-1} ß_{m,m} \\
-\alpha_{m,m-2} &= a_{-2} ß_{m,m} + a_{-1} ß_{m,m-1} \\
&\vdots \\
-\alpha_{m,0} &= a_{-m} ß_{m,m} + a - (_{m-1}) + ß_{m,m-1} + \ldots\ldots \qquad + a_{-1} ß_{m,1}
\end{aligned}\right\}$$

(2.3)

Eliminating the α's from (2,2) and (2,3) gives the equations for the ß's. These are

$$\left.\begin{aligned}
a_0 + a_{-1} ß_{m,1} + a_{-2} ß_{m,2} + \ldots\ldots\ldots\ldots\ldots + a_{-m} ß_{m,m} &= 0 \\
a_1 + a_0 ß_{m,1} + a_{-1} ß_{m,2} + \ldots\ldots\ldots\ldots\ldots + a_{-(m-1)} ß_{m,m} &= 0 \\
&\vdots \\
a_{m-1} + a_{m-2} ß_{m,1} + a_{m-3} ß_{m,2} + \ldots\ldots\ldots\ldots\ldots + a_0 ß_{m,m} &= 0
\end{aligned}\right\} \quad (2.4)$$

Now $\Delta_{m,m+1}(z)=\{P_m(z)Q_{m+1}(z)-P_{m+1}(z)Q_m(z)\}$ is a polynomial in z Whose degree is at most 2m. Using (2.1) we have

$$\begin{aligned}
\frac{\Delta_{m,m+1}(z)}{Q_m(z)Q_{m+1}(z)} &= f_{m,m}(z) - f_{m+1,m+1}(z) \\
&= \{f_{m,m}(z) - f(z)\} - \{f_{m+1,m+1}(z) - f(z)\} \\
&= 0(z^{m}, z^{-(m+1)})
\end{aligned}$$

Consequently $\Delta_{m,m+1}(z)$ consists of a single term in z^m This

term can be determined by considering the term in z^m in $Q_{m+1}(z)\{P_m(z)-Q_m(z)f(z)\}$ i.e. in $Q_m(z)f(z)$. Hence

$$\Delta_{m,m+1}(z) + \{a_m + a_{m-1}\beta_{m,1} + \ldots + a_0\beta_{m,m}\}z^m = 0 \tag{2.5}$$

We define the determinant $D_{r,s}$ by

$$D_{r,s} \equiv \begin{vmatrix} a_r & a_{r+1} & \cdot & \cdot & \cdot & a_{r+s} \\ a_{r-1} & a_r & \cdot & \cdot & \cdot & a_{r+s-1} \\ \cdot & \cdot & & & & \\ \cdot & \cdot & & & & \\ \cdot & \cdot & & & & \\ a_{r-s} & a_{r-s+1} & \cdot & \cdot & \cdot & a_r \end{vmatrix} \tag{2.6}$$

where $r = 0, \pm 1, \pm 2, \ldots$ and $s = 0,1,2,\ldots$ • For $s < 0$, $D_{r,s} \equiv 1$. These determinants are known as Hankel determinants and occur in the theory of continued fractions which correspond to power series expansions at one point. [See for example Henrici (1958)]. Using this notation we find from (2.4) and (2.5) that

$$\Delta_{m,m+1} = (-1)^{m+1} \frac{D_{0,m} z^m}{D_{-1,m-1}} \tag{2.7}$$

As regards $\Delta_{m,m+2}(z)$ this is a polynomial of degree $(2m+1)$ at most,

and proceeding as before we find that

$$\frac{\Delta_{m,m+2}(z)}{Q_m(z)Q_{m+2}(z)} = o(z^m, z^{-(m+1)}).$$

Hence $\Delta_{m,m+2}(z)$ consists of two terms only and is of the form $(A^m z^m + A_{m+1}, z_{m+1})$ where the A's are constants. Clearly $A_m z^m$ is just $\Delta_{m,m+1}(z)$ and the remaining term is calculated by considering the

term in z^{m+1} in $Q_{m+2}(z)\{p_m(z) - Q_m(z)f(z)\}$. This is

$$\beta_{m+2,m+2} C_m z^{m+1}$$

where

$$C_m \equiv a_{-1} + a_{-2}\beta_{m,1} + \ldots\ldots \qquad + a_{-(m+1)}\beta_{m,m}.$$

Eliminating the ß's from this latter result and (2.4) yields

$$C_m = D_{-1,m} \big/ D_{-1,m-1} \qquad (2.8)$$

for m= 0,1,2, ... Solving (2.4) for $\beta_{m,m}$ gives us

$$\beta_{m,m} = (-1)^m D_{0,m-1} \big/ D_{-1,m-1} \qquad (2.9)$$

for m= 1,2,3, ...

Using the results (2.7) to (2.9) and (1.9) we find that

$$p_{m+1} = \frac{D_{0,m}}{D_{-1,m-1}} z \Bigg/ \frac{D_{0,m-1}}{D_{-1,m-2}}, \quad q_{m+1} = 1 - \frac{D_{0,m}}{D_{-1,m}} z \Bigg/ \frac{D_{0,m-1}}{D_{-1,m-1}}. \qquad (2.10)$$

for m= 1,2,3,..., For m =0 we find that the formulae in (2.10) still hold, save that p_1. is constant and independent of z. Thus the continued fraction which has for its successive convergents

$f_{1,1}$, $f_{2,2}$,...., is

$$\frac{n_1}{1+d_1 z+} \quad \frac{n_2 z}{1+d_2 z+\ldots} \qquad (2.11)$$

where

$$n_m = \frac{D_{0,m-1}}{D_{-1,m-2}} \Bigg/ \frac{D_{0,m-2}}{D_{-1,m-3}}, \qquad d_m = -\frac{D_{0,m-1}}{D_{-1,m-1}} \Bigg/ \frac{D_{0,m-2}}{D_{-1,m-2}}$$

for m = 1,2,3, ...

<u>Case (ii) Matching unequal numbers of terms of each series</u>

A continued fraction of the form (2.11) can be constructed using much the same method as before, although slight differences occur.

Since in the rational function approximations (1.3) an even number of parameters is available we consider approximations of the form $f_{m+2r,m}$. To obtain a continued fraction whose elements

can be expressed in a simple way, we consider firstly the sequence of convergents

$$f_{1,1}, f_{2,2}, \ldots, f_{m,m}, f_{m+1,m-1}, f_{m+2,m}, \ldots, f_{m+2r-1,m-1}, f_{m+2r,m}, \ldots\ldots$$

for r = 1,2,3, …… .

This sequence leads to the continued fraction

$$\frac{n_1}{1+d_1z+}\;\frac{n_2z}{1+d_2z+}\;\frac{n_mz}{1+d_mz+}\;\frac{n_{m+1}z}{1}{}_{+}\;\frac{n_{m+2}z}{1}{}_{+}\;\ldots\ldots \tag{2.12}$$

where

$$n_r = \frac{D_{0,r-1}}{D_{-1,r-2}}\bigg/\frac{D_{0,r-2}}{D_{-1,r-3}}\;,\quad d_r = -\frac{D_{0,r-1}}{D_{-1,r-1}}\bigg/\frac{D_{0,r-2}}{D_{-1,r-2}}$$

for $r = 1,2,3,\ldots\ldots m$, and

$$n_{m+2r+1} = \frac{D_{r-1,m+r-1}}{D_{r-2,m+r-2}}\bigg/\frac{D_{r-1,m+r-2}}{D_{r-2,m+r-3}}\;.\; n_{m+2r} = -\frac{D_{r-1,m+r-1}}{D_{r-1,m+r-2}}\bigg/\frac{D_{r-1,m+r-2}}{D_{r-2,m+r-2}}$$

for r = 1,2,3, ………

From the m^{th} partial quotient onwards, the continued fraction takes the form of a conventional continued fraction which corresponds to a power series expansion at z = 0. In fact setting m = 0 in (2.12) we obtain (save for z in the first partial numerator) the continued fraction which corresponds to a single power series at z = 0.

In considering the approximation $f_{m+2r,m}$ we could also take the sequence

$$f_{1,0},\; f_{2,0}\; \ldots\ldots,\; f_{2r,0}\;\; f_{2r+1,1},\; \ldots\ldots,\; f_{2r+m,m}, \ldots\ldots\ldots$$

This sequence gives rise to the continued fraction

$$\frac{n_1}{1+}\;\frac{n_2z}{1+}\ldots\ldots+\frac{n_{2r}z}{1+}\;\frac{n_{2r+1}z}{1+d_{2r+1}z}+\ldots\ldots\ldots\frac{n_{2r+m}z}{1+d_{2r+m}z}+\ldots\ldots\ldots \tag{2.13}$$

where

$$n_{2s-1} = \frac{D_{s-1,s-1}}{D_{s-2,s-2}} \Big/ \frac{D_{s-1,s-2}}{D_{s-2,s-3}}, \qquad n_{2s} = -\frac{D_{s,s-1}}{D_{s-1,s-2}} \Big/ \frac{D_{s-1,s-1}}{D_{s-2,s-2}}$$

for $s = 1,2,\ldots.r$, and

$$n_{2r+m} = \frac{D_{r,r+m-1}}{D_{r-1,r+m-2}} \Big/ \frac{D_{r,r+m-2}}{D_{r-1,r+m-3}}, \qquad d_{2r+m} = -\frac{D_{r,r+m-1}}{D_{r-1,r+m-1}} \Big/ \frac{D_{r,r+m-2}}{D_{r-1,r+m-2}}$$

for $m = 1,2,3, \ldots\ldots\ldots\ldots\ldots$

Assuming that the expansion (1.1) is known, the result (2.12) is of use when at $z = \infty$ it is either difficult to determine the series or there is a good reason for not proceeding beyond a certain term in the series. Such situations can occur when the series at $z = \infty$ is asymptotic.

The result (2.13) may be of use when either a_0 or a_{-1} is zero. Hitherto we have assumed that both $a_0, a_{-1}, \pm 0$ so that $n_1 = a_0$ and $d_1 = -a_0 / a_{-1}$ are both non zero. Even if this condition does not hold simple modifications are often available. Thus if $a_0 = 0$ but $a_1, a_2 \neq 0$

we could consider the function $g(z) = [f(z) - a_1 z\} / z^2$. The expansions

for $g(z)$ for $|z|$ small and for $|z|$ large have leading terms a_2 and $-a_1/z$ respectively, so that for $g(z)$ continued fractions may be developed as before. Clearly, provided *any* two consecutive values of the sequence $\{a_n\}$, $n = 0, \pm 1, \pm 2, \ldots$, are both non zero a suitable $g(z)$ may be constructed.

On the other hand we could proceed as follows and form a continued fraction of the form (2.13) for the function $f(z)/z$ in the case when $a_0 = 0$, but $a_1, a_2 \neq 0$. Thus, we have

$$\frac{n_1}{1} + \frac{n_2 z}{1} + \frac{n_3 z}{1 + d_3 z} + \ldots\ldots\ldots\ldots$$

where the first two terms of the series

$$\frac{f(z)}{z} = a_1 + a_2 z + a_3 z^2 + \ldots\ldots$$

have been fitted before any of those of

$$\frac{f(z)}{z} = -\{\frac{a_{-1}}{z^2} + \frac{a_{-2}}{z^3} + ...\}$$

This method is more direct than the former.

3. Algorithm for the continued fractions

For continued fractions which correspond to a single power series, the well-known q-d algorithm [Rutishauser (1954), Henrici (1958)] is available for obtaining the partial numerators. It is found that for continued fractions that correspond to power series at two points, an algorithm- essentially the q-d algorithm- can be generated.

We start by assuming that in (1.1) and (1.2) $a_r \neq 0$, $r = 0, \pm 1, \pm 2, ...$ and by defining

$$\left.\begin{array}{l} f^{(r)}(z) \equiv [f(z) - \{a_0 + a_1 z + ... + a_{r-1} z^{r-1}\}]/z^r \\ \text{for} \quad r = 1,2,3,.......f^{(0)}(z) \equiv f(z), \text{ and} \\ f^{(-r)}(z) = [f(z) + \{\frac{a_{-1}}{z} + \frac{a_{-2}}{z^2} + ... + \frac{a_{-r}}{z^r}\}]z^r \end{array}\right\} \quad (3.1)$$

Clearly $f^{(r)}(z)$ possesses the expansions

$$f^{(r)}(z) = a_r + a_{r+1} z + a_{r+2} + z^2 +$$

for $|z|$ small and

$$f^{(r)}(z) = -\left\{\frac{a_{r-1}}{z} + \frac{a_{r-2}}{z^2} + ... + \frac{a_0}{z^r} + \frac{a_{-1}}{z^{r+1}} + ...\right\}$$

for $|z|$ large. Consequently for $f^{(r)}(z)$ we can forra the continued fraction

$$\frac{a_r}{1 + d_1^{(r)} z} {}_{+} \frac{n_2^{(r)} z}{1 + d_2^{(r)} z} {}_{+} \frac{n_3^{(r)} z}{1 + d_3^{(r)} z} {}_{+..........} \quad (3.2)$$

for $r = 0, \pm 1, \pm 2, ...$

Now the continued fraction (3.2) can be regarded as the even part of

$$\left.\begin{aligned}&\frac{a_r}{1}+\frac{d_1^{(r)}z}{1}-\frac{l_2^{(r)}}{1}+\frac{m_2{}^{(r)}z}{1}-\frac{l_3^{(r)}}{1}+\frac{m_3{}^{(r)}z}{1}-\\&\text{where}\\&l_i{}^{(r)}=n_i^{(r)}/\{n_i^{(r)}+d_{i-1}^{(r)}\},\quad m_i{}^{(r)}=d_i^{(r)}d_{i-1}^{(r)}/\{n_i^{(r)}+d_{i-1}^{(r)}\},\end{aligned}\right\}\qquad(3.3)$$

for r = 0, + 1, + 2, .. and 1 = 2,3,4 ...

The odd part of (3.3) is

$$a_r-\frac{a_r d_1^{(r)}z}{1-l_2^{(r)}+d_1^{(r)}z}+\frac{l_2^{(r)}m_2^{(r)}z}{1-l_3^{(r)}+m_1^{(r)}z}$$

Thus the function $\{f^{(r)}(z)-a_r\}/z$ generates the continued fraction

$$-\frac{a_r d_1^{(r)}}{1-l_2^{(r)}+d_1^{(r)}z}+\frac{l_2^{(r)}m_2^{(r)}z}{1-l_z^{(r)}+m_2^{(r)}z}\qquad(3.4)$$

But $\{f^{(r)}(z)-a_r\}/z=f^{(r+1)}(z)$ which has the continued fraction

$$\frac{a_{r+1}}{1+d_1^{(r+1)}z}+\frac{n_2^{(r+1)}z}{1+d_2^{(r+1)}z}+\frac{n_3^{(r+1)}z}{1+d_3^{(r+1)}z}+\ldots\ldots\ldots\ .\qquad(3.5)$$

By an equivalence transformation (3.4) can be brought into the form (3.5) and equating the various coefficients leads to

$$\left.\begin{aligned}n_{i+1}^{(r)}+d_i^{(r)}&=n_i^{(r+1)}+d_i^{(r+1)}\\n_{i+1}^{(r)}+d_{i+1}^{(r+1)}&=n_{i+1}^{(r+1)}+d_i^{(r)}\end{aligned}\right\}\qquad(3.6)$$

for i = 2,3,4, ... and r = 0,± 1, ± 2 ,

Now directly from (3.2) we have

$$d_1^{(r)}=-a_r/a_{r-1}\quad,\qquad r=0,\pm 1,\ \pm 2,\ ..\qquad(3.7)$$

and equating the constant terms of (3.4) and (3.5) and using (3.7) we fund that

$$n_2^{(r)} + d_1^{(r)} = d_1^{(r+1)}$$

Thus if we define $n_1^{(r)} \equiv 0$ for r = 0 , ± 1, ± 2, .. and calculate

$d_1^{(r)}$ by (3.7), then (3.6) may be used to calculate the subsequent n's and d.s. The rules of formulation follow the characteristic pattern of the q-d algorithm.

$$\begin{array}{ccccccccc}
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\
0 & d_1^{(-4)} & n_2^{(-4)} & d_2^{(-4)} & n_3^{(-4)} & d_3^{(-4)} & n_4^{(-4)} & d_4^{(-4)} & \cdots \\
0 & d_1^{(-3)} & n_2^{(-3)} & d_2^{(-3)} & n_3^{(-3)} & d_3^{(-3)} & n_4^{(-3)} & d_4^{(-3)} & \cdots \\
0 & d_1^{(-2)} & n_2^{(-2)} & d_2^{(-2)} & n_3^{(-2)} & d_3^{(-2)} & n_4^{(-2)} & d_4^{(-2)} & \cdots \\
0 & d_1^{(-1)} & n_2^{(-1)} & d_2^{(-1)} & n_3^{(-1)} & d_3^{(-1)} & n_4^{(-1)} & d_4^{(-1)} & \cdots \\
0 & d_1^{(0)} & n_2^{(0)} & d_2^{(0)} & n_3^{(0)} & d_3^{(0)} & n_4^{(0)} & d_4^{(0)} & \cdots \\
0 & d_1^{(1)} & n_2^{(1)} & d_2^{(1)} & n_3^{(1)} & d_3^{(1)} & n_4^{(1)} & d_4^{(1)} & \cdots \\
0 & d_1^{(2)} & n_2^{(2)} & d_2^{(2)} & n_3^{(2)} & d_3^{(2)} & n_4^{(2)} & d_4^{(2)} & \cdots \\
0 & d_1^{(3)} & n_2^{(3)} & d_2^{(3)} & n_3^{(3)} & d_3^{(3)} & n_4^{(3)} & d_4^{(3)} & \cdots \\
0 & d_1^{(4)} & n_2^{(4)} & d_2^{(4)} & n_3^{(4)} & d_3^{(4)} & n_4^{(4)} & d_4^{(4)} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots &
\end{array}$$

We refer to the above arrangement as the n-d array. The inset block shows how from $d_1^{(r)}$ the subsequent n's and d's are calculated.

The continued fraction which fits equal numbers of terms of (1.1) and (1.2) is in this notation.

$$\frac{a_0}{1+d_1^{(0)}z_+}\;\frac{n_2^{(0)}z}{1+d_2^{(0)}z_+}\;\frac{n_3^{(0)}z}{1+d_3^{(0)}z_{+\ldots\ldots}} \qquad (3.8)$$

the appropriate n's and d's lying along the central row of the n-d array. Given say the firat four terms of (1.1) and (1.2), i.e. a_r for r = -4,-3,..,3 we calculate the entries which lie inside the outer block in the n-d array. These include the first four partial quotients of the continued fraction (3.8). The n-d array may be used for other purposes and in other ways.

Clearly the continued fractions corresponding to all the functions $f^{(r)}(z)$ are available.

Further since in (3.6) the individual n's and d's enter linearly, the entries in the n-d array could be generated from initial conditions other than those given in (3.7). For example if f(z) possesses a single series expansion, namely (1,1), we can construct (in our notation) the corresponding continued fraction

$$\frac{a_0}{1} + \frac{n_2^{(0)} z}{1} + \frac{n_3^{(0)}}{1} + \ldots\ldots\ldots\ldots$$

This problem is solved, conventionally, by a direct application of the q-d algorithm. Here the problem oan be solved by setting

$$d_1^{(r)} = - a_r/a_{r-1},\ r=1,2,3\ \ldots \quad \text{and} \quad d_i^{(o)}\ o,\ i = 1,2,3\ \ldots$$

Equation (3.6) then enable the entries in the n-d array, on and below the $n^{(o)}$, $d^{(o)}$ row, to be completed in the usual way. If needed, the entries above the $n^{(o)}$,$d^{(o)}$ row can be obtained by solving (3.6) for $n_{i+1}^{(r)}$ and $d_i^{(r)}$ in terms of the other n's and d's in the equations.
It is found that

$$\left(n_{i+1}^{(r+1)} + d_{i+1}^{(r+1)}\right)d_i^{(r)} = \left(n_i^{(r+1)} + d_i^{(r+1)}\right)d_{i+1}^{(r+1)}$$

$$\left(n_{i+1}^{(r+1)} + d_{i+1}^{(r+1)}\right)n_{i+1}^{(r)} = \left(n_i^{(r+1)} + d_i^{(r+1)}\right)n_{i+1}^{(r+1)}$$

We can also use (3.6) to construct continued fractions which fit a finite number of terms of, say, (1.2) and all the terms of (1.1). In this problem we are given a_r for r = - m,-(m-l), -(m-2), ..., and we set

$$d_1^{(r)} = - a_r/a_{r-1} \quad , \qquad r = - (m-1), -(m-2), \ldots ,$$

and

$$d_i^{(0)} = 0 \quad , \qquad i = (m+1), (m+2), \ldots .$$

In the resulting continued fraction the $(m+2r)^{th}$ convergent is $f_{m+2r,m}$ and the $(m+2r+l)^{th}$ convergent is $f_{m+2r+1,m-1}$ as in (2.12).
With suitable initial conditions the continued fraction (2.13), and others, may be generated from the n-d array.
The algorithm (3.6) can be reconciled with the results of previous section as follows. In determinant form we see from (2.11) that

$$n_i^{(r)} = \frac{D_{r,i-1}}{D_{r-1,i-2}} \bigg/ \frac{D_{r,i-2}}{D_{r-1,i-3}} \quad , \qquad d_i^{(r)} = \frac{D_{r,i-1}}{D_{r-1,i-1}} \bigg/ \frac{D_{r,i-2}}{D_{r-1,i-2}}$$

i = 1,2,3, ... , r = 0, ± 1, ± 2, ...

Substituting these results into the first equation in (3.6) and rearranging leads to

$$\frac{D_{r,i-2}\,D_{r,i} - D_{r,i-1}^2}{D_{r-1,i-1}\,D_{r+1,i-1}} = \frac{D_{r,i-3}\,D_{r,i} - D_{r,i-2}^2}{D_{r-1,i-2}\,D_{r+1,i-2}}$$

for i = 2,3,4, ..., r = 0,± 1, ± 2, .. •
Now the Hankel determinants $D_{r,s}$ have the known property that

$$D_{r,i-1}^2 - D_{r,i-2}\,D_{r,i} + D_{r-1,i+1}\,D_{r+1,i-1} = 0$$

[See for example Henrici (1958)], and thus the first equation of (3.6) is established. On substituting the determinants into the second equation of (3.6) it reduces to an identity. The particular case i = 1 of (3.6) is easily verified. We have thus demonstrated the connection between the algorithm (3.6) and the determinant results of Section 2.

4. Effects of zero coefficients in the series.

In the n-d array the entries which are affected by the coefficient a_1, lie between the solid lines in the following table

$$
\begin{array}{ccccccc}
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\
0 & d_1^{(-3)} & n_2^{(-3)} & d_2^{(-3)} & n_3^{(-3)} & d_3^{(-3)} & --- \\
0 & d_1^{(-2)} & n_2^{(-2)} & d_2^{(-2)} & n_3^{(-2)} & d_3^{(-2)} & --- \\
0 & d_1^{(-1)} & n_2^{(-1)} & d_2^{(-1)} & n_3^{(-1)} & d_3^{(-1)} & --- \\
0 & d_1^{(0)} & n_2^{(0)} & d_2^{(0)} & n_3^{(0)} & d_3^{(0)} & --- \\
0 & d_1^{(1)} & n_2^{(1)} & d_2^{(1)} & n_3^{(1)} & d_3^{(1)} & --- \\
0 & d_1^{(2)} & n_2^{(2)} & d_2^{(2)} & n_3^{(2)} & d_3^{(2)} & --- \\
0 & d_1^{(3)} & n_2^{(3)} & d_2^{(3)} & n_3^{(3)} & d_3^{(3)} & --- \\
0 & d_1^{(4)} & n_2^{(4)} & d_2^{(4)} & n_3^{(4)} & d_3^{(4)} & --- \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots &
\end{array}
$$

If $a_1 = 0$ and all other a's are non zero we have $d_1^{(1)} = 0$ and $d_1^{(2)} = \infty$ • To calculate subsequent entries we set $a_1 = € \rightarrow$ say and then let $\varepsilon \rightarrow 0$ in the affected elements. The results are

$$n_2^{(0)} = -d_1^{(0)} \qquad n_3^{(0)} = (d_2^{(0)} + a_2/a_0 d_1^{(0)})$$

$$d_1^{(0)} = 0 \quad n_2^{(1)} = \infty \qquad d_2^{(1)} = \infty \quad n_3^{(1)} = (d_1^{(3)} + a_2/a_0 d_1^{(0)}) \tag{4.1}$$

$$d_1^{(0)} = 0 \quad n_2^{(1)} = \infty \qquad d_2^{(1)} = \infty \quad n_3^{(1)} = 0 \qquad d_3^{(2)} = (n_2^{(3)} d_1^{(3)} - a_2/a0) \big/ n_3^{(1)}$$

$$d_2^{(3)} = -n_2^{(3)} n_3^{(3)} = -d_2^{(3)} \qquad d_3^{(3)} = -(a_2/a_0) n_3^{(3)} \big/ (n_2^{(3)} d_1^{(3)} - a_2/a_0)$$

All other entries may be calculated in the usual way.
If the continued fraction is constructed directly from the series (1-1) and (1.2) using determinants, the condition $a_1 = 0$ and all other a's $\neq 0$ does not give rise to any difficulty. The trouble occurs in the subsidiary continued fractions in the n-d array as indicated by (4.l).

It is clear from (4.1) that zero coefficients make calculations for the n-d array very cumbersome and it is certainly easier for numerical purposes to modify f(z).

If some of the coefficients - other than a_0 or a_{-1} - in the expansions (1.1) and (1.2) of f(z) are zero, we consider the function

$$g(z) \equiv f(z) - K/(-1+z) \qquad (4.2)$$

where the constant K is chosen such that the coefficients in the corresponding expansions of g(z) are non-zero. The continued fraction of the form (1.12) for g(z) can be calculated from the n-d algorithm in the usual way. The continued fraction for f(z) can thus be expressed as the sum of the continued fraction for g(z) and the rational function K/(l+z). To construct the continued fraction for f(z) in the form (1.12) we proceed as follows.
Suppose the continued fraction for f (z) is

$$\frac{n_1}{1+d_1 z} + \frac{n_2 z}{1+d_2 z} + \ldots\ldots + \frac{n_m z}{1+d_m z} + \ldots \qquad (4.3)$$

with convergents to N_m / D_m, $m = 1,2,3 \ldots$, and that for g(z) is

$$\frac{p_1}{1+q_1 z} + \frac{p_2 z}{1+q_2 z} + \ldots\ldots + \frac{p_m z}{1+q_m z} + \ldots \qquad (4.4)$$

with convergent P_m/Q_m, $m = 1,2,3, \ldots$

The polynomials N_m, P_m are of degree (m-1) in z and D_m, Q_m of degree m in z.

Define

$$\left.\begin{array}{llll} \alpha_m \equiv p_1 p_2 .. p_m & , & \alpha_m \equiv n_1 n_2 ..... n_m \\ ß_m \equiv p_1 p_2 .. p_m & , & \delta_m \equiv d_1 d_2 ..... d_m \end{array}\right\} \qquad (4.5)$$

and λ_m , $/\mu_m$ to be respectively the coefficients of *z* in the polynomials Q_m ,D_m

Thus

$$\left.\begin{array}{l} \lambda_{m+1} = \lambda_m + q_{m+1} + p_{m+1}, \quad m = 1,2,3,..... \\ \lambda_1 = q_1 \\ \textit{whilst} \\ \mu_{m+1} = \mu_m + d_{m+1} + n_{m+1}, \quad m = 1,2,3,..... \\ \mu_1 = d_1 \end{array}\right\} \qquad (4.6)$$

We suppose that all the p's and q's in (4.4) are known and that the n's and d's in (4.3) are chosen such that

$$\frac{p_m}{q_m} + \frac{k}{1+z} - \frac{N_m}{D_m} = 0(z^m, z^{-(m+1)})$$

or

$$\frac{(1+z)(P_m D_m - N_m Q_m) + KQ_m D_m}{(1+z)Q_m D_m} = 0(z^m, z^{-(m+1)})$$

The numerator on L.H.S. is of degree 2m at most and the denominator is of degree (2m+l), Hence for R.H.S. to hold the numerator of L.H.S. consists of a single term in z^m Let this be $C_m z^m$ where C_m is a constant which depends on m but not on *z*. Thus

$$(1+z)\left(\frac{P_m}{Q_m} - \frac{N_m}{D_m}\right) + K \equiv \frac{C_m z^m}{Q_m D_m} \qquad (4.7)$$

Using this result and that

$$P_{m+1}\,Q_m - P_m Q_{m+1} \equiv (-1)^m \alpha_{m+1} z^m$$

we find

$$\frac{C_{m+1}z}{Q_{m+1}D_{m+1}} \equiv \frac{C_m}{Q_m D_m} + (-1)^m (1+z)\left\{\frac{\alpha_{m+1}}{Q_m Q_{m+1}} - \frac{y_{m+1}}{D_m D_{m+1}}\right\} \tag{4.8}$$

From (4.8) by equating coefficients of z^r for $r = 0, 1, -2m$ we have

$$0 = C_m + (-1)^m (\alpha_{m+1} - y_{m+1})$$

$$C_{m+1} = -(\lambda_m + \mu_m)C_m + (-1)^m \{\alpha_{m+1}(1-\lambda_m - \lambda_{m+1}) - y_{m+1}(1-\mu_m - \mu_{m+1})\}$$

$$0 = \frac{C_m}{\beta_m \delta_m} + (-1)^m \left\{\frac{\alpha_{m+1}}{\beta_m \beta_{m+1}} - \frac{y_{m+1}}{\delta_m \delta_{m+1}}\right\}$$

Thus

$$\left.\begin{aligned} y_{m+1} - \alpha_{m+1} &= y_m(1+\lambda_{m-1} - \mu_m) - \alpha_m(1+\mu_m - 1 - \lambda_m) \\ \frac{y_{m+1}\beta_m}{\delta_{m+1}} - \frac{\alpha_{m+1}\delta_m}{\beta_{m+1}} &= y_{m+1} - \alpha_{m+1} \end{aligned}\right\} \tag{4.9}$$

for m = 1,2,3, ...
These equations together with (4.5) and (4.6) and the initial values n_1, d_1 given by

$$n_1 = P_1 + K$$

$$\frac{n_1}{d_1} = \frac{P_1}{q_1} + K$$

are sufficient to determine the subsequent ns and d's.

5. Error of the approximations

The chief aim of the present paper has been the construction of the continued fractions. The problem of estimating the error between a given function and the m^{th} convergent of its continued fraction has not been fully investigated. However from the preceding work several useful results are available.

Firstly the order of the error between a function and a particular convergent follows immediately.

Since the approximation $f_{m,m}$ agrees with m terms of each of (1.1) and (1.2) the terms in z^m and $z^{-(m+1)}$ will be the same for both

$\{f-f_{m,m}\}$ and $\{f_{m+1,m+1}-f_{m,m}\}$. For the continued fraction for f(z)

$$\frac{n_1}{1+d_1 z}+\frac{n_2 z}{1+d_2 z}+\ldots\ldots+\frac{n_m z}{1+d_m z}+\ldots \tag{5.1}$$

with convergents $P_m(z)/Q_m(z)$, we have

$$f_{m,m}=P_m(z)/Q_m(z) \quad , \quad \Delta_{m,m+1}(z)=(-1)^m n_1 n_2 \ldots\ldots n_{m+1} z^m \quad ,$$

and

$$Q_m(z) = 1 + \ldots\ldots\ldots + (d_1 d_2 \ldots d_m) z^m \cdot$$

Hence

$$f_{m+1,m+1}-f_{m,m}=\frac{\Delta_{m+1,m}(z)}{Q_m(z)Q_{m+1}(z)}$$

$$=\frac{(-1)^m n_1,n_2\ldots\ldots\ldots\ldots n_{m+1}\, z^m}{1+\ldots\ldots\ldots \qquad +(d_1 d_2\ldots d_m)^2 d_{m+1} z^{2m+1}}$$

Thus for $|z|$ small

$$|f-f_{m,m}|=|n_1 n_2\ldots n_{m-1}|\,|z|^m \quad \{1+0(|z|)\}$$

and for $|z|$ large

$$|f-f_{m,m}|=\left|\frac{n_1 n_2\ldots n_{m+1}}{(d_1 d_2\ldots\ldots d_m)^2}d_{m+1}\right|\frac{1}{|z|^{m+1}}\{1+0(\frac{1}{|z|})\} \tag{5.2}$$

for m = 1,2,3,......

In determinant notation these results become respectively

$$|f-fm,m|=\left|\frac{D_{0,m}}{D_{-1,m-1}}\right||z|^m\ \{1+0(|z|)\}$$

$$|f-fm,m|=\left|\frac{D_{-1,m}}{D_{0,m-1}}\right|\frac{1}{|z|^{m+1}}\ \{1+0(\frac{1}{|z|})\} \tag{5.3}$$

for m= 1,2,3,......

The next results are all concerned with error bounds on the positive real axis of z. If all the coefficients of $Q_m(z)$, m=1,2,3,..., are positive then f(z) will be free of singularities on $x \geq 0$. All the coefficients of $Q_m(z)$ will be positive under the various sufficien conditions

$$\left.\begin{array}{lll} \text{(i)} & d_m > 0, & n_{m+1} > 0 \\ \text{(ii)} & d_m > 0, & d_{m+1} + n_{m+1} \geq 0 \\ \text{(iii)} & d_m > 0, & d_m + n_{m+1} \geq 0 \end{array}\right\} \quad (5.4)$$

for m = 1,2,3,,, . We shall restrict ourselves to these cases.

Case (i) is evident and cases (ii) and (iii) can be established by induction- Further for continued fractions in which $Q_m(x) > 0$ for $x \geq 0$ the quantity $|f_{m+1,m+1} - f_{m,m}|$ will have a single maximum in the range

$x \geq 0$. A rough estimate of the position of this maximum can be found by equating the leading terms for x small and x large in (5.2). This gives

$$x_{max} \approx [(d_1 d_2 \ldots . d_m)^2 d_{m+1}]^{-\frac{1}{(2m+1)}} \quad (5.5)$$

or in the determinants

$$x_{max} \approx [(\frac{D_{-1,m} D_{-1,m-1}}{D_{0,m} D_{0,m-1}}]^{\frac{1}{(2m+1)}}$$

If the conditions of case (i) hold for the continued fraction (5.1). then for $x > 0$ by a well known result which is easily verified the successive convergents $P_m(x)/Q_m(x)$ lie on opposite sides of f(x),and

$$|f - \frac{P_m}{Q_m}| < |\frac{P_{m+1}}{Q_{m+1}} - \frac{P_m}{Q_m}|$$

or

$$|f - f_{m,m}| < |\frac{n_1 n_2 \ldots \ldots . n_{m+1} x^m}{Q_m Q_{m+1}} \quad (5.6)$$

and an estimate of the maximum error can be obtained by setting $x = x_{max}$ of (5.5) in this result.

To find corresponding results for cases (ii) and (iii) we start with the continued fraction

$$R_m = \frac{n_{m+1}x}{1+d_{m+1}x} \quad + \quad \frac{n_{m+2}x}{1+d_{m+2}x} \tag{5.7}$$

and observe that as $x \to \infty$, $R_m \to n_{m+1}/d_{m+1}$,, whilst as $x \to 0$, $R_m \to n_{m+1}x$.

Define $\overline{P}_m, \overline{Q}_m$ by

$$\left.\begin{aligned} \overline{P}_m &\equiv d_{m+1}P_m + n_{m+1}P_{m-1} \\ \overline{Q}_m &\equiv d_{m+1}Q_m + n_{m+1}Q_{m-1} \\ &\equiv (Q_{m+1} - Q_m)/x \end{aligned}\right\} \tag{5.8}$$

we show that for $x > 0$ and the conditions of case (ii) P_m/Q_m and $\overline{P}_m/\overline{Q}_m$ lie on opposite sides of f(x).

Now

$$f - \frac{P_m}{Q_m} = \frac{P_m + R_m P_{m-1}}{Q_m + R_m Q_{m-1}} - \frac{P_m}{Q_m}$$

$$= \frac{(-1)^m n_1 n_2 \ldots\ldots n_m R_m z^{m-1}}{Q_m(Q_m + R_m Q_{m-1})} \quad ,$$

and similarly

$$f - \frac{\overline{P}_m}{\overline{Q}_m} = \frac{(-1)^m (d_{m+1}R_m - n_{m+1}) n_1 n_2 \ldots\ldots n_m x^{m-1}}{\overline{Q}_m (Q_m + R_m Q_{m-1})}$$

We show that $(n_{m+1} - d_{m+1} R_m)$ and R_m have the same sign for $x > 0$.

Expressed in continued fraction form

$$n_{m+1} \quad - d_{m+1} - \quad R_m \quad = n_{m+1}\left[1-\frac{d_{m+1}x}{1+d_{m+1}x_{+}}\,\frac{n_{m+2}x}{1+d_{m+2}x_{+\ldots\ldots}}\right]$$

If the continued fraction in square brackets on R.H.S. has convergents U_r/V_r r= 1,2,3 .. with $U_1= d_{m+1}x$, $V_1 = 1+d_{m+1}{}^{x}$, then under the conditions of (ii) all coefficients of V_r are positive and it can be shown inductively that $V_r - U_r \geq 1$, r = 1,2,3, ..
Thus the sign of (n_{m+1} - d_{m+1} . R_m) and R_m is the same, being determined by the sign of n_{m+1} , and consequently P_m/Q and $\overline{P}_m / \overline{Q}_m$ lie on opposite sides of f. Using (5.8) this yields the result

$$| f - f_{m,m} | < \left|\frac{P_m}{Q_m} - \frac{\overline{P}_m}{\overline{Q}_m}\right| = \frac{| n_1 n_2 .....n_{m+1} | x^m}{Q_m (Q_{m+1} - Q_m)} \qquad (5.9)$$

For Case (iii) we define P^*_m, Q^*_m by

$$\left.\begin{aligned} P^*_m &= P_m + n_{m+1}x\, P_{m-1} \\ Q^*_m &= Q_m + n_{m+1}xQ_{m-1} \\ &= Q_{m+1} - d_{m+1}xQ_{m-1} \end{aligned}\right\} \qquad (5.10)$$

In this case we fund that P_m / Q_m and P^*_m / Q^*_m lie on opposite sides of f and that

$$| f - f_{m,m} | < \frac{| n_1 n_2 .....n_{m+1} | x^m}{Q_m (Q_{m+1} - d_{m+1}xQ_m)} \qquad (5.11)$$

An estimate of the maximum error for each case may be obtained by setting x = x_{max} of (5.5) in (5.9) and (5.11).

We turn now to the continued fraction

$$\frac{n_1}{1+d_1z}+\quad\frac{n_2}{1+d_2z}+\ldots\quad+\frac{n_m}{1+d_mz}+\quad\frac{n_{m+1}z}{1}+\quad\frac{n_{m+2}z}{1}+\ldots.. \qquad (5.12)$$

Here

$Q_{m+2r}\ (z) = 1 + .. \qquad + (d_1d_2\ldots\ d_m n_{m+2}\, n_{m+4\ldots.} n_{m+2r})z^{m+r}$.

For this continued fraction it is easier to consider the convergents $f_{m+2r,m}$.

We find that

$$\begin{aligned} f_{m+2r+2,m}-f_{m+2r,m} &= \Delta_{m+2r+2,m+2r}\ /Q_{m+2r}Q_{m+2r+2} \\ &= \Delta_{m+2r+1,m+2r}\ /Q_{m+2r}Q_{m+2r+2} \\ &= \frac{(-1)^m n_1n_2\ldots..n_{m+2r+1}z^{m+2r}}{1+\ldots\ldots+(d_1d_2..d_m n_{m+2}\ldots.n_{m+2r})^2 n_{m+2r+2}z^{2m+2r+1}} \end{aligned}$$

and so for $|z|$ small

$$\left.\begin{aligned} &|f-f_{m+2r,m}| = |n_1n_2..n_{m+2r+1}|\,|z|^{m+2r}\ \{1+0(|z|)\} \\ &\text{and so for } |z| \text{ large} \\ &|f-f_{m+2r,m}| = \left|\frac{n_1n_2..n_{m+2r+1}}{(d_1d_2..d_m n_{m+2}..n_{m+2r})^2 n_{m+2r+2}}\right| \frac{1}{|z|^{m+1}}\{1+0(\frac{1}{|z|})\} \end{aligned}\right\} \qquad (5.13)$$

for r= 1,2,3, ...

These are also expressible simply in terms of determinants, being respectively

$$\left.\begin{aligned} |f-f_{m+2r,m}| &= \left|\frac{D_{r,m+r}}{D_{r-1,m+r-1}}\right| |z|^{m+2r}\ \{1+0(|z|)\} \\ |f-f_{m+2r,m}| &= \left|\frac{D^2_{r,m+r}}{D_{r,m+r-1}D_{r+1,m+r}}\right| \frac{1}{|z|^{m+1}}\{1+0(\frac{1}{|z|})\} \end{aligned}\right\} \qquad (5.14)$$

for r = 1,2,3, ...

If in (5.12) all the n's and d's are positive we have for r =1,2,3 ...

$$| f - f_{m+r,m} | < \frac{(n_1 n_2 .....n_{m+1}) x^{m+r}}{Q_{m+r} Q_{m+r+1}} \qquad (5.15)$$

but for this case. corresponding to (5.5), x_{max} does not have a simple expression.

On the other hand if the conditions of (iii) hold for the first part of the continued fraction (5.12) i.e. $d_r > 0$, $d_r + n_{r+1} \geq 0$ for r=1,2,...m, and subsequently $n_{m+2r} > 0$, $n_{m+2r+1} < 0$ for r==1,2,3.. such that $n_{m+2r} + n_{m+2r+1} \geq 0$ all the denominator nolynoinials of (5.12) have all coefficients positive. For these conditions $f_{m+2r,m}$ and $f_{m+2r+2,m}$ lie on opposite sides of f and

$$\left. \begin{aligned} & | f - f_{m+2r,m} | < \frac{(n_1 n_2 .....n_{m+2r+1}) x^{m+2r}}{Q_{m+2r} Q_{m+2r+2}} \\ & \text{with} \\ & x_{max} \approx [(d_1 d_2 ..d_m n_{m+2} ..n_{m+2r+2})^2 n_{m+2r+2}]^{-\frac{1}{2m+2r+1}} \end{aligned} \right\} \qquad (5.16)$$

for r =1,2,3 ...

<u>Remarks</u>

So far we have considered expansions about the points z=0 and $z = \infty$. Suitable expansions about the finite points w=0 and w = 1 can be brought into the previous form by z=w/(l-w). Corresponding to (2.12) we would have the continued fraction

$$\frac{n_1(1-w)}{1+(d_1-1)w_+} \frac{n_2 w(1-w)}{1+(d_2-1)w_{+...+}} \frac{n_m w(1-w)}{1+(d_3-1)w_+} \frac{n_{m+1}w}{1_+} \frac{n_{m+2}w}{1_+} \frac{n_{m+3}w}{1_+} \frac{n_{m+4}w}{1_{+.......}}$$

Also, when fitting unequal numbers of terms we have chosen to fit more of(1.1) then (1.2). Clearly if required we could arrange to fit more terms of (1.2) than (1.1), for example by replacing z by 1/z.

Furthermore we have assumed that the two series (1.1) and (1.2) are expansions of the same function. In fact the methods outlined in Sections 2 and 3 are both purely formal methods for transforming the series into continued fractions. Thus they would work just as well if the series (1.1) and (1.2) arose from independent functions. Such continued fractions are not considered here.

Applications

Ex.1 For certain functions which possess expansions of the appropriate type, it is often easier to construct the continued fractions using known properties of the function rather than using the formal method. Murphy (1971) derives by elenientary means the result

$$\frac{1}{\sqrt{(1+z^2)}} = \frac{1}{1+z-}\,\frac{z}{1+z-}\,\frac{z/2}{1+z-}\,\frac{z/2}{1+z-}$$

for real $z \geq 0$, An investigation of the convergence of this continued fraction [Wall (1948) p 391 shows that the result holds for z-complex provided the z-plane is cut from -i to + i along that part of the unit circle which lies in the left hand half plane.

w'e also find that for $m \geq 3$

$$\frac{1}{\sqrt{(1+z^2)}} = \underbrace{\frac{1}{(1+z)-}\,\frac{z}{(1+z)-}\,\frac{z/2}{(1+z)-}\cdots-\frac{z/2}{(1+z)-}}_{m\ \text{terms}}\,\frac{z/2}{1-}\,\frac{z/2}{1+}\,\frac{z/2}{1-}$$

in the z - plane cut along the imaginary axis from i to $i\infty$ and from -i to $-i\infty$. Using the previous notation with $f = (1+z^2)^{-\frac{1}{2}}$ the convergents of the above continued fraction are

$$f_{1,1},\ f_{2,2},\ldots,\ f_{m,m},\ f_{m+1,m-1},\ f_{m+2,m}\ f_{m+2r-1,m-r}, f_{m+2r,m}, \ldots$$

for $r = 1,2,3, \ldots$.

The continued fractions corresponding to m= 1 ,2 are easily formed.

For the above continued fractions it is possible to express exactly the error between the function and a given convergent [Murphy (1971)].

In particular on the positive real axis the error has a single maximum. If we consider the error $|f - f_{M+r,M-r}|$ where 2M, fixed, is the total number of terms fitted we find that the maximum error is least when r=0. When r =0 the position of the maximum error is at $x\approx 1$ and as expected moves to the right as r increases. This type of behaviour is also indicated by the estimates of x_{max} given in (5.5) and (5.16).

This example indicates that in forming rational function approximations to f (z) with known power series at z =0 it may be worth taking into account any known behaviour of f(z) at z =∞.

<u>Ex.2</u> The function $\cot^{-1} z$ possesses expansions of the required type

$$\cot^{-1} z = \pi/2 - z + z^3/3 - z^5/5 + \ldots\ldots\ldots\ldots\ldots, |z| < 1$$

$$= \frac{1}{z} - \frac{1}{3z^3} + \frac{1}{5z^5} - \qquad\qquad , |z| > 1$$

It is found that $\cot^{-1} z$ can be expressed as a continued fraction in the form

$$\cot^{-1} z = \frac{n_1}{1 + d_1 z_+} \frac{n_2 z}{1 + d_2 z_{+\ldots\ldots+}} \frac{n_m z}{1 + d_m z_{+\ldots\ldots}}$$

the result holding in the z-plane cut from -i to + i along the unit circle in the left hand plane and where the first 10 values for the n's and d's are

1	1.570 796 327	1.570 796 327
2	-0.934 176 554	0.934 176 554
3	-0.500 334 865	0.979 385 145
4	-0.505 199 058	0.992 512 632
5	-0.504 693 025	0.996 725 229
6	-0.503 610 746	0.998 311 322
7	-0.502 739 835	0.999 015 076
8	-0.502 115 905	0.999 373 255
9	-0.501 672 261	0.999 575 460
10	-0.501 350 824	0.999 698 696

The coefficients have been calculated using both the modified n-d method and the determinant method.

Noting that condition (ii) of (5.4) holds, the result (5.9) shows that the 10th convergent of this continued fraction is sufficient to provide an approximation which gives an accuracy of 7 decimal places to $\cot^{-1} x$ everywhere on the positive real axis. Moreover for this purpose it is necessary only to know the n's and d's correct to the number of decimals to the left of the dotted steps in the tables, i.e.

$$\cot^{-1} x = \frac{1.570\ 7963}{1 + 1.570\ 7963x} - \frac{0.934\ 1766x}{1 + 0.934\ 1766x} - \frac{0.500\ 335x}{+1 + 0.979\ 385x} \ldots\ldots - \frac{0.5x}{1 + x}$$

The figures to the right of the dotted steps are brought into use in calculating $\cot^{-1}$ z for z not on the positive real axis.

The continued fraction for cot z has also been constructed using the fact that the differential coefficient of $\cot^{-1}$ z is a rational function of z. This method leads to recurrence relationships between the n's and d's, but these relationships did not prove useful for numerical purposes.

Noting in this example the close connection between the coefficients in the two series expansions of $\cot^{-1}$ z, D.M.Drew (Mathematics Department, Brunel University) considered the function f(z)s (π-4- $\tan^{-1}$ z)/π(1-z) which has series expansions of the form

$$f(z) = A_0 + A_1 z + A_2 z^2 + \qquad , \quad |z|<|,$$

$$= \frac{A_0}{z} + \frac{A_1}{z^2} + \frac{A_2}{z^3} + \qquad , \quad |z|>|$$

The only singularities of this function are the branch points $z = \pm i$, there being no pole at $z = 1$. The merit of this function is that in the continued fraction of type (2.11) $d_m = 1$, m=1,2,3, ... as is clear from considerations of symmetry. The resulting continued fraction for f(z) is

$$\frac{n_1}{1+z+} \; \frac{n_2 z}{1+z+----}$$

where the first 10 n's are

n	**n_m**
1	**1.000 000 000**
2	**-0.726 760 455**
3	**-0.521 301 151**
4	**-0.511 469 660**
5	**-0.506 931 221**
6	**-0.504 582 129**
7	**-0.503 238 733**
8	**-0.502 406 203**
9	**-0.501 856 712**
10	**-0.501 475 638**

In this example the 10th convergent can produce f(x) to 9 decimal places everywhere on the positive real axis. For this purpose the values of n_m need only be the corrected values to the left of the dotted steps.

The continued fraction holds in the z-plane out from -i to +i along the part of the unit circle lying in the left hand half plane.

It is noticed that in all the examples considered so far $n_m = -\frac{1}{2}$ or $n_m \to -\frac{1}{2}$ as $m \to \infty$ (save for n_1) and $d_m=1$ or $d_m \to 1$ as $m \to \infty$.

This phenomenon is connected with the fact the examples all have the same branch points, namely z=± i. This connection will not be pursued here.

Ex,3. As an example of a function expanded about two finite points, consider $\cos\frac{1}{2}\pi w$ and the points w=0 and w=1. Setting z =w/(l-w) and

$f(z) = \cos,\left(\frac{\pi}{2}\frac{z}{1+z}\right)$ we have near z =0

$$f(z) = 1 - \frac{1}{2!}\left(\frac{\pi}{2}\frac{z}{1+z}\right)^2 + \frac{1}{4!}\left(\frac{\pi}{2}\frac{z}{1+z}\right)^4$$

and near $z = \infty$

$$f(z) = \left(\frac{\pi}{2}\frac{z}{1+z}\right) + \frac{1}{3!}\left(\frac{\pi}{2}\frac{z}{1+z}\right)^3 +$$

both of which may be developed in the conventional form. These lead to the continued fraction

$$\cos\frac{1}{2}\pi w = \frac{n_1(1-w)}{1+(d_1-1)w} + \frac{n_2 w(1-w)}{1+(d_2-1)w} + \frac{n_3 w(1-w)}{1+(d_3-1)w} + \ldots$$

where the first five n's and d's are

m	n_m	d_m
1	1.000 000 000	0.636 619 772
2	-0.636 619 772	1.751 938 394
3	0.185 953 899	0.709 222 139
4	-0.061 975 437	1.359 472 097
5	0.029 110 532	0.800 776 250

In this case rounding errors entered the calculations for the n's and d's earlier than in the previous examples. On the other hand one only needs the first five convergents to obtain 9D places accuracy on the range $0 \leq w \leq 1$, and for this purpose one need only use the corrected values of n and d to the left of the dotted steps. Further n's and d's suggested that the n's were tending to and oscillating about 0 whilst the d's were tending to and oscillating about 1.

Since $\cos\frac{1}{2}\pi w$ has *no* singularities, save at $w = \infty$, one would expect the continued fraction to converge for all finite complex w.

<u>Ex 4.</u> The Error Function Erf(z), defined by

$$\mathrm{Erf}(z) \equiv \frac{2}{\sqrt{\pi}}\int_0^z e^{-t^2}dt$$

has its only singularity at z - ∞. Near z = ∞, the asymptotic expansion

$$\mathrm{Erf}(z) \sim 1 - \frac{2}{\sqrt{\pi}}e^{-z^2}\{\frac{1}{2z} - \frac{2!}{1!(2z)^3} + \frac{4!}{2!(2z)^5} - \ldots\ldots\}$$

holds for |arg z| < 3π/4.

This expansion is not of the required form, but the function $f(z) \equiv \frac{\sqrt{\pi}}{2}e^{z^2}(1-\mathrm{Erf}(z)) \equiv \frac{\sqrt{\pi}}{2}e^{z^2}$ Erfc(z) possesses suitable expansions near z = 0 and z =∞. Thus

$$f(z) = \frac{\sqrt{\pi}}{2} - z + z^2\frac{\sqrt{\pi}}{2} - \frac{2}{3}z^3 + .$$

for |z| small and

$$f(z) \sim \frac{1}{2z} - \frac{2!}{1!(2z)^3} + \frac{4!}{2!(2z)^5}$$

for |z| large, and | arg z | < 3π/4.

These lead to a continued fraction of the form (2.11) with the first 3 n's and d's given by

m	n_m	d_m
1	0.886 226 925	1.772 453 851
2	-0.644 074 684	0.644 074 684
3	-0.219 838 875	0.512 201 755
4	-0.172 428 334	0.435 068 863
5	-0.145 805 541	0.384 273 998
6	-0.128 527 432	0.347 808 652
7	-0.115 961 794	0.319 361 904
8	-0.094 377 846	0.262 185 218

In this example, on the positive real axis the maximum error of the m^{th} convergent moves to the right as m increases. For the 8^{th}

convergent the maximum error occurs at $x \approx 2$ and we find that the 6th convergent produces an accuracy of 8D places over the whole positive real axis. For this purpose we need only use the corrected values to the left of the dotted steps.

This continued fraction in fact, holds for all z but since the asymptotic expansion of f(z) for |z| large changes for the range $3\pi/4 \leq \arg z \leq 5\pi/4$ convergence will be slow in this region.

However, since

$$\text{Erfc}(z) + \text{Erfc}(-z) = 2$$

we need only use the continued fraction for $|\arg z| \leq \pi/2$.

The closely allied Dawson's integral

$$F(z) = e^{-z^2} \int_0^{z} e^{t^2} dt$$

has a particularly neat continued fraction of the form (2.11). The only singularity of F(z) is at $z = \infty$. Near $z = 0$

$$F(z) = z - \frac{2z^3}{3} + \frac{4z^5}{13} - \frac{8z^7}{105} +$$

and F(z) has the asymptotic expansion

$$F(z) \sim \frac{1}{2z} + \frac{1}{4z^3} + \frac{3}{8z^5} + \frac{15}{(16z)^7} +$$

for $|z|$ large and $|\arg z| < \pi/4$.

Clearly F(z)/z possesses expansions of the required form in the variable z^2. The continued fraction is found to be

$$F(z) = \frac{z}{1 + 2z^2 -} \quad \frac{4z^2}{3 + 2z^2 -} \quad \frac{8z^2}{5 + 2z^2 -} \cdots \frac{4(m-1)z^2}{-2m - 1 + 2z^2 -} \cdots$$

which holds for all z although for the region $\pi/4 < |\arg z| < 3\pi/4$ convergence will be slow. Even on the real axis convergence is slow but here this is partly compensated by knowing all the terms of the continued fraction. The maximum error of the m^{th} convergent on the real axis occurs at $x \approx \pm\sqrt{m/e}$. Taking $m = 11$ so that $x \approx 2$ the maximum error is < 0.0002. This continued fraction was given by Wynn (1959)•

Ex. 5 Finally, we consider an example of a continued fraction of type (2.12) arising from the hypergeometric function F(a, b; c; z). This function has branch points at $z \approx 1$ and $z = \infty$. For $|z| < 1$

$$F(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}$$

where $(a)_n \Gamma(a+n)/\Gamma(a)$ and $c \neq 0, -1, -2, \ldots..$ In the z- plane cut from 1 to ∞ along the positive real axis we have for $|z| \geq 1$

$$F(a;b;c;z) = \frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)} (-z)^{-a} F(a,1+a-c;1+a-b;z^{-1})$$

$$+\frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)} (-z)^{-b} F(b,1+b-c;1+b-a;z^{-1}) \quad ,$$

[See for example Copson (1944)p 257].

Thus the particular function $F(1, \beta+1; \gamma+1; -z)$ has its singularities at $z = -1$ and $z = \infty$ and is single valued in the z-plane cut between these two points along the negative real axis. Near $z = 0$ it has a power series of the usual type. Near $z = \infty$, we see from the previous result that the first series has the required type but that the second series, for general β, does not. Assuming $R1\,\beta > 0$, if we choose $m = [R1\,\beta] + 1$ for $R1\,\beta$ not an integer ([ ] integer part of) and $m = R1\,\beta$ for $R1\,\beta$ an integer we could construct a continued fraction of the form (2.12) using m terms of the expansion near $z = \infty$. It is found that

$$F(1,ß+1;y+1;-z)=\frac{y}{y+ßz+}\;\frac{(y-ß)z}{(y+1)+(ß-1)z+}\;\frac{2(y-ß+1)z}{(y+2)+(ß-2)z+..}\;\frac{(m-1)(y-ß+m-2)z}{+(y+m-1)+(ß-m+1)z+}$$

$$\frac{m(y-ß+m-1)z}{(y+m)}+\frac{y(ß+1)z}{(y+m+1)}+\ldots\ldots+\frac{r(y-ß+r-1)z}{(y+2r-m)}+\frac{(y+r-m)(ß+r-m+1)z}{(y+2y-m+1)}+\ldots\ldots$$

where r ≥ m, y = 1-1,-2.... . This continued fraction holds for all z in the z plane cut from –1 to -∞ along the negative real axis.

This result is useful for R1 ß>0 with m chosen as above. For m =0 and a slight modification of the first term the formula reduces to the continued fraction of Gauss (Wall 1948 Ch.XIII).

Assuming R1b≥R1a it is clear that the function $(1+z)^{a-1}$ F(a,b; c;- z) exhibits near z = 0 and z = ∞ properties similar to those of the case just considered. Consequently we can develop it - at least numerically - in a continued fraction of the type (2.12).

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<u>References</u>

Copson, E.T. (1935) Theory of Functions of a Complex Variable.
Oxford University Press

Copson, E.T. (1965) Asymptotic Expansions, Cambridge Tracts in
Mathematics and Mathematical Physics.
Cambridge University Press

Henrici, P. (1958) The Quotient-Differenee Algorithm.
Nat.Bur.Stand.Appl,Math.Series 49

Murphy, J.A. (1971) Certain Rational Function Approximations to $(1+x^2)^{-\frac{1}{2}}$.
J. Inst.Maths.Applies.7

Perron, 0, (1970) Die Lehre von den Kettenbruchen.
Teubner, Stuttgart.

Rutishauser, H. (1950 Der Quotienten-Differenzen - Algorithmus ,
Z.Angew. Math.Physik 5.

Wall, H.S. (1948) Analytic Theory of Continued Fractions.
Van Nostrand.

Wynn, P. (1959) Converging Factors for Continued Fractions .
Numerische Mathematik Vol.1.